Phys 103

Chapter 10

Rotation of a Rigid Object About a Fixed Axis

By

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LECTURE OUTLINE

10.1 Angular Position, Velocity, and Acceleration

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Rotational Motion with

Constant Angular Acceleration

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Introduction

When an extended object such as a wheel rotates about its axis, the motion cannot be analyzed by treating the object as a particle because at any given time different parts of the object have different linear velocities and linear accelerations. We can, however, analyze the motion by considering an extended object to be composed of a collection of particles, each of which has its own linear velocity and linear acceleration.

In dealing with a rotating object, analysis is greatly simplified by assuming that the object is rigid. A rigid object is one that is nondeformable—that is, the relative locations of all particles of which the object is composed remain constant. All real objects are deformable to some extent; however, our rigid-object model is useful in many situations in which deformation is negligible.

10.1 Angular Position, Velocity, and Acceleration

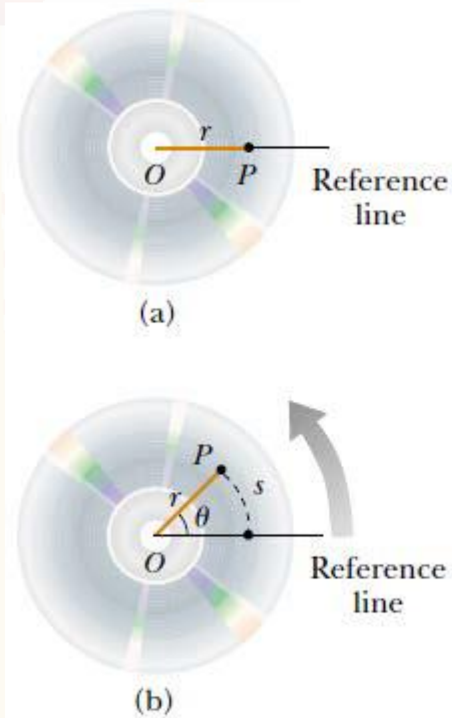
Angular Position

Consider a particle at P is at a fixed distance r from the origin and rotates about it in a circle of radius r.

The particle moves through an arc of length s, as in Figure . The arc length *s* is related to the angle θ through the relationship:

$$s = r\theta \ \rightarrow \theta = \frac{s}{r}$$

Note the dimensions of θ in Equation $\theta = \frac{s}{r}$. Because θ is the ratio of an arc length and the radius of the circle, it is a pure number. However, we commonly give θ the artificial unit radian (rad).



10.1 Angular Position, Velocity, and Acceleration

Angular Speed

As a particle travels from position 1 to position 2 in a time interval Δ, the reference line of length *r* sweeps out an angle Δθ= θf-θi. This quantity Δθis defined as the ***angular displacement*** of the rigid object:

$$\Delta\theta = \theta_f - \theta_i$$

We define the average angular speed as:

$$\bar{\omega} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

the instantaneous angular speed is:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Angular speed has units of radians per second (rad/s)

10.1 Angular Position, Velocity, and Acceleration

Angular Acceleration

The average angular acceleration of a rotating rigid object is defined as:

$$\bar{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

the instantaneous angular acceleration is defined as:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Angular acceleration has units of radians per second squared (rad/s2)

When a rigid object is rotating about a *fixed axis*, every particle on the object rotates through the same angle in a given time interval and has the same angular speed and the same angular acceleration.

That is, the quantities θ, ω, and αcharacterize the rotational motion of the entire rigid object as well as individual particles in the object.

10.2 Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

In our study of linear motion, we found that the simplest form of motion to analyze is motion under constant linear acceleration.

Likewise, for rotational motion about a fixed axis, the simplest motion to analyze is motion under constant angular acceleration.

Rotational Motion about a fixed axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
${\omega_f}^2 = {\omega_i}^2 + 2\alpha(\theta_f - \theta_i)$	${v_f}^2 = {v_i}^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Notice that these expressions are of the same mathematical form as those for linear motion under constant linear acceleration with the change: $x \to \theta, \omega, a \ \to \alpha$

10.3 Angular and Linear Quantities

With $\theta_i = 0, \alpha = constant\ \omega_f = \omega_i + at,\ \ \theta_f = \boldsymbol{\omega_i t} + \frac{\mathbf{1}}{\mathbf{2}}\alpha \boldsymbol{t^2}\ \ and\ \ \omega_f{}^2 = \omega_i{}^2 + 2\alpha\theta_f$

We shall find relations between linear and angular quantities:

$$\because \omega_f = \omega_i + at$$

$$\because v = \frac{ds}{dt} = \frac{d(r\theta)}{dt} = r\frac{d\theta}{dt} = r\omega$$

$$\because a_t = \frac{dv}{dt} = \frac{d(r\omega)}{dt} = r\frac{d\omega}{dt} = r\alpha$$

$$\because a_c = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \frac{r^2\omega^2}{r} = r\omega^2$$

$$\therefore a = \sqrt{a_t{}^2 + a_c{}^2} = \sqrt{(r\alpha\ )^2 + (r\omega^2)^2} = \sqrt{r^2\alpha^2 + r^2\omega^4}$$

$$\therefore a = r\sqrt{\alpha^2 + \omega^4}$$

a_t: tangential acceleration, a_c: central acceleration, a: total acceleration

10.3 Angular and Linear Quantities

Example 10.1 Rotating Wheel

- A wheel rotates with a constant angular acceleration of 3.50 rad/s^2.

(A) If the angular speed of the wheel is 2.00 rad/s at $t_i = 0$, through what angular displacement does the wheel rotate in 2.00 s?

$$\because \quad \theta_i = 0$$

$$\therefore \Delta\theta = \theta_f = (2)(2) + \frac{1}{2}(3.5)(2)^2 = 11\,rad = 630^o$$

(B) Through how many revolutions has the wheel turned during this time interval?

$$\Delta\theta = 630^o \left(\frac{1 \text{ rev}}{360^o}\right) = 1.75 \text{ rev}$$

(C) What is the angular speed of the wheel at t = 2.00 s?

$$\because \omega_f = \omega_i + \alpha t$$

$$\therefore \omega_f = 2 + (3.5)(2) = 9\,rad/s$$

10.4 Rotational Kinetic Energy

Let us consider an object as a collection of particles and assume that it rotates about a fixed z axis with an angular speed ***ω*** If the mass of the ithparticle is $\boldsymbol{m_i}$ and its tangential speed is $\boldsymbol{v_i}$, its kinetic energy is:

$$K_i = \frac{1}{2} m_i {v_i}^2$$

$$\because v_i = r_i \omega$$

$$\therefore K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i {v_i}^2 = \frac{1}{2} \sum_i m_i {v_i}^2 = \frac{1}{2} \sum_i m_i (r_i \omega\ )^2$$

$$= \frac{1}{2} \sum_i m_i {r_i}^2 \omega^2 = \frac{1}{2} \left(\sum_i m_i {r_i}^2 \right) \omega^2$$

define the moment of inertia I as:

$$I = \left(\sum_i m_i {r_i}^2 \right), so \ \ \therefore K_R = \frac{1}{2} I\ \omega^2$$

10.4 Rotational Kinetic Energy

Example 10.3 The Oxygen Molecule

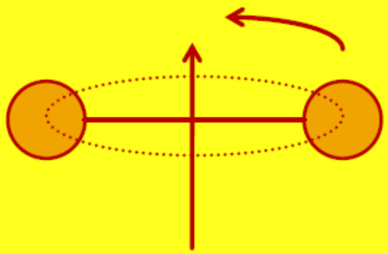
▶Consider an oxygen molecule (O_2) rotating in the xy plane about the z axis. The rotation axis passes through the center of the molecule, perpendicular to its length. The mass of each oxygen atom is $2.66\text{x}10^{-26}$ kg, and at room temperature the average separation between the two atoms is d = $1.21\text{x}10^{-10}$ m. (The atoms are modeled as particles.)

(A) *Calculate the moment of inertia of the molecule about the z axis.*

$$\because\ I=\sum_i m_i r_i^2 = m\left(\frac{d}{2}\right)^2 = m\left(\frac{d}{2}\right)^2 = \frac{1}{2}md^2 \qquad (1)$$

$$\therefore I = \frac{1}{2}(2.66\times10^{-26})(1.21\times10^{-10})^2 = 1.95\times10^{-46}\,kg.m^2 \qquad (2)$$

This is a very small number, consistent with the Minuscule masses and distances involved



Example 10.4 Four Rotating Objects

▶Four tiny spheres are fastened to the ends of two rods of negligible mass lying in the xy plane. We shall assume that the radii of the spheres are small compared with the dimensions of the rods.

▶*(A) If the system rotates about the y axis with an angular speed ω, find the moment of inertia and the rotational kinetic energy about this axis.*

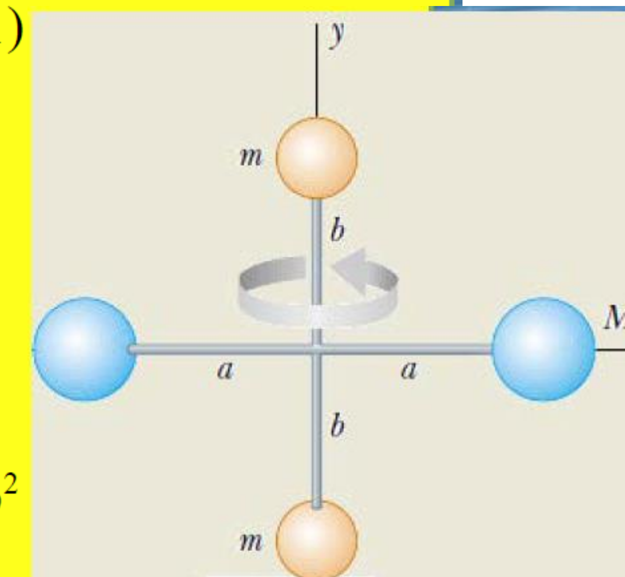
$$\because\ I_y = \sum_i m_i r_i^{\ 2} = Ma^2 + Ma^2 + m(0) + m(0) = 2Ma^2 \quad (1)$$

$$\therefore K_R = \frac{1}{2} I_y \omega^2 = \frac{1}{2}(2Ma^2)\omega^2 = Ma^2\omega^2 \quad (2)$$

▶*(B) Same but in the xy plane about the z axis*

$$I_z = \sum_i m_i r_i^{\ 2} = Ma^2 + Ma^2 + mb^2 + mb^2 = 2Ma^2 + 2mb^2$$

$$\therefore K_R = \frac{1}{2} I_z \omega^2 = \frac{1}{2}(2Ma^2 + 2mb^2)\omega^2 = (Ma^2 + 2mb^2)\omega^2$$



Lecture Summary

If a particle moves in a circular path of radius *r through an angle θ(measured in radians),* the arc length it moves through is ***s = r*$\boldsymbol{\theta}$*.***

The angular position of a rigid object is defined as the angle θbetween a reference line attached to the object and a reference line fixed in space. The angular displacement of a particle moving in a circular path or a rigid object rotating about a fixed axis is $\Delta\theta = \theta_f - \theta_i$.

The instantaneous angular speed of a particle moving in a circular path or of a rigid object rotating about a fixed axis is: $\boldsymbol{\omega = d\theta/dt}$

The instantaneous angular acceleration of a particle moving in a circular path or a rotating rigid object is: $\boldsymbol{\alpha = d\omega/dt}$

When a rigid object rotates about a fixed axis, every part of the object has the same angular speed and the same angular acceleration.

Lecture Summary

If an object rotates about a fixed axis under constant angular acceleration, one can apply equations of kinematics that are analogous to those for linear motion under constant linear acceleration:

With $\theta_i = 0, \alpha = constant$

$$\omega_f = \omega_i + at, \qquad \theta_f = \omega_i t + \frac{1}{2} at^2 \; and \;\; \omega_f{}^2 = \omega_i{}^2 + 2a\theta_f$$

Relationships between linear and rotational quantities:

$$s = r\theta, v = r\omega \quad and \quad a_t = r\alpha$$

The moment of inertia of a system of particles is defined as:

$$I = \left(\sum_i m_i r_i{}^2\right), so \;\; \therefore K_R = \frac{1}{2} I\, \omega^2$$

Thank You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