

PHYS 507
Lecture 12: Maxwell's Equations

Dr. Vasileios Lempesis

Electrodynamics before Maxwell

- The following relations represent electrodynamics when Maxwell started his work:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

Gauss's law

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Ampère's law

- But there is an inconsistency here in Ampère's law!

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J}$$

This is zero!

This is not
always zero!

How Maxwell restored the inconsistency?

- Maxwell modified the last relation as follows:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- This was done because

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial (\epsilon_0 \nabla \cdot \mathbf{E})}{\partial t} = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

- Such modification changes nothing, as far as magnetostatics is concerned: when \mathbf{E} is constant, we still have $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. The additive term is difficult to be detected in laboratory experiments. However, it plays an important role in the propagation of e/m waves.

The importance of the extra term

- The physical meaning of this extra term is that as a changing magnetic field induces an electric field (Faraday's law), so

A changing electric field induces a magnetic field

- Maxwell called his extra term the displacement current

$$\mathbf{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Maxwell's equations

<i>i)</i>	$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$	Gauss's law
<i>ii)</i>	$\nabla \cdot \mathbf{B} = 0$	no name
<i>iii)</i>	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	Faraday's law
<i>iv)</i>	$\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$	Ampere's law with Maxwell's correction

Maxwell's equations tell you how charges produce fields. The force law tells you how fields affects charges

Maxwell's equations are differential equations, so they must be supplemented by suitable boundary conditions: fields must be zero at large distances from a localized charge distribution.

Magnetic charge

- The symmetry of Maxwell's equations is spoiled by the charge density ρ . This symmetry would have been restored if we had assumed the existence of a "magnetic charge" density ρ_m and a current of magnetic charge \mathbf{J}_m .

<i>i)</i>	$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho_e$	Gauss's law
<i>ii)</i>	$\nabla \cdot \mathbf{B} = \mu_0 \rho_m$	no name
<i>iii)</i>	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \mathbf{J}_m$	Faraday's law
<i>iv)</i>	$\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}_e$	Ampere's law with Maxwell's correction

- But such quantities have not been found. \mathbf{B} and \mathbf{E} are not on equal footing. Dirac, though, showed that the existence of magnetic charge would explain why electric charge is quantized.

Maxwell's equations in matter

- When we work with materials that are subject to electric and magnetic polarization there is a more convenient way to write them. This is so because inside polarized matter there will be accumulations of “bound” charge and current over which you exert no direct control.
- From the static case we know that electric polarization produces a bound charge density and a magnetic polarization results in a bound current.

$$\rho_b = -\nabla \cdot \mathbf{P}$$

$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

Maxwell's equations in matter

- There is just one new feature to consider in the **non-static** case. Any **change** in the electric polarization involves a flow of (bound) charge \mathbf{J}_p which must be included in the total current. This current density is given by:

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$$

- We must not confuse polarization current with the **bound** current \mathbf{J}_b . The later is associated with **magnetization** of the material and involves spin and orbital motion of electrons. \mathbf{J}_p , by contrast, is the result of the linear motion of charge when the electric polarization changes.
- The definition of the polarization current density is consistent with the continuity equation.

Maxwell's equations in matter

$$i) \quad \nabla \cdot \mathbf{D} = \rho_f$$

$$ii) \quad \nabla \cdot \mathbf{B} = 0$$

$$iii) \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$iv) \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Displacement
current \mathbf{J}_d

Boundary Conditions

- In general, the fields \mathbf{E} , \mathbf{B} , \mathbf{D} and \mathbf{H} will be discontinuous at a boundary between two different media, or at a surface that carries charge density σ or current density \mathbf{K} . The explicit form of these discontinuities can be deduced from Maxwell's equations in their integral form

$$\left. \begin{array}{l} \text{(i)} \quad \oint_S \mathbf{D} \cdot d\mathbf{a} = Q_{f,\text{enc}} \\ \text{(ii)} \quad \oint_S \mathbf{B} \cdot d\mathbf{a} = 0 \end{array} \right\} \text{over any closed surface } S$$

$$\left. \begin{array}{l} \text{(iii)} \quad \oint_P \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \oint_S \mathbf{B} \cdot d\mathbf{a} \\ \text{(iv)} \quad \oint_S \mathbf{H} \cdot d\mathbf{l} = I_{f,\text{enc}} + \frac{d}{dt} \oint_S \mathbf{D} \cdot d\mathbf{a} \end{array} \right\} \text{for any surface } S \text{ bounded by the closed loop } P$$

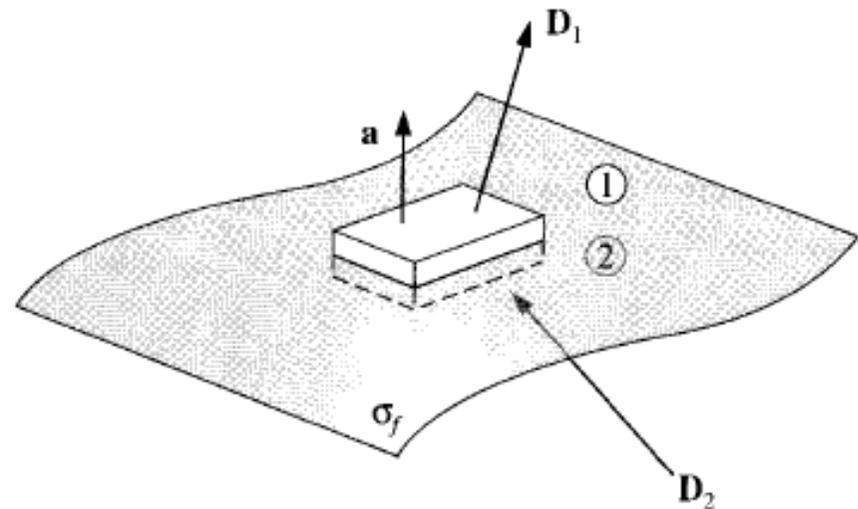
Boundary Conditions

- The component of \mathbf{D} that is perpendicular to the interface is discontinuous in the amount:

$$D_1^\perp - D_2^\perp = \sigma_f$$

- With identical reasoning we have for the magnetic field \mathbf{B} :

$$B_1^\perp - B_2^\perp = 0$$



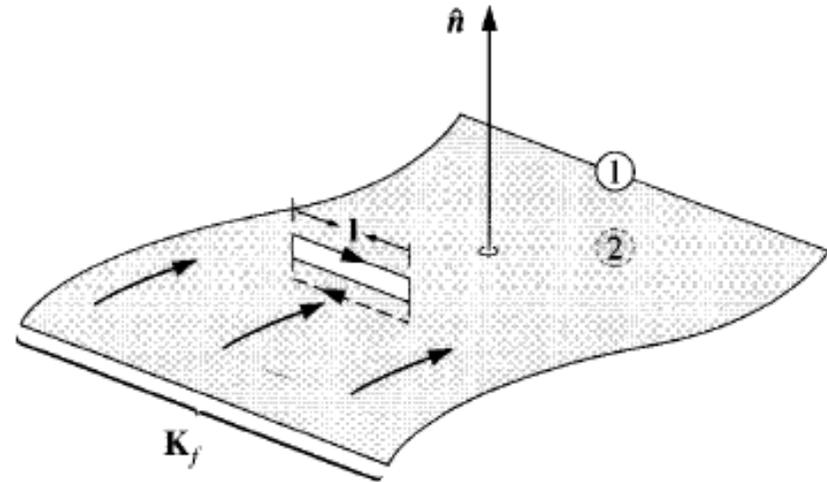
Boundary Conditions

- For the electric field \mathbf{E} we get a continuity for the parallel components:

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$$

- While the parallel components of \mathbf{H} are discontinuous:

$$\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$$



Boundary Conditions

- All the previous equations are the general boundary conditions for electrodynamics. In the case of **linear** media, they can be expressed in terms of **E** and **B** alone:

$$\begin{array}{ll} \text{(i)} \quad \varepsilon_1 E_1^\perp - \varepsilon_2 E_2^\perp = \sigma_f & \text{(iii)} \quad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \\ \text{(ii)} \quad B_1^\perp - B_2^\perp = 0 & \text{(iv)} \quad \frac{1}{\mu_1} \mathbf{B}_1^\parallel - \frac{1}{\mu_2} \mathbf{B}_2^\parallel = \mathbf{K}_f \times \hat{\mathbf{n}} \end{array}$$

- In particular, if there is no free charge or free current at the interface, then

$$\begin{array}{ll} \text{(i)} \quad \varepsilon_1 E_1^\perp - \varepsilon_2 E_2^\perp = 0 & \text{(iii)} \quad \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \\ \text{(ii)} \quad B_1^\perp - B_2^\perp = 0 & \text{(iv)} \quad \frac{1}{\mu_1} \mathbf{B}_1^\parallel - \frac{1}{\mu_2} \mathbf{B}_2^\parallel = 0 \end{array}$$