PHYS 301
HANDOUT 6
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1. Calculate the integral of the function $f(z) = z^*$ a) on a circle centered at the origin of the axes and having radius equal to one, b) for a contour from $z = 0$ to $z = 1$ and then $z = 1+i$, c) on a line segment from $z = 0$ to $z = 1+i$.

(Ans: a) $2\pi i$, b) $1+i$ c) 1)

2. Calculate the integral of the function $f(z) = 1/z$ a) on any simple closed contour which does not include the origin of axes b) on a closed loop which contains the origin of axes.

(Ans: a) 0, b) $2\pi i$)

3. Calculate the integral of the function $f(z) = z^n$ (where n is an integer).

4. Calculate the integral of the function $f(z) = 1/z^n$ (where n is an integer different than zero).

(Ans: 0)

5. Calculate the integral

$$I = \int_C \frac{e^\zeta}{\zeta^n} d\zeta$$

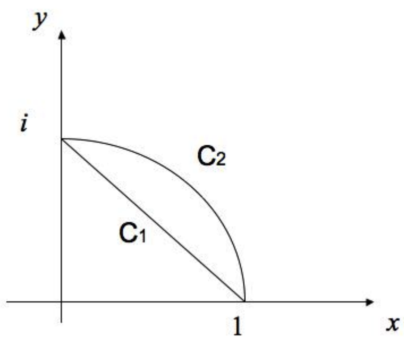
along a contour which contains but does not pass through the point $z = 0$.

(Ans: $2\pi i$)

6. Calculate the integral (Ver. 53)

$$I = \int_C \frac{P(\zeta)}{(\zeta - z)^{n+1}} d\zeta \, .$$

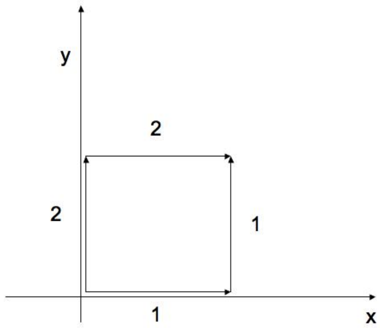
7. Evaluate the integral $\int (1/z)dz$ along the two paths show in the figure below.



8. Verify that

$$\int_{0,\,0}^{1,\,1} z^* dz$$

depends on the path by evaluating the integral for the two paths shown in the figure below. Recall that $f(z) = z^*$ is not an analytic function of z and that Cauchy's integral theorem therefore does not apply



9. Show that

$$\int_C \frac{dz}{z^2 + z} = 0$$

in which the contour C is a circle with radius larger than 1.
Hint. Direct use of the Cauchy integral theorem is illegal. Why? The integral may be evaluated by transforming to polar coordinates.

10. Show that

$$\int_C (z - z_0)^n \, dz = \begin{cases} 2\pi i, & n = -1 \\ 0, & n \neq -1 \end{cases}$$

where the contour C encircles the point $z = z_0$ in a positive (counterclockwise) sense. The exponent n is an integer.

11. Evaluate

$$\int_C \frac{dz}{z^2 - 1} dz$$

where C is the circle with radius 2 and centred at the origin.

12. Evaluate the line integral

$$\int_C \frac{(z^3 + z^2 + z + 1)}{z^4} dz$$

where C is the lower quarter centered at 0 joining $\frac{-1-i}{\sqrt{2}}$ and $\frac{-1+i}{\sqrt{2}}$ in the positive (counterclockwise) sense.

(Ans: $\frac{-2\sqrt{2}}{3} + i\left(\frac{\pi}{2} - 1\right)$)

13. Evaluate the integral

$$\frac{1}{2\pi i}\int_C \frac{e^z}{z-2}dz$$

if C is the circumference (a) $|z| = 3$, (b) $|z| = 1$. (Sch. 134)

(Ans: (a) e^2, (b) 0)

14. Evaluate the integral

$$\int_C \frac{\sin(3z)}{z+\pi/2}dz$$

if C is the circumference. (Sch. 134)

(Ans: $2\pi i$)

15. Evaluate the integral

$$\int_C \frac{e^{3z}}{z-\pi i}dz$$

if C is the circumference (a) $|z-1| = 4$, (b) the ellipse $|z-2|+|z+2| = 6$. (Sch. 134)

(Ans: (a) $-2\pi i$, (b) 0)

16. Evaluate the integral

$$\frac{1}{2\pi i}\int_C \frac{\cos(\pi z)}{z^2 - i}dz$$

on an orthogonal which has its vertices at the points (a) $2 \pm i$, $-2 \pm i$ (b) $-i$, $2-i$, $2+i$, i. (Sch. 134)

(Ans: (a) 0, (b) -1/2)

17. Show that

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}dz = \sin t$$

if $t > 0$ and C is the circle $|z| = 2$. (Sch. 134)

18. Assuming that $f(z)$ is analytic on and within a closed contour C and that the point z_0 is within C, show that

$$\int_C \frac{f'(z)}{(z-z_0)}dz = \int_C \frac{f(z)}{(z-z_0)^2}dz$$

19. You know that $f(z)$ is analytic on and within a closed contour C. You suspect that the nth derivative $f^{(n)}(z_0)$ is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{\left(z - z_0\right)^{n+1}} dz .$$

20. If $f(z) = \left(z + 6\right)/(z^2 - 4)$ show that the integral $\int_C f(z)dz$ is:

a) 0 if the contour is the circle $x^2 + y^2 = 1$,

b) $4\pi i$ if C is the circle $\left(x - 2\right)^2 + y^2 = 1$ and

c) -$2\pi i$ if C is the circle $\left(x + 2\right)^2 + y^2 = 1$.

21. Let a polynomial $P(z)$ of degree *n*, having *n* simple roots, none of which lies on a simple closed contour *C*. Calculate the integral,

$$I = \frac{1}{2\pi i}\int_C \frac{P'(z)}{P(z)} dz$$