PHYS 507
Lecture 6: Electrostatics

Special Techniques

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Introduction

- Finding the electric field and even the potential using their integral definitions is quite a difficult task specially in problems where conductors are involved and maybe the charge density ρ is not known. Since the charge is free to move around the only thing we know is the total charge or perhaps the potential of each conductor.
- We then recast the problem in the form of the Poisson's or Laplace's equations:

$$\nabla^2 V(\mathbf{r}) = \rho / \varepsilon_0 \qquad \nabla^2 V(\mathbf{r}) = 0$$

- These equations together with the proper boundary conditions are equivalent to the integral definitions of the potential.

Laplace's Equation-a

- In the case where the density is zero the Poisson equation is reduced in the Laplace equation which in Cartesian coordinates takes the form:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- This formula is of fundamental importance in electrostatics and appears also in diverse fields of physics like gravity, heat, magnetism and the study of soap bubbles.
- Its solutions are called **harmonics**.

Laplace Equation in one dimension

- In the case of one dimension Laplace equation is

$$\frac{\partial^2 V}{\partial x^2} = 0 \Rightarrow V(x) = mx + b$$

where *m* and *b* are calculated by boundary conditions.

- $V(x)$ is the average of $V(x+a)$ and $V(x-a)$ for any *a*.

$$V(x) = \frac{1}{2}\left[V(x+a) + V(x-a)\right]$$

- Laplace equation tolerates **no local maxima** or **minima.**

Laplace Equation in two dimensions

- In two dimensions Laplace equation takes the form of a partial differential equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

- Its solutions are called harmonic functions and have the following properties:
- A) The value of the V at any point (x, y) is the average of those *around* the point. More precisely if you draw a circle of any radius R about this point.

$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{sphere}} V \, dl$$

- B) V has no local maxima or minima

Laplace Equation in three dimensions

- In three dimensions Laplace equation takes the form of a partial differential equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

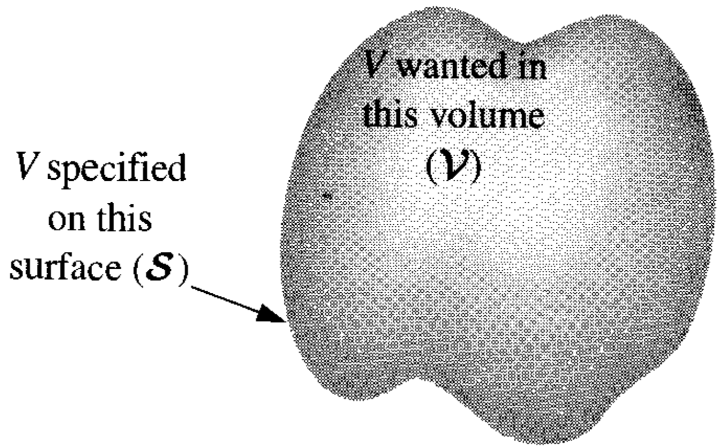
- Its solutions are called harmonic functions and have the following properties:
- A) The value of the V at any point ($x, y,,z$) is the average of those *around* the point. More precisely if you draw a **sphere** of any radius R about this point.

$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{sphere}} V \, da$$

- B) V has no local maxima or minima

Boundary Conditions and Uniqueness Theorem

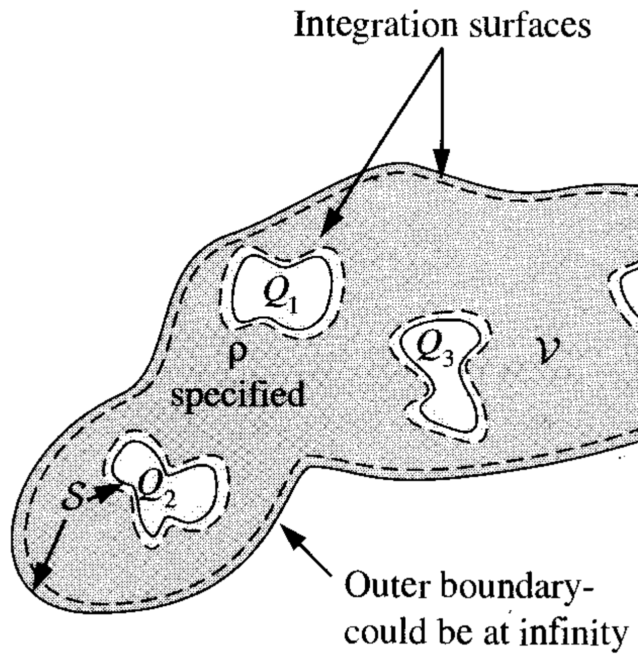
- **First Uniqueness Theorem:** The solution to Laplace's equation in some volume **V** is uniquely determined if the potential *V* is specified on the boundary surface *S*.



The potential in a volume **V** is uniquely specified if a) the charge density throughout the volume and b) the value of *V* on all boundaries are specified

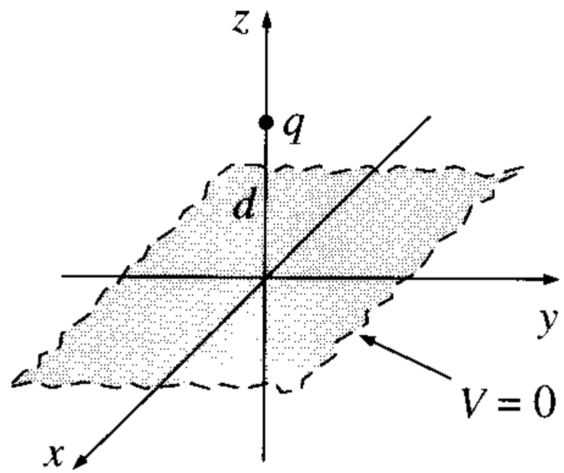
Conductors and the Second Uniqueness Theorem

- **Second Uniqueness Theorem:** In a volume **V** surrounded by conductors and containing a specified charge density *ρ*, the electric field is uniquely determined if the total charge on each conductor is given. (The region as a whole can be bounded by another conductor, or else unbounded)



The Method of Images-a

- Suppose a charge q above a grounded surface. What is the potential above the surface?
- It is not given by $(1/4\pi\varepsilon_0)q/r$ since there is a negative induced charge on the surface. This induced charge contributes to the total potential.
- How, then can we calculate the potential?



The Method of Images-b

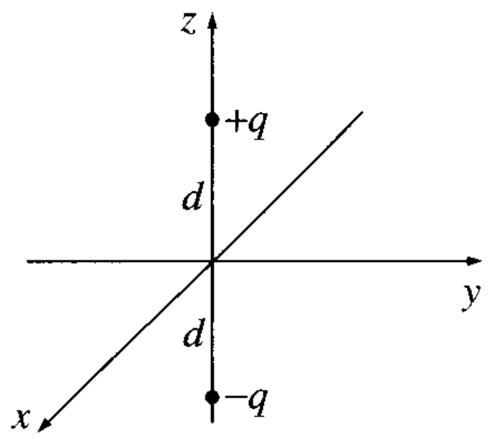
- From a mathematical point of view our problem is to solve the Poisson equation in the region $z > 0$, with a single point charge q at $(0, 0, d)$, subject to the boundary conditions:

1. $V = 0$ when $z = 0$ (since the plane is grounded.
2. $V \rightarrow 0$ far from the charge.

- From the first uniqueness theorem it is guaranteed that there is only one function that meets these requirements. We can guess it by a trick or clever thinking.

The Method of Images-c
The trick

- Forget the original problem. Now think the problem shown in figure: Two opposite charges at symmetric positions with respect to the origin.
- The potential of this configuration is given to the right. And satisfies both conditions of the previous slide!



$$V(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

The Method of Images-c
The trick

- The trick configuration reproduces the correct potential for the original configuration for $z >= 0$ (not for $z < 0$, but we do not care for this!)
- Note the crucial role played by the uniqueness theorem.
- But the story has not finished yet!
- Now that we know the potential we can find the surface charge density as well!

The Method of Images-d
The trick

- We use the following relation for the normal derivative of the potential with respect to the surface:

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$$

- The normal direction to the surface is the *z*-direction so:

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$$

- By doing the calculation we get:

$$\sigma(x,y) = \frac{-qd}{2\pi\left(x^2 + y^2 + d^2\right)^{3/2}}$$

What do you expect to get if you integrate the surface charge density over the whole surface?