PHYS 507
Lecture 7: Electrostatics

Special Techniques: Part B

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Separation of Variables -a

- The method of **separation of variables**, is applicable in circumstances where the potential (V) or the charge density (σ) is specified on the boundaries of some region, and we are asked to find the potential in the interior. The strategy is:
- *We look for solutions that are the products of functions, each of which depends on only one of the coordinates.*
- The success of this method hinged on two extraordinary properties of the separable solutions: **completeness** and **orthogonality**.

Separation of Variables -b

- A set of functions $f_n(y)$ is said to be **complete** if any other function $f(y)$ can be expressed as a linear combination of them:

$$f(y) = \sum_{n=1}^{\infty} C_n f_n(y)$$

- A set of functions $f_n(y)$ is said to be **orthogonal** if the integral of the product of any of two different members of the group is zero:

$$\int_0^a f_n(y) f_m(y)\, dy = 0 \quad \text{for} \quad n \neq m$$

Legendre Functions

- Legendre functions or Legendre polynomials are the solutions of Legendre's differential equation that appear when we separate the variables of Helmholtz' equation, Laplace equation or Schrodinger equation using spherical coordinate.
- They are solutions of the following differential equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right]+n(n+1)P_n(x)=0$$

- **The constant *n* is an integer (n=0,1,2,…).**
- $P_n(x)$ **converge only when** $|x|<1$

Legendre Functions

- Legendre functions (or polynomials) is a power series solution of Legendre differential equation about the origin ($x = 0$).

•The series solution should converge to be terminated in order to meet the physical requirements. The series solution converges when $|x| < 1$ is satisfied.

• The Legendre polynomials can be expressed as Rodrigues' Formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Legendre Polynomial

plots

$$P_0(x) = 1$$

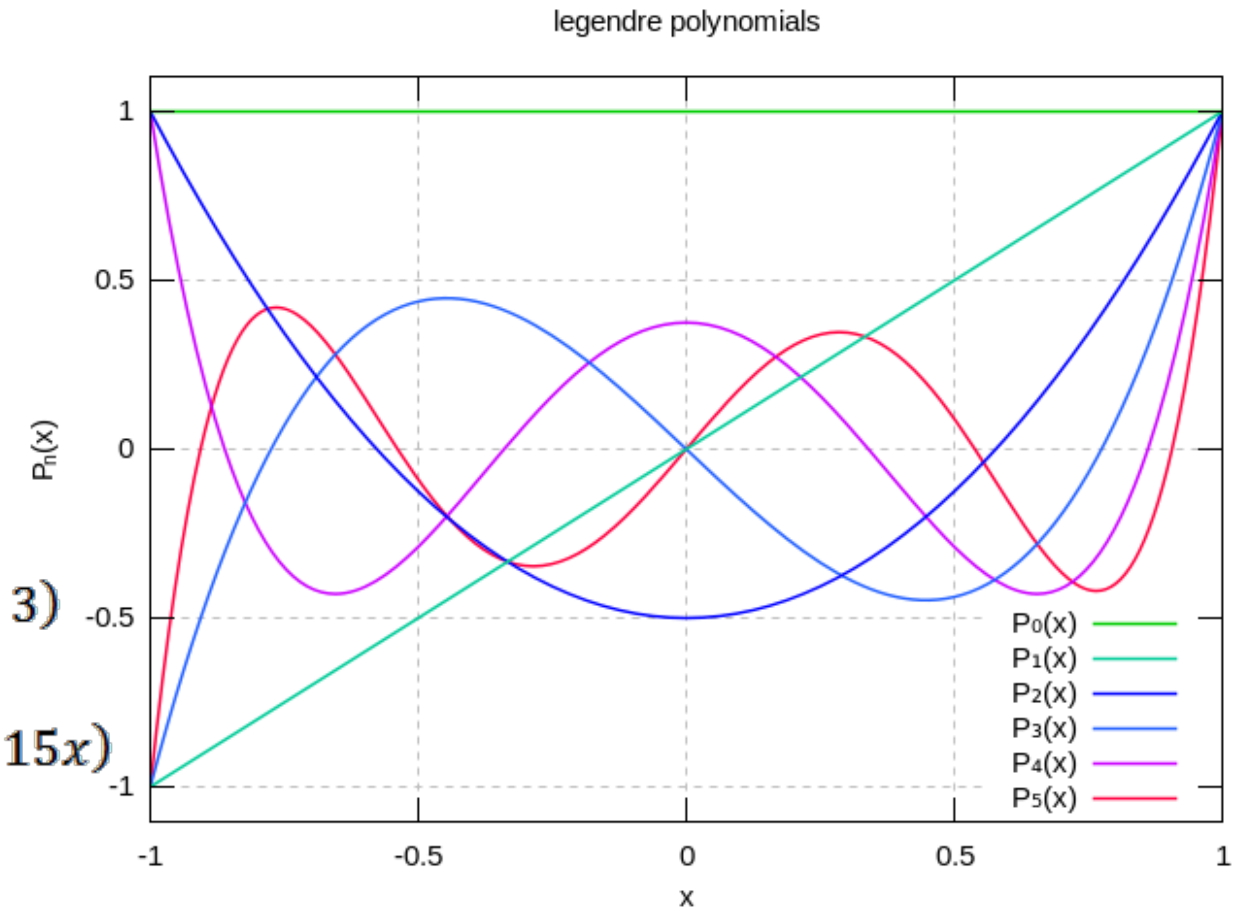
$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$



Legendre Polynomial

The generating function

The *generating* function of Legendre polynomials:

$$g(t,x) = (1 - 2xt + t^2)^{-1/2}$$

Which has an important application in **electric multipole expansions**. If we expand this function as a binomial series if $|t| < 1$ we obtain

$$g(t,x) = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Legendre Polynomials

The recurrence function

the Legendre polynomials obey the three term recurrence relations:

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$$

$$(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x)$$

But these relations are valid for $n = 1,2,3,\ldots$

Legendre Polynomials

Special Properties

- Some special values

$$P_n(1) = 1$$

$$P_n(-1) = (-1)^n$$

and

$$\left.\begin{aligned} P_{2n}(0) &= (-1)^n \frac{(2n-1)!!}{(2n)!!} \\ P_{2n+1}(0) &= 0 \end{aligned}\right\} \quad \text{for } n = 0,1,2,\ldots$$

Where $n!! = \begin{cases} n(n-2)(n-4)\ldots 1 \text{ if } n \text{ is odd} \\ n(n-2)(n-4)\ldots 2 \text{ if } n \text{ is even} \\ 1 \quad\quad\quad\quad \text{if } n = 0 \end{cases}$ (called double factorial)

Legendre Polynomials

Special Properties

- The Parity property:(with respect to $x = 0 \;, \theta = \pi/2$)

$$P_n(-x) = (-1)^n P_n(x)$$

$$P_n(\cos(\pi - \theta)) = (-1)^n P_n(\cos(\theta))$$

If n is odd the parity of the polynomial is odd, but if it is even the parity of the polynomial is even.

- Upper and lower Bounds for $P_n(\cos(\theta))$

$$|P_n(\cos(\theta))| \leq P_n(1) = 1$$

Legendre Polynomials

Orthogonality

* Legendre's equation is self-adjoint. Which satisfies Sturm-Liouiville theory where the solutions are expected to be orthogonal to satisfying certain boundary conditions. Legendre polynomials are a set of orthogonal functions on (-1,1).

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{n,m}$$

where

$$\delta_{n,m} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Legendre Polynomials

legendre Series

- According to Sturm-Liouville theory that Legendre polynomial form a complete set.

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

- The coefficients a_n are obtained by multiplying the series by $P_m(x)$ and integrating in the interval [-1,1]

$$a_n = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) dx = 0$$

Multipole Expansion-a

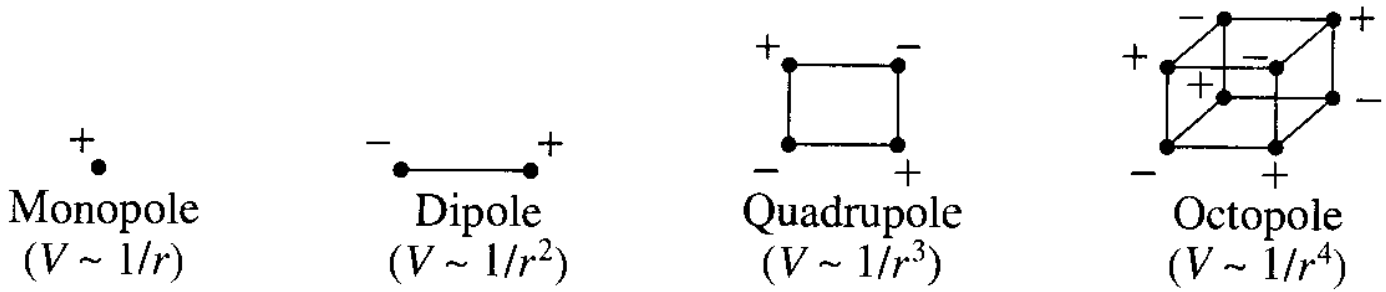
- If you are far away from a localized charge distribution of total charge Q then you approximate at large distances the potential with that of a point charge:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

- But what if this charge is zero? The answer is not as straightforward as it looks. This is where the method of **multipole expansion** is used in electrostatics.

Multipole Expansion-b

- It can be shown that the total potential of the charge configurations shown in the figure has the spatial dependence which is is written below the corresponding configuration:



- Note that in all these configurations the total charge is **zero**!

Multipole Expansion-c

- The method of multipole expansion proves that for an arbitrary localized charge distribution there is an expansion of the total potential in powers of $(1/r)$ given by the relations:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int\left(r'\right)^n P_n\left(\cos\theta'\right)\rho\left(r'\right)d\tau'$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int\rho\left(r'\right)d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho\left(r'\right)d\tau' + \frac{1}{r^3}\int\left(r'\right)^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho\left(r'\right)d\tau' + \ldots\right]$$

The monopole term

- Ordinarily the multipole expansion is dominated (at large *r*) by the **monopole** term:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}, \qquad Q = \int \rho \, d\tau$$

- If the charge is point-like it is the only term. The other multipole terms vanish

The dipole term

- In the case where the total charge is zero the dominant term is the **dipole** term.

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r' \cos\theta' \rho\left(r'\right) d\tau'$$

- This term can be written with the help of **dipole moment** $\mathbf{p} \equiv \int \mathbf{r}'\rho\left(r'\right) d\tau'$ as:

$$V_{dip}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

Dipole moments are vectors and they add accordingly!

Origin of coordinates in multipole expansion

- A point-like charge at the origin is a "pure" monopole.
- A point-like charge shifted from the origin is **no longer** a monopole!
- For instance the point charge in the figure has a dipole moment,

$$\mathbf{p} = qd\hat{\mathbf{y}}$$