PHYS 404
HANDOUT 12-*Fourier Series*

1. Obtain a Fourier series for the function $f(x)$ defined as follows:

$$f(t) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases} \qquad \text{period } 2\pi$$

(Adv. Math. p. 313)

2. Obtain a Fourier series for the function given by

$$f(t) = \begin{cases} 1 + (2x/\pi) & -\pi < x \leq 0 \\ 1 - (2x/\pi) & 0 \leq x < \pi \end{cases} \qquad \text{period } 2\pi$$

(Adv. Math. p. 314)

3. An alternating current after passing through a rectifier has the form

$$i = \begin{cases} I_0 \sin\theta & 0 < \theta \leq \pi \\ 0 & \pi \leq \theta < 2\pi \end{cases} \qquad \text{period } 2\pi$$

Express it in a Fourier series.

(Adv. Math. p. 315)

4. Expand $\sin^2 x$ in the range $0 < x < \pi$, (i) in a sine series, (ii) in a cosine series.

(Adv. Math. p. 319)

5. Find a sine series to represent the trapezoidal function :

$$f(x) = \begin{cases} 4x/l & 0 \leq x < l/4 \\ 1 & l/4 \leq x < 3l/4 \\ 4(1 - x/l) & l/4 \leq x < l \end{cases}$$

(Adv. Math. p. 322)

6. Find the Fourier series for the function:

$$f(t) = \begin{cases} -1 & -T/2 < t < 0 \\ 1 & 0 < t < T/2 \end{cases} \qquad \text{period } T$$

(Sch. p. 6)

7. Find the Fourier series for the function:

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ t/\pi & 0 < t < \pi \end{cases} \qquad \text{period } 2\pi$$

(Sch. p. 7)

8. Find the Fourier series for the function:

$$f(t)=\begin{cases}1+(4t/T) & -T/2<t\le 0\\ 1-(4t/T) & 0\le t<T/2\end{cases}\qquad \text{period } T$$

(Sch. p. 14)

9. Find the Fourier series for the function:

$$f(t)=\begin{cases}0 & -T/2<t\le 0\\ A\sin\omega_0 t & 0\le t<T/2\end{cases}\qquad \text{period } T$$

(Sch. p. 15)

10. Expand $f(t)=\sin^5 t$ in Fourier series.

(Sch. p. 16)

11. In the analysis of a complex waveform (ocean tides. Earthquakes, musical tones, etc.) it might be more convenient to have the Fourier series written as:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(nx-\theta_n\right).$$

Show that this is equivalent to a Fourier series with

$$\begin{aligned}a_n&\to a_n\cos\theta_n, & a_n^2&\to a_n^2+b_n^2\\ b_n&\to a_n\sin\theta_n, & \tan\theta_n&=b_n/a_n\end{aligned}.$$

12. A function $f(x)$ is expanded in an exponential Fourier series

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{inx}$$

If this function is real, what restriction is imposed on the coefficients c_n?

13. Apply the summation technique to show that

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\begin{cases}(\pi-x)/2, & 0<x\le\pi\\ -(\pi+x)/2, & -\pi\le x<\pi\end{cases}.$$

14. Find a Fourier series to represent x in the range $(-\pi,\ \pi)$.

(Adv. Math. p. 311)

15. Find the complex Fourier series of the sawtooth function defined by

$$f(t)=At/T,\qquad 0<t<T,\quad \text{period } T.$$

(Sch. p. 47)

16. Find the complex Fourier series of the rectified sine wave periodic functions defined by

$$f(t) = A\sin \pi t, \qquad 0 < t < 1, \quad \text{period 1.}$$

(Sch. p. 48)

17. Show that a time displacement τ in a periodic function has no effect on the magnitude spectrum, but changes the phase spectrum.

(Sch. p. 50)

18. Expand the function

$$f(x) = x^2, \qquad -\pi < x < \pi,$$

and show that it is related to the so called Riemann zeta function.

(Arf. p. 772)

19. Expand the function

$$f(x) = \begin{cases} 1 & x^2 < x_0^2 \\ 0 & x^2 < x_0^2 \end{cases}$$

in the interval $\left[-\pi, \pi\right]$.

20. Find the Fourier series representation of

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$

From your Fourier series show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

(Arf. p. 777)

21. Show that the integration of the Fourier expansion of $f(x) = x$, $-\pi < x < \pi$, leads to

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \left(-1\right)^{n+1} n^{-2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$$

(Arf. p. 780)

22. Prove the power content relation.