INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS

Rene Dennemeyer
California State College at San Bernardino

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CONTENTS

EXAMPLE 1-11 $(z^2 - 2yz - y^2)p + (xy + xz)q = xy - xz$
The subsidiary equations are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + xz} = \frac{dz}{xy - xz}$$

Since $Q/(y + z) = R/(y - z)$, it follows that $dy/(y + z) = dz/(y - z)$. Hence $y\,dy - z\,dz = z\,dy + y\,dz$. Integration yields the first integral

$$u(x,y,z) = z^2 - y^2 + 2zy = c_1$$

Also, $xP + yQ + zR = 0$, so $x\,dx + y\,dy + z\,dz = 0$. A second independent integral is

$$v(x,y,z) = x^2 + y^2 + z^2 = c_2$$

The general solution is

$$F(z^2 - y^2 + 2xy,\ x^2 + y^2 + z^2) = 0$$

Characteristic Curves

From a geometric as well as an analytic viewpoint there are many connections between the system of ordinary differential equations (1-39) and the partial differential equation (1-38). Of importance in this regard is the vector field

$$\mathbf{V}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k} \tag{1-44}$$

called the *characteristic vector field* associated with Eq. (1-38). From each point in the region $\mathscr{T}$ there emanates a characteristic vector. Let $z = \varphi(x,y)$ be a solution of Eq. (1-38) in $\mathscr{R}$. Geometrically the solution is visualized as a smooth surface S lying in $\mathscr{T}$, called an *integral surface*. Let $\mathbf{n}$ be the normal vector to S [see Eq. (1-14)]. If $\mathbf{V}$ is the characteristic vector at a point on S, then

$$\mathbf{V} \cdot \mathbf{n} = \frac{Pp + Qq - R}{(p^2 + q^2 + 1)^{1/2}} = 0$$

Accordingly $\mathbf{V}$ lies in the tangent plane to S at each point on S. Conversely, if S is a smooth surface with equation $z = \varphi(x,y)$, and if at each point on S the characteristic vector $\mathbf{V}$ at that point lies in the tangent plane to S, then S is an integral surface, and $z = \varphi(x,y)$ is a solution of Eq. (1-38) (see Fig. 1-2).

A smooth curve C lying in $\mathscr{T}$ and such that the characteristic vector $\mathbf{V}$ at each point on C is tangent to C is called a *characteristic curve* of Eq. (1-38). Accordingly the characteristic curves are just the field lines of the vector field $\mathbf{V}(x,y,z)$. A necessary and sufficient condition for a smooth curve C to be a characteristic curve is that the subsidiary equations (1-39) hold along C. This follows from the fact that a set of direction numbers of the tangent to C is the set of differentials dx, dy, dz, and the subsidiary equations state that these direction numbers are proportional to the set of direction numbers P, Q, R of the vector $\mathbf{V}$; that is, the tangent has the direction of $\mathbf{V}$.

The characteristic curves constitute a two-parameter family of space curves. Exactly one characteristic passes through each point of $\mathcal{T}$. To see this, take x as the independent variable in Eq. (1-40). The system (1-40) satisfies the hypotheses of the fundamental existence theorem for a system of first-order equations (see Ref. 2). Thus, if x_0 is fixed and y_0, z_0 are chosen values, then there exists a unique solution

$$y = y(x,y_0,z_0) \qquad z = z(x,y_0,z_0) \tag{1-45}$$

in a neighborhood of x_0 such that

$$y(x_0,y_0,z_0) = y_0 \qquad z(x_0,y_0,z_0) = z_0 \tag{1-46}$$

In xyz space each function in Eq. (1-45) represents a cylinder, and relation (1-46) states that these cylinders intersect at (x_0,y_0,z_0). In a neighborhood of the point (x_0,y_0,z_0) the cylinders intersect in a smooth curve C which passes through (x_0,y_0,z_0). Since the system (1-40) is satisfied along C, it follows that C is a characteristic curve. By the uniqueness of solution of the system (1-40) subject to conditions (1-46), C is the unique characteristic through (x_0,y_0,z_0). Now regard y_0 and z_0 as parameters. Then the family of curves defined in Eq. (1-45) constitute a two-parameter family of characteristic curves such that exactly one member passes through each point of $\mathcal{T}$. Moreover every characteristic curve of the partial differential equation belongs to this family. Note that the preceding statements imply that distinct characteristic curves cannot intersect.

If S is an integral surface of Eq. (1-38) and C is a characteristic curve, either C does not intersect S, or else C is embedded in S. To show this, suppose $z = \varphi(x,y)$ is the equation of the integral surface S. Let C be a characteristic curve which intersects S at (x_0,y_0,z_0). In a neighborhood of $x = x_0$ there is a unique solution $y = y(x)$ of the differential equation

$$\frac{dy}{dx} = \frac{Q[x,y,\varphi(x,y)]}{P[x,y,\varphi(x,y)]} \tag{1-47}$$

such that $y(x_0) = y_0$. The space curve C' defined by the equations $y = y(x)$, $z = \varphi[x,y(x)]$ passes through the point (x_0,y_0,z_0) and lies in the surface S. Also

$$\frac{dz}{dx} = \varphi_x + \varphi_y y' = \varphi_x + \varphi_y \frac{Q}{P} = \frac{P\varphi_x + Q\varphi_y}{P} = \frac{R}{P}$$

Thus the functions which define C' satisfy the system (1-40); moreover $y(x_0) = y_0$, and $z(x_0) = z_0$. But the functions which define the characteristic C satisfy exactly the same conditions. There can be but one set of functions $y(x)$, $z(x)$ with these properties. Accordingly the characteristic C and the embedded curve C' are one and the same.

Geometric and Analytic Justification of Lagrange's Method

A heuristic geometric argument based on the preceding result may be given in support of the fact that under the present hypotheses on P, Q, R each solution of Eq. (1-38) is defined by a relation of the form in Eq. (1-43), at least in the small. Let S be an integral surface of Eq. (1-38), and let (x_0,y_0,z_0) be a point on S. Choose a smooth curve Γ such that (1) Γ lies in S and passes through (x_0,y_0,z_0), (2) Γ is *noncharacteristic*; i.e., the tangent to Γ is nowhere parallel to the characteristic vector $\mathbf{V}$ along Γ. Through each point of Γ there passes exactly one characteristic C, and C is embedded in S. Also, the tangent to such a characteristic C does not coincide with the tangent to Γ at the point where C intersects Γ. In this manner a subfamily of the family of characteristics is singled out, and in a neighborhood of (x_0,y_0,z_0) this subfamily of characteristic curves generates, or sweeps out, the integral surface S (see Fig. 1-3). The particular subfamily in question is obtained from the two-parameter family of all characteristics by imposing a functional relation $F(c_1,c_2) = 0$ on the parameters c_1, c_2 appearing in Eq. (1-42).

Assume that

$$u(x,y,z) = c_1 \qquad v(x,y,z) = c_2 \tag{1-48}$$

are functionally independent integrals of the subsidiary equations in $\mathscr{T}$. Let (x_0,y_0,z_0) be a fixed but otherwise arbitrarily chosen point of $\mathscr{T}$. Then in a neighborhood of (x_0,y_0,z_0) the surfaces

$$u(x,y,z) = u(x_0,y_0,z_0) \qquad v(x,y,z) = v(x_0,y_0,z_0) \tag{1-49}$$

intersect in a smooth curve C, which passes through the point. The curve C is the characteristic passing through (x_0,y_0,z_0). Choose x as the parameter

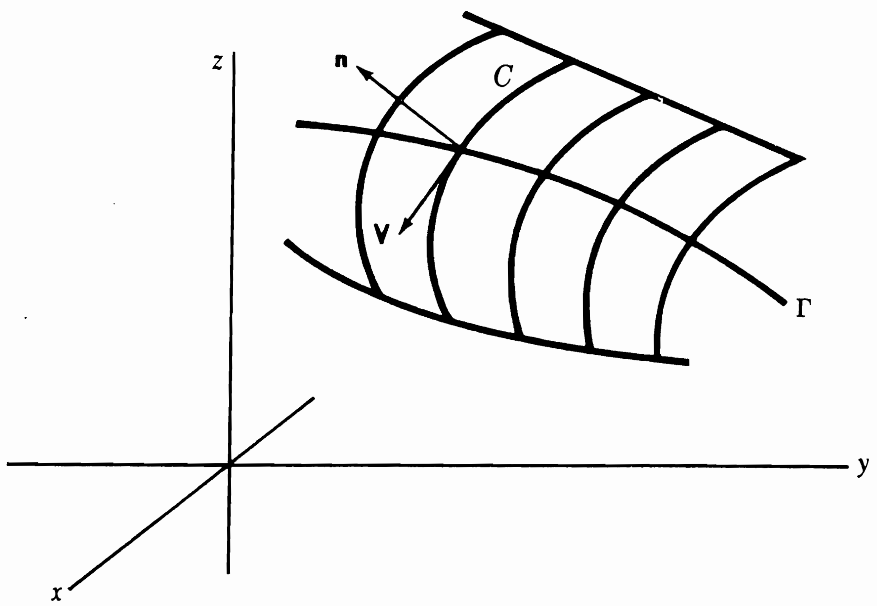

Figure 1-3

along C; then differentiation of Eqs. (1-49) yields

$$u_x + u_y \frac{dy}{dx} + u_z \frac{dz}{dx} = 0 \qquad v_x + v_y \frac{dy}{dx} + v_z \frac{dz}{dx} = 0$$

Hence, from the subsidiary equations (1-40), it follows that

$$Pu_x + Qu_y + Ru_z = 0 \qquad Pv_x + Qv_y + Rv_z = 0 \tag{1-50}$$

These equations hold simultaneously along C. In vector form they are

$$\mathbf{V} \cdot \nabla u = 0 \qquad \mathbf{V} \cdot \nabla v = 0$$

Thus (1-50) is just a restatement of the fact that at each point of C the characteristic vector $\mathbf{V}$ is perpendicular to each of the vectors ∇u, ∇v and so is parallel to $\nabla u \times \nabla v$. Recall $\nabla u \times \nabla v$ has the same direction as the tangent to C at the point.

The term general solution applied to the relation (1-43) can be justified analytically. Let u, v be as described in the preceding paragraph, and let F be any continuously differentiable function such that $F(u,v) = 0$ implicitly defines a function

$$z = \varphi(x,y)$$

having continuous partial derivatives in some neighborhood of a point (x_0,y_0,z_0) of $\mathcal{T}$. Exactly as in Example 1-4, differentiation of $F = 0$ with respect to x and y and elimination of F lead to the equation

$$\frac{\partial(u,v)}{\partial(y,z)} \varphi_x + \frac{\partial(u,v)}{\partial(z,x)} \varphi_y = \frac{\partial(u,v)}{\partial(x,y)} \tag{1-51}$$

Now

$$\frac{\partial(u,v)}{\partial(z,x)} = \frac{Q}{P} \frac{\partial(u,v)}{\partial(y,z)} \qquad \frac{\partial(u,v)}{\partial(x,y)} = \frac{R}{P} \frac{\partial(u,v)}{\partial(y,z)} \tag{1-52}$$

The preceding relations are obtained from the simultaneous equations (1-50). If the expressions for the Jacobians are inserted in Eq. (1-51), the result is

$$P\varphi_x + Q\varphi_y = R \tag{1-53}$$

Hence φ is a solution of the partial differential equation (1-38). It is emphasized here that the preceding results are local, since the reasoning is based upon the existence theorem for the system of ordinary differential equations (1-40). Thus the properties have been shown to hold only in some neighborhood of a point (x_0,y_0,z_0) of $\mathcal{T}$.

EXAMPLE 1-12 $xzp + yzq = -(x^2 + y^2)$

From Example 1-9 two independent integrals of the corresponding subsidiary equations are

$$u(x,y,z) = \frac{y}{x} = c_1 \qquad v(x,y,z) = x^2 + y^2 + z^2 = c_2$$

If c_1, c_2 are assigned positive values, the corresponding surfaces are a plane and a sphere. In the first octant these surfaces intersect in a segment of a circle (see Fig. 1-4). To show directly that the curve of intersection is a characteristic, recall that the tangent to C has the direction of the vector $\nabla u \times \nabla v$. Now

$$\nabla u = \left(\frac{-y}{x^2}\right)\mathbf{i} + \frac{1}{x}\mathbf{j} \qquad \nabla v = 2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

so that

$$\nabla u \times \nabla v = \frac{2[xz\mathbf{i} + zy\mathbf{j} - (x^2 + y^2)\mathbf{k}]}{x^2} = \frac{2\mathbf{V}}{x^2}$$

Every characteristic in the first octant is a segment of such a circle. Suppose F is a function such that $F(u,v) = 0$ can be solved for z to obtain

$$z = \varphi(x,y) = \left[f\left(\frac{y}{x}\right) - (x^2 + y^2)\right]^{1/2}$$

where φ is continuously differentiable. Then φ satisfies

$$xz\varphi_x + yz\varphi_y = -(x^2 + y^2)$$

1-6 CAUCHY PROBLEM FOR QUASILINEAR FIRST-ORDER EQUATIONS

A fundamental problem in the study of ordinary differential equations is to determine a solution of a first-order equation $y' = f(x,y)$ which passes

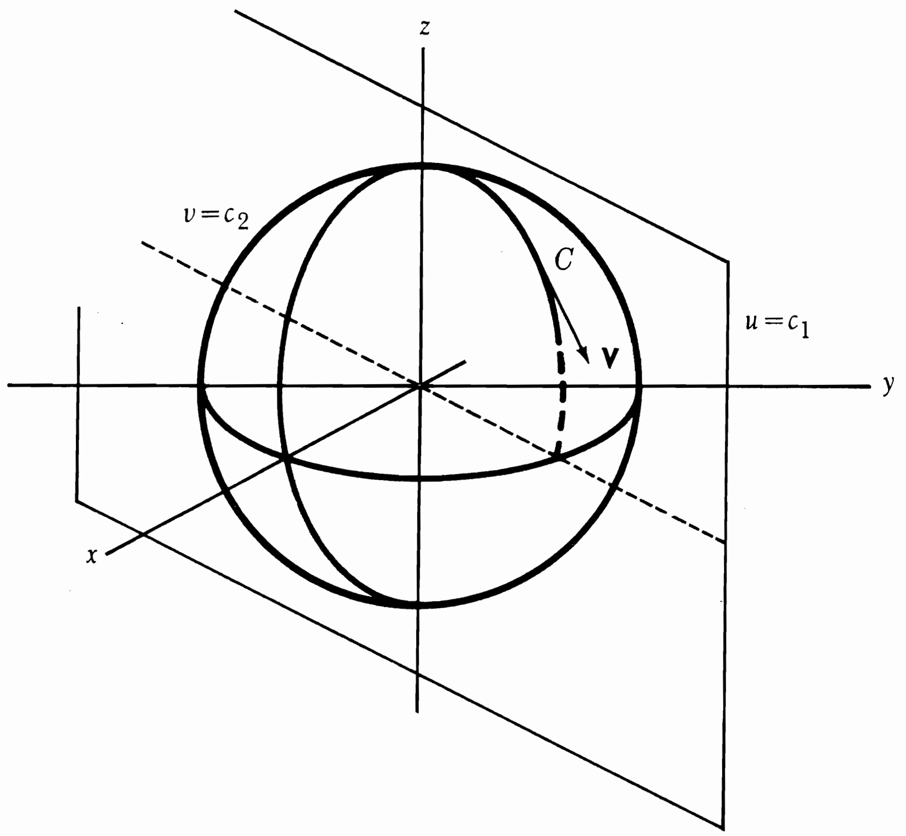

Figure 1-4

through a prescribed point in the xy plane. Under quite general conditions a unique solution to the problem exists. An analogous problem in the study of first-order partial differential equations in two independent variables x, y is to determine an integral surface such that the surface passes through a prescribed curve in xyz space. Such a problem is termed a *Cauchy problem*. In this section a method of solving the Cauchy problem for Eq. (1-38) is described.

Before proceeding with the general case consider the following example.

EXAMPLE 1-13 Find a solution $z = \varphi(x,y)$ of $yp - xq = 0$ such that $\varphi(x,0) = x^4$. The problem is to find an integral surface which passes through the curve Γ defined by the simultaneous equations $z = x^4$ and $y = 0$. This curve lies in the xz plane. By the method of Lagrange one obtains the general solution of the differential equation as

$$z = f(x^2 + y^2)$$

where f is arbitrary. Every integral surface is a surface of revolution about the z axis. The condition that such a surface contains Γ is $f(x^2) = x^4$. Thus $f(t) = t^2$. The solution of the Cauchy problem is $z = (x^2 + y^2)^2$ (see Fig. 1-5). There is but one surface of revolution about the z axis which contains Γ, and so the solution obtained is unique.

In general there may or may not exist a solution of the Cauchy problem for Eq. (1-38). There is also the possibility that infinitely many distinct solutions exist. All three cases are illustrated, with the aid of the partial

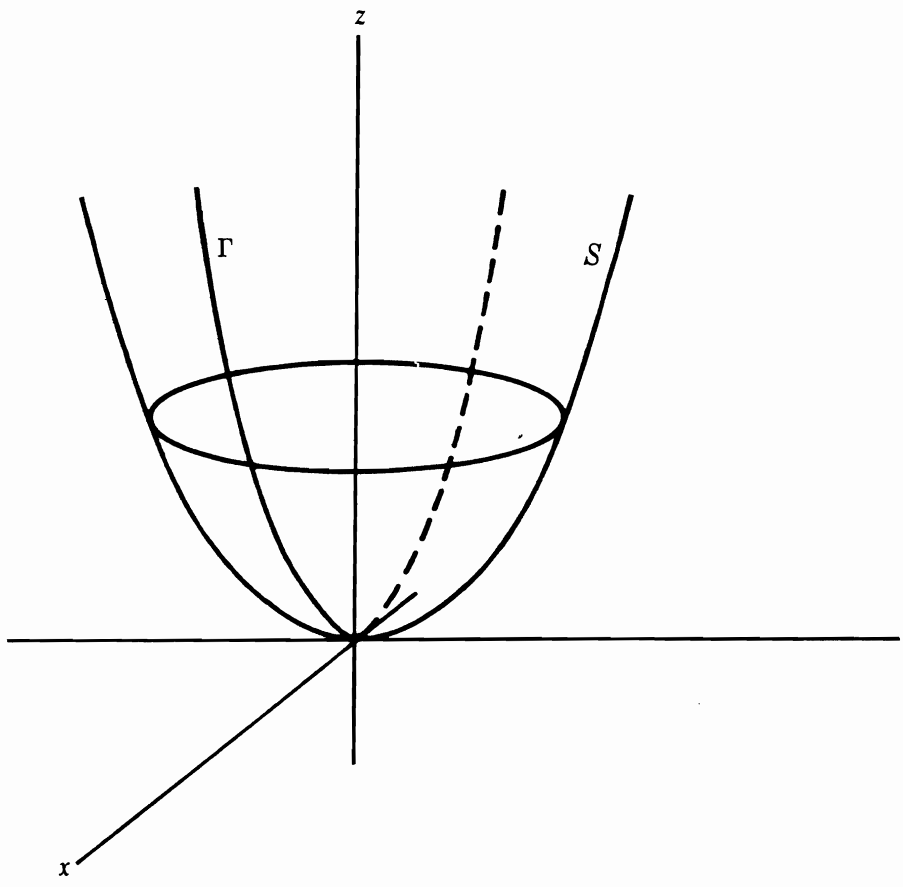

Figure 1-5

differential equation of the preceding example. If Γ is the curve stated in the example, there exists exactly one solution. Suppose instead Γ is the circle $x^2 + y^2 = 1$, $z = 1$. Choose any function $h(t)$ such that $h(1) = 1$ and consider the function $z = h(x^2 + y^2)$. The corresponding surface is an integral surface which contains Γ. Clearly there are infinitely many solutions in this case. Observe that the given curve Γ is now a characteristic curve of the partial differential equation. At the opposite extreme of circumstances let Γ be the ellipse $x^2 + y^2 = 1$, $z = y$. If $z = f(x^2 + y^2)$ is a solution of the Cauchy problem, then on the circle $x^2 + y^2 = 1$ one has $z = f(1)$, a constant. However this is incompatible with the requirement that $z = y$ whenever $x^2 + y^2 = 1$. Thus no solution exists. Note that in the last case the given curve is such that its projection on the xy plane coincides with the projection on the xy plane of a characteristic curve, but Γ itself is noncharacteristic.

A Method of Solution

Let Γ be a given smooth curve defined parametrically by

$$x = f(t) \qquad y = g(t) \qquad z = h(t) \tag{1-54}$$

for $a < t < b$. Assume also that Γ is noncharacteristic. To construct an integral surface of Eq. (1-38) which contains Γ one can proceed as follows. Let $u = c_1$, $v = c_2$ be two independent integrals of the subsidiary equations (1-39). Write down the equations

$$u[f(t),g(t),h(t)] = c_1 \qquad v[f(t),g(t),h(t)] = c_2 \tag{1-55}$$

Eliminate t from the pair of equations so as to derive a functional relation

$$F(c_1,c_2) = 0 \tag{1-56}$$

between c_1 and c_2. Then the solution of the Cauchy problem is

$$F[u(x,y,z),v(x,y,z)] = 0 \tag{1-57}$$

EXAMPLE 1-14 Find an integral surface of

$$(y + xz)p + (x + yz)q = z^2 - 1$$

which passes through the parabola $x = t$, $y = 1$, $z = t^2$.

The subsidiary equations are

$$\frac{dx}{y + xz} = \frac{dy}{x + yz} = \frac{dz}{z^2 - 1}$$

Since $P + Q = (x + y)(z + 1)$ and $P - Q = (x - y)(z - 1)$,

$$\frac{dx + dy}{x + y} = \frac{dz}{z - 1} \qquad \frac{dx - dy}{x - y} = \frac{dz}{z + 1}$$

Two independent integrals of this system are

$$u = \frac{z - 1}{x + y} = c_1 \qquad v = \frac{z + 1}{x - y} = c_2$$

Note that the characteristic curves here are the straight lines determined by the intersection of the planes

$$z - 1 - c_1(x + y) = 0 \qquad z + 1 - c_2(x - y) = 0$$

The given curve Γ is noncharacteristic. Equation (1-55) takes the form

$$\frac{t^2 - 1}{t + 1} = c_1 \qquad \frac{t^2 + 1}{t - 1} = c_2$$

From the first equation, $t = c_1 + 1$. Insert this into the second equation; then $(c_1 + 1)^2 = c_1c_2$. The integral surface which contains Γ has the equation

$$\left(\frac{z - 1 + x + y}{x + y}\right)^2 + 1 = \frac{z^2 - 1}{x^2 - y^2}$$

An Existence and Uniqueness Theorem

From a geometric viewpoint a functional relation $F(c_1,c_2) = 0$ imposed on the arbitrary constants c_1, c_2 singles out a one-parameter subfamily of the two-parameter family of characteristic curves in Eq. (1-42). Analytically, the condition that the resulting surface contains the given curve Γ is that the relation

$$F\{u[f(t),g(t),h(t)],v[f(t),g(t),h(t)]\} = 0 \tag{1-58}$$

holds identically in t, for $a < t < b$. But if F is obtained by eliminating t between the pair of equations in (1-55), then clearly the identity holds. The success of the method hinges on the ability to eliminate t in Eq. (1-55). The following local existence and uniqueness theorem for the Cauchy problem shows that this is always possible provided Γ is noncharacteristic.

THEOREM 1-1 Let $\mathscr{T}$ be a region of xyz space and $\mathscr{R}$ the projection of $\mathscr{T}$ on the xy plane. Let the following properties hold: (1) the coefficients P, Q, R in Eq. (1-38) are continuously differentiable functions of x, y, z, and P, Q do not vanish in $\mathscr{T}$; (2) Γ is a given space curve lying in $\mathscr{T}$ and is defined parametrically by Eq. (1-54), where the functions f, g, h are invertible and have continuous first derivatives; (3) $[f'(t)]^2 + [g'(t)]^2 \neq 0$, $a < t < b$; (4) (x_0,y_0,z_0) is a point on Γ corresponding to $t = t_0$; (5) $P(x_0,y_0,z_0)g'(t_0) - Q(x_0,y_0,z_0)f'(t_0) \neq 0$. Then there exists a neighborhood N of (x_0,y_0) in $\mathscr{R}$, a neighborhood $|t - t_0| < \delta$ of t_0, and a unique function $z = \varphi(x,y)$ such that φ is a solution of Eq. (1-38) on N and

$$h(t) = \varphi[f(t),g(t)]$$

holds identically in t for $|t - t_0| < \delta$.

Proof The proof is based on the following property of the system (1-40). Under the present hypotheses on P, Q, and R there exists a two-parameter family of solutions

$$y = y(x,c_1,c_2) \qquad z = z(x,c_1,c_2) \tag{1-59}$$

of Eq. (1-40), where the functions y and z are continuous and have continuous first partial derivatives with respect to the parameters c_1, c_2 in a certain range of values of these parameters which includes y_0, z_0. Moreover

$$y(x_0,c_1,c_2) = c_1 \qquad z(x_0,c_1,c_2) = c_2 \tag{1-60}$$

for each pair of values c_1, c_2. The proof of this property is given in Ref. 2. From Eq. (1-60) one obtains

$$\frac{\partial y}{\partial c_1} = 1 \qquad \frac{\partial y}{\partial c_2} = 0 \qquad \frac{\partial z}{\partial c_1} = 0 \qquad \frac{\partial z}{\partial c_2} = 1$$

at $x = x_0$. Hence the Jacobian $\partial(y,z)/\partial(c_1,c_2) \neq 0$ in some neighborhood of (x_0,y_0,z_0). In turn this implies that Eq. (1-59) can be solved for c_1, c_2 to obtain the functions

$$c_1 = u(x,y,z) \qquad c_2 = v(x,y,z) \tag{1-61}$$

and the functions u and v have the property that

$$y_0 = u(x_0,y_0,z_0) \qquad z_0 = v(x_0,y_0,z_0)$$

The Jacobian $\partial(u,v)/\partial(y,z)$ is different from zero in some neighborhood of (x_0,y_0,z_0) since it is the reciprocal of $\partial(y,z)/\partial(c_1,c_2)$. With u and v constructed in this manner, define the functions $c_1(t)$ and $c_2(t)$ by

$$c_1(t) = u[f(t),g(t),h(t)] \qquad c_2(t) = v[f(t),g(t),h(t)]$$

Then $c_1(t_0) = y_0$, and $c_2(t_0) = z_0$. Moreover c_1 and c_2 have continuous first derivatives with respect to t in some neighborhood of $t = t_0$, and, by the chain rule,

$$\frac{dc_1}{dt} = \nabla u \cdot \frac{d\mathbf{r}}{dt} \qquad \frac{dc_2}{dt} = \nabla v \cdot \frac{d\mathbf{r}}{dt} \tag{1-62}$$

where

$$\frac{d\mathbf{r}}{dt} = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Recall that the vector $d\mathbf{r}/dt$ is tangent to Γ. Now at least one of the values $c_1'(t_0)$, $c_2'(t_0)$ is different from zero. For suppose that $c_1'(t_0) = c_2'(t_0) = 0$. Then at $t = t_0$, $x = x_0$, $y = y_0$, $z = z_0$ the vector $d\mathbf{r}/dt$ is perpendicular to the vector ∇u and also perpendicular to the vector ∇v. Since the Jacobian $\partial(u,v)/\partial(y,z) \neq 0$, at (x_0,y_0,z_0) the vector $\nabla u \times \nabla v$ has a nonzero x component and so is not the zero vector at that point. Also $d\mathbf{r}/dt$ is not the zero vector at $t = t_0$, by virtue of hypothesis 3. Hence $d\mathbf{r}/dt$ has the same direction as $\nabla u \times \nabla v$. But $\nabla u \times \nabla v$ has the same direction as the characteristic vector $\mathbf{V}$ (see Sec. 1-5). Thus Γ is characteristic at (x_0,y_0,z_0). This, however, is impossible by virtue of hypothesis 5, which asserts that $\mathbf{V} \times d\mathbf{r}/dt \cdot \mathbf{k} \neq 0$ at that point. Thus at least one of the derivatives $c_1'(t_0)$, $c_2'(t_0)$ must be different from zero. For definiteness assume $c_1'(t_0) \neq 0$. By a basic theorem of calculus the equation $c_1 = c_1(t)$ can be solved in a neighborhood of t_0 to obtain the inverse function $t = t(c_1)$, and moreover $t(y_0) = t_0$. Substitute $t(c_1)$ for t in the equation $c_2 = c_2(t)$. Then

$$c_2 = c_2[t(c_1)] = \psi(c_1)$$

With the function ψ constructed in this manner consider the equation

$$v(x,y,z) - \psi[u(x,y,z)] = 0 \tag{1-63}$$

Clearly, if in this relation x, y, z are replaced by $f(t)$, $g(t)$, $h(t)$, an identity in t results. In particular this implies the equation is satisfied by $x = x_0$, $y = y_0$, $z = z_0$. Observe that the left member of Eq. (1-63) is continuously differentiable in a neighborhood of (x_0,y_0,z_0). Now it is asserted that

$$v_z - \psi'(u)u_z \neq 0 \tag{1-64}$$

at (x_0,y_0,z_0). To show this consider that

$$\psi'(c_1) = \frac{dc_2}{dc_1} = \frac{dc_2/dt}{dc_1/dt} = \frac{\nabla v \cdot d\mathbf{r}/dt}{\nabla u \cdot d\mathbf{r}/dt}$$

If $v_z - \psi'(u)u_z = 0$ at (x_0,y_0,z_0), then

$$v_z \nabla u \cdot \frac{d\mathbf{r}}{dt} - u_z \nabla v \cdot \frac{d\mathbf{r}}{dt} = 0$$

Recall the expression for the triple product $(\nabla u \times \nabla v) \times d\mathbf{r}/dt$. Then the preceding equation takes the form

$$(\nabla u \times \nabla v) \times \frac{d\mathbf{r}}{dt} \cdot \mathbf{k} = 0$$

Hence

$$\mathbf{V} \times \frac{d\mathbf{r}}{dt} \cdot \mathbf{k} = 0$$

which contradicts hypothesis 5. Thus (1-64) holds. In turn (1-64) implies that Eq. (1-63) can be solved in a neighborhood of (x_0,y_0,z_0) to obtain a continuously differentiable function $z = \varphi(x,y)$ such that $z_0 = \varphi(x_0,y_0)$. In fact there is a neighborhood of t_0 such that

$$h(t) = \varphi[f(t),g(t)]$$

hold identically in t. The proof that φ is a solution of Eq. (1-38) in a neighborhood of (x_0,y_0,z_0) follows from the discussion in the paragraph of Sec. 1-5 which precedes Example 1-12.

Under the hypothesis that Γ is noncharacteristic there can be at most one solution of the Cauchy problem in a neighborhood of (x_0,y_0,z_0). For if two distinct integral surfaces of Eq. (1-38) contain Γ near (x_0,y_0,z_0), then in a sufficiently small neighborhood of (x_0,y_0,z_0), Γ is the unique curve of intersection of these surfaces. But then Γ is a characteristic.

PROBLEMS

Sec. 1-2

1 Classify each of the following equations to the extent of the definitions given in Sec. 1-2.

(a) $u_{xy} + x^2u_{yy} + (\cos y)u_x = (\tan xy)u + x^2y^2$

(b) $\left(\dfrac{\partial\varphi}{\partial x}\right)^2 \dfrac{\partial^2\varphi}{\partial x^2} + y^2\dfrac{\partial^2\varphi}{\partial y^2} + \varphi = 0$

(c) $x^2r + yt - zp + (1 - z^3)q = x$ **(d)** $x^2p - yz^2q = 0$

(e) $u_yu_{xxy} + u_xu_{yy} - u_z^2 + xy^2 = z$ **(f)** $pq = z$

(g) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ Laplace's equation

(h) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ Cauchy-Riemann equations

Sec. 1-3

2 **(a)** Show that each sphere in the family $x^2 + y^2 + z^2 + c_1z = 0$ intersects each sphere in the family $x^2 + c_1x + y^2 + z^2 = 0$ orthogonally.

(b) Let G be a given differentiable function of the variables x, y, z such that $G(x,y,z) = c$ defines a surface for each value of the constant c in a certain range of values and so a family of surfaces. Let $F(x,y,z) = 0$ implicitly define a differentiable function $z = f(x,y)$ and so a surface S. If S intersects orthogonally each surface in the above family of surfaces, show that the quasilinear equation $pG_x + qG_y = G_z$ must be satisfied.

3 Eliminate the arbitrary functions which appear and obtain a differential equation of lowest order.

(a) $z = xf(y)$ **(b)** $u = e^{-x}F(x - 2y)$ **(c)** $G(x^2 + y^2 + z^2, z) = 0$

(d) $u = x^nF\left(\dfrac{y}{x}, \dfrac{z}{x}\right)$ n a positive integer

(e) $ze^{-x^2} + \psi(x^2 + y^2) = 0$

(f) $z = f(y)\cos x + g(y)\sin x$ **(g)** $u = f(x - ct) + g(x + ct)$

(h) $u = f_1(y - mx) + xf_2(y - mx) + x^2f_3(y - mx)$

4 Consider the relation $\Phi(x^2 + y^2, y^2 + z^2, u^2 + xy) = 0$, where Φ is an arbitrary function. Let $F(x,y,z,u) = x^2 + y^2$, $G(x,y,z,u) = y^2 + z^2$, and $H(x,y,z,u) = u^2 + xy$. Let x, y, z be independent variables, and suppose for each choice of Φ the equation $\Phi(F,G,H) = 0$ implicitly defines u as a function of x, y, z. Follow the method of Example 1-4 and differentiate $\Phi = 0$ with respect to x, y, z so as to eliminate Φ and derive the first-order quasilinear equation

$$2yzuu_x - 2xzuu_y + 2xyuu_z = z(x^2 - y^2)$$

5 Consider the relation $\Phi(F_1, \ldots, F_n) = 0$, where Φ is arbitrary and $F_i(x_1, \ldots, x_n, u)$, $i = 1, \ldots, n$, are n differentiable functions. Let $x_1, \ldots, x_n$ be independent variables, and suppose for each choice of Φ the equation $\Phi = 0$ implicitly defines u as a function of $x_1, \ldots, x_n$. Differentiate $\Phi = 0$ successively with respect to $x_1, x_2, \ldots, x_n$ and so obtain the n equations

$$\sum_{i=1}^{n} \Phi_i(F_{ix_j} + F_{iu}u_{x_j}) = 0 \qquad j = 1, \ldots, n$$

where Φ_i means $\partial\Phi/\partial F_i$, F_{ix_j} means $\partial F_i/\partial x_j$, etc. Eliminate Φ and show that a quasilinear equation

$$\sum_{k=1}^{n} A_k(x_1, \ldots, x_n, u)\frac{\partial u}{\partial x_k} = G(x_1, \ldots, x_n, u)$$

results.

6 Eliminate the arbitrary constants $a, b, c, \ldots$ which appear and obtain a partial differential equation of lowest order.

(a) $z = e^{ax+by}$

(b) $a(x^2 + y^2) + bz^2 = 1$

(c) $z = bx^a y^{1-a}$

(d) $z = ax^2 + 2bxy + cy^2$

(e) $u = Ae^{ax}\cos ay$

(f) $u = A\cos ax \cos at$

(g) $u = Ae^{-a^2t}\cos ax$

(h) $az + b = a^2x + y$

(i) $z = ax + by + a^2 + b^2$

7 Find a partial differential equation of lowest order satisfied by each surface in the given family surfaces.

(a) All planes through the point (1,0,0) not perpendicular to the xy plane.

(b) All spheres of unit radius.

(c) The family of all tangent planes to the surface $z = xy$.

Sec. 1-4

8 Hold one independent variable constant and integrate with respect to the remaining variable so as to obtain a solution involving an arbitrary function. Verify that your answer is correct by substituting into the differential equation.

(a) $q = x^2 + y^2$ **(b)** $p = \sin \dfrac{x}{y}$ **(c)** $z_x + xz = x^3 + 3xy$

9 Let $Ap + Bq = 0$ be a first-order equation, where A, B are constants. Assume a solution of the form $z = f(ax + by)$, f arbitrary and a, b constants. Substitute into the differential equation to determine suitable values of a, b.

10 Obtain the general solution.

(a) $3p - 4q = x^2$

(b) $p - 3q = \sin x + \cos y$

(c) $5p + 4q + z = x^3 + 1 + 2e^{3y}$

(d) $p + 2q - 5z = \cos x + y^3 + 1$

(e) $p - aq = e^{mx} \cos by$ a, m, b constants

11 Make the change of independent variables $\xi = \log x$, $\eta = \log y$ and reduce the differential equation to one with constant coefficients. Obtain the general solution.

(a) $4xp - 2yq = 0$

(b) $2xp + 3yq = \log x$

(c) $xp - 7yq = x^2y$

(d) $8xp - 5yq + 4z = x^2 \cos x$

(e) $axz_x + byz_y + cz = x^2 + y^2$

12 Follow the method of the text in Example 1-8 and obtain the general solution in the form (1-37).

(a) $xyp - x^2q + yz = 0$

(b) $yp - xq = 0$

(c) $(x + a)p + (y + b)q + cz = 0$ a, b, c constants

13 Let $Lu = Au_x + Bu_y + Cu_z + Du = G$ be a linear first-order equation in three independent variables x, y, z and dependent variable u, where A, B, C, D are constants and $G(x,y,z)$ is a given function. Assume $A \neq 0$. To obtain a solution of the homogeneous equation $Lu = 0$ which involves an arbitrary function assume a solution of the form $u = e^{-Dx/A}f(ax + by + cz)$, substitute into $Lu = 0$, and show that the constants a, b, c must satisfy $Aa + Bb + Cc = 0$ if the assumed form satisfies $Lu = 0$ for arbitrary choice of f. Conversely, if a, b, c are chosen so that the preceding equation holds, then $u = e^{-Dx/A}f(ax + by + cz)$ is a solution for arbitrary (differentiable) f. If $A = 0$ but $B \neq 0$, one can proceed similarly with $u = e^{-Dx/B}f(ax + by + cz)$, etc. Find a solution involving an arbitrary function for each of the following equations.

(a) $2u_x - u_y + 4u_z + u = 0$

(b) $u_x - 4u_z + 7u_z - u = x + y + z + 1$

Sec. 1-5

14 Obtain the general solution.

(a) $p + xq = z$

(b) $xp + yq = nz$ n constant

(c) $(x + z)p + (y + z)q = 0$

(d) $(y + x)p + (y - x)q = z$

(e) $zp + yq = x$

(f) $(x + y)(p - q) = z$

(g) $yp - xq = x^3y + xy^3$

(h) $(mz - ny)p + (nx - lz)q = ly - mx$ l, m, n constants

(i) $x^2p + y^2q = axy$ $a \neq 0$, constant

(j) $(y^3x - 2x^4)p + (2y^4 - x^3y)q = 9z(x^3 - y^3)$

(k) $x(y - z)p + y(z - x)q = z(x - y)$

(l) $yp - xq + z + x^2 + y^2 - 1 = 0$

(m) $p - 2q = 3x^2 \sin (y + 2x)$

(n) $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

(o) $\cos y \dfrac{\partial z}{\partial x} + \cos x \dfrac{\partial z}{\partial y} = \cos x \cos y$

(p) $(z + e^x)p + (z + e^y)q = z^2 - e^{x+y}$

(q) $xp + yq = 2xy(a^2 - z^2)^{1/2}$

15 Refer to Prob. 2**b** and find the general form of the equation of all surfaces orthogonal to the given family of surfaces.

(a) $x^2 + y^2 + z^2 = 2ax$ (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + z^2 = c^2$

16 (a) Let $\Phi(x,y,z)$ be a continuously differentiable function and let c be a fixed constant such that $\Phi = c$ defines a smooth surface S. Show that the family of all smooth space curves C orthogonal to S must satisfy the system of equations

$$\frac{dx}{\Phi_x} = \frac{dy}{\Phi_y} = \frac{dz}{\Phi_z}$$

(b) The velocity potential Φ of a stationary velocity field of fluid flow is $\Phi(x,y,z) = xy + xyz^2$. The velocity field is $\mathbf{V} = \nabla\Phi$. The *trajectories* of the field are the curves C such that the tangent to the curve at each point has the direction of $\mathbf{V}$ at the point, i.e., the field lines. Find the trajectories.
(c) The potential Φ of an electrostatic field is $\Phi(x,y,z) = 1/x + 1/y + 1/z$. The electric field is $\mathbf{E} = -\nabla\Phi$. Find the field lines of the $\mathbf{E}$ field.

17 Find the general form of the equation of all surfaces such that the tangent plane to each point on the surface passes through the fixed point $(0,0,a)$.

18 Let $x_1, \ldots, x_n$ be independent variables in a region $\mathscr{R}$ of n-dimensional space, and let

$$\sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} = 0 \qquad (1)$$

be a linear equation with coefficients P_i which are continuously differentiable and such that they are not simultaneously zero in $\mathscr{R}$. A function $u = \varphi(x_1, \ldots, x_n)$ is said to be a *solution of* (1) in $\mathscr{R}$ if φ is continuously differentiable and (1) holds identically in $\mathscr{R}$. The system of ordinary differential equations

$$\frac{dx_1}{P_1} = \cdots = \frac{dx_n}{P_n} \qquad (2)$$

is called the *subsidiary system* of (1). These equations define an $(n-1)$-parameter family of curves in n space, called the *characteristic curves* of (1). If x_n is chosen as the independent variable, then (2) can be written in the form

$$\frac{dx_1}{dx_n} = \frac{P_1}{P_n} \cdots \frac{dx_{n-1}}{dx_n} = \frac{P_{n-1}}{P_n} \qquad (3)$$

The general solution of the system (3) is of the form

$$x_i = x_i(x_n, c_1, \ldots, c_{n-1}) \qquad i = 1, \ldots, n-1$$

where the c_j are arbitrary constants. If these are solvable for the c_j's, the general solution of (2) can be written as $u_i(x_1, \ldots, x_n) = c_i$, $i = 1, \ldots, n-1$, where the $n-1$ functions u_i are functionally independent in $\mathscr{R}$. Each relation $u_i = c_i$ is called an *integral* of the subsidiary equations (2). For fixed c_j the equation $u_j(x_1, \ldots, x_n) = c_j$ defines a hypersurface (of dimension $n-1$) in n space. For each fixed set of values $c_1, \ldots, c_n$ the $n-1$ hypersurfaces $u_i = c_i$, $i = 1, \ldots, n-1$, intersect in a characteristic curve C in n space. Through each point $(x_1^{(0)}, \ldots, x_n^{(0)})$ of $\mathscr{R}$ there passes one, and only one, characteristic C. Each function $u_j(x_1, \ldots, x_n)$ satisfies (1) in $\mathscr{R}$. For given a point in $\mathscr{R}$ there exist constants

$c_1, \ldots, c_n$ such that a characteristic C passes through the point, and so equations (2) hold at the point. Differentiate the equation $u_j = c_j$; then

$$0 = du_j = \sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i} dx_i = \sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i}\left(\frac{P_i}{P_n} dx_n\right) = \frac{dx_n}{P_n} \sum_{i=1}^{n} P_i \frac{\partial u_j}{\partial x_i}$$

If $u_1, \ldots, u_{n-1}$ are functionally independent in $\mathscr{R}$, the *general solution* of (1) is $u = f(u_1, \ldots, u_{n-1})$, f arbitrary. For each choice of f this defines a solution of (1), since

$$\sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} = \sum_{i=1}^{n} P_i\left(\sum_{j=1}^{n-1} \frac{\partial f}{\partial u_j} \frac{\partial u_j}{\partial x_i}\right) = \sum_{j=1}^{n-1} \frac{\partial f}{\partial u_j}\left(\sum_{i=1}^{n} P_i \frac{\partial u_j}{\partial x_i}\right) = 0$$

Conversely, let u be a solution of (1) in $\mathscr{R}$. Then the n equations

$$\sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} = 0 \qquad \sum_{i=1}^{n} P_i \frac{\partial u_j}{\partial x_i} = 0$$

$j = 1, \ldots, n-1$, hold simultaneously in $\mathscr{R}$. Since the P_i have values different from zero at each point of $\mathscr{R}$, it follows that the Jacobian

$$\frac{\partial(u, u_1, \ldots, u_{n-1})}{\partial(x_1, \ldots, x_n)} = 0$$

in $\mathscr{R}$. This implies that (locally at least) there exists a function f such that $u = f(u_1, \ldots, u_{n-1})$.

Let v be a particular solution of the linear equation

$$\sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} + Ru = 0 \tag{4}$$

where $R(x_1, \ldots, x_n)$ is continuously differentiable, in $\mathscr{R}$. Let $u_1, \ldots, u_{n-1}$ be $n-1$ functionally independent integrals of the subsidiary equations (2). Then the *general solution* of (4) is $u = vf(u_1, \ldots, u_{n-1})$, f arbitrary. For if f is a given function, then $u_{x_i} = v_{x_i} f + v f_{x_i}$. Hence

$$\sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} = f \sum_{i=1}^{n} P_i \frac{\partial v}{\partial x_i} + v \sum_{i=1}^{n} P_i \frac{\partial f}{\partial x_i}$$

$$= f(-Rv) = -Ru$$

so u is a solution of (4). Conversely, if u is a solution of (4) and $w = u/v$, then $w_{x_i} = (vu_{x_i} - uv_{x_i})/v^2$, so that

$$\sum_{i=1}^{n} P_i \frac{\partial w}{\partial x_i} = \frac{1}{v} \sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} - \frac{u}{v^2} \sum_{i=1}^{n} P_i \frac{\partial v}{\partial x_i}$$

$$= \frac{1}{v}(-Ru) - \frac{u}{v^2}(-Rv) = 0$$

Thus w is a solution of (1). It follows that $w = f(u_1, \ldots, u_{n-1})$ for some f, and so $u = vf(u_1, \ldots, u_{n-1})$. The *general solution* of the inhomogeneous equation

$$\sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} + Ru = G \tag{5}$$

where G is a given continuously differentiable function, is $u = u_p + u_h$, where u_p is a particular solution of (5) and u_h is the general solution of (4). Find the general solution of each of the following.

(a) $P_1u_x + P_2u_y + P_3u_z + Ru = 0$ $\quad P_1, P_2, P_3, R$ constants
(b) $(y + z)u_x + (z + x)u_y + (x + y)u_z = 0$
(c) $3u_x + 5u_y - u_z = \cos y - 2e^{-z}$
(d) $u_x - u_y + 7u_z + u = Ae^{ax+by+cz}$ $\quad A, a, b, c$ constants
(e) $(x + z)u_x + yu_y - 2u_z = 4e^z$
(f) $xzu_x + yzu_y - (x^2 + y^2)u_z = zy^2 \sin y$
(g) $(z - y)u_x + yu_y - zu_z = y(x + z) - y^2$
(h) $(\tan x)u_x + (\tan y)u_y + u_z = \sin z$
(i) $xu_x + yu_y + zu_z = nu$ $\quad n$ constant
(j) $\sqrt{x}u_x + \sqrt{y}u_y + \sqrt{z}u_z + u = x + y$
(k) $\dfrac{\partial u}{\partial x_1} + x_1\dfrac{\partial u}{\partial x_2} + x_1x_2\dfrac{\partial u}{\partial x_3} + x_1x_2x_3\dfrac{\partial u}{\partial x_4} = 0$
(l) $x_1\dfrac{\partial u}{\partial x_1} + x_1x_2\dfrac{\partial u}{\partial x_2} + x_1x_2x_3\dfrac{\partial u}{\partial x_3} + x_1x_2x_3x_4\dfrac{\partial u}{\partial x_4} + u = x_1^2x_2^2$

19 A first-order quasilinear equation in n independent variables $x_1, \ldots, x_n$ and dependent variable u has the form

$$\sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} = R \tag{1}$$

where the functions are assumed to be continuously differentiable functions of the x_i as well as u, and such that the P_i do not vanish in some region $\mathscr{T}$ of $(x_1, \ldots, x_n, u)$ space. Let $\mathscr{R}$ be the projection of $\mathscr{T}$ onto the hyperplane $u = 0$. A *solution* $u = \varphi(x_1, \ldots, x_n)$ of (1) on $\mathscr{R}$ is a continuously differentiable function of the x_i such that if φ and its derivatives are substituted into (1), an identity results. Assume $w = \psi(x_1, \ldots, x_n, u)$ is a solution of the linear homogeneous equation

$$\sum_{i=1}^{n} P_i \frac{\partial w}{\partial x_i} + R\frac{\partial w}{\partial u} = 0 \tag{2}$$

in the $n + 1$ independent variables $x_1, \ldots, x_n, u$. Assume also that $w_u \neq 0$ and that $w = 0$ implicitly defines a continuously differentiable function $u = \varphi(x_1, \ldots, x_n)$. Then $u_{x_j} = -w_{x_j}/w_u$, $j = 1, \ldots, n$. Substitute these expressions into the left-hand side of (1):

$$\sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} = \sum_{i=1}^{n} P_i\left(\frac{-w_{x_j}}{w_u}\right) = -\left[\frac{\sum_{i=1}^{n} P_i(\partial w/\partial x_j)}{w_u}\right] = R$$

where (2) has been used. Thus u is a solution of (1). The problem of solving (1) is reduced to solving (2). From the results of Prob. 17 the general solution of (2) is $w = f(w_1, \ldots, w_n)$, where $w_1, \ldots, w_n$ are n functionally independent integrals of the subsidiary equations

$$\frac{dx_1}{P_1} = \cdots = \frac{dx_n}{P_n} = \frac{du}{R} \tag{3}$$

Then the *general solution* of (1) is $f(w_1, \ldots, w_n) = 0$, f arbitrary. Obtain the general solution of the following.

(a) $u_x + xu_y + xyu_z = xyzu$

(b) $xu_x + (z + u)u_y + (y + u)u_z = y + z$

(c) $xu_x + yu_y + zu_z = u + \dfrac{xy}{z}$

(d) $(s - x)u_x + (s - y)u_y + (s - z)u_z = s - u$, $s = x + y + z + u$

Sec. 1-6

20 Determine the integral surface which passes through the given curve.

(a) $p + q = z$; $z = \cos x$, $y = 0$

(b) $z(p - q) = y - x$; $x = 1$, $z = y^2$

(c) $xp - yq = 0$; $x = y = z = t$

(d) $(x + z)p + (y + z)q = 0$; $x = 1 - t$, $y = 1 + t$, $z = t$

(e) $x^2p + y^2q = z^2$; $x = t$, $y = 2t$, $z = 1$

(f) $xzp + yzq + xy = 0$; $xy = a^2$, $z = h$

(g) $2xzp + 2yzq = z^2 - x^2 - y^2$; $x + y + z = 0$, $x^2 + y^2 + z^2 = a^2$

(h) $(y - z)p + (z - x)q = x - y$; $x = t$, $y = 2t$, $z = 0$

(i) $x(x^2 + y^2)p + 2y^2(xp + yq - z) = 0$; $x^2 + y^2 = a^2$, $z = h$

(j) $(x^2 + y^2)p + 2xyq = xz$; $x = a$, $y^2 + z^2 = a^2$

(k) $x(y - z)p + y(z - x)q = z(x - y)$; $x = y = z = t$

(l) $z(x + z)p - y(y + z)q = 0$; $x = 1$, $y = t$, $z = \sqrt{t}$

(m) $yp - xq = 2xyz$; $x = t$, $y = t$, $z = t$

(n) $(y^2 - x^2 + 2xz)p + 2y(z - x)q = 0$; $x = 0$, $y^2 + 4z^2 = 4a^2$

(o) $p \sec x + aq = \cot y$; $z(0,y) = \sin y$

21 Find a surface orthogonal to the sphere $x^2 + y^2 + z^2 - 2ax = 0$, $a > 0$, and passing through the line $y = x$, $z = h$, $0 < h < a$.

22 In Eq. (1-38) let the coefficients P, Q, R be analytic functions in $\mathcal{T}$, that is, differentiable any number of times and such that the Taylor's series expansion about a point of $\mathcal{T}$ converges. Let Γ be a curve described by Eq. (1-54), where f, g, h are analytic functions of t. Let Γ lie in $\mathcal{T}$, and let (x_0,y_0,z_0) be a point on Γ corresponding to $t = t_0$. We wish to determine a solution of the Cauchy problem in a neighborhood by means of a power-series expansion. The problem is whether or not the given data determine the coefficients in the expansion

$$z = z_0 + p_0(x - x_0) + q_0(y - y_0) + \frac{1}{2!}\left[\varphi_{xx}|_0\,(x - x_0)^2 + \varphi_{xy}|_0\,(x - x_0)(y - y_0) + \varphi_{yy}|_0\,(y - y_0)^2\right] + \cdots$$

about (x_0,y_0,z_0). The condition that $z = \varphi(x,y)$ contain Γ near (x_0,y_0,z_0) is

$$h(t) = \varphi[f(t),g(t)] \tag{1}$$

Differentiate (1); then

$$pf'(t) + qg'(t) = h'(t) \tag{2}$$

Set $t = t_0$; then

$$p_0 f'(t_0) + q_0 g'(t_0) = h'(t_0) \tag{3}$$

From Eq. (1-38),

$$P_0 p_0 + Q_0 q_0 = R_0 \tag{4}$$

Accordingly, if the determinant

$$\Delta = Pg' - Qf' \tag{5}$$

is not zero at t_0, then (3) and (4) determine p_0 and q_0 uniquely. In this event successive differentiation of (2) and Eq. (1-38) determines the higher derivatives of φ uniquely also. Note that the condition $\Delta \neq 0$ at t_0 is hypothesis 5 of Theorem 1-1 and ensures that Γ is noncharacteristic at (x_0,y_0,x_0). If $\Delta = 0$ at $t = t_0$, either Eqs. (2) and (4) are dependent, or else they are incompatible. The first possibility occurs if, and only if, Γ is characteristic at (x_0,y_0,z_0). In this event there are infinitely many distinct integral surfaces containing Γ. If the second possibility occurs, no solution of the Cauchy problem exists. In each of the following Cauchy problems determine whether or not a solution exists. If a unique solution exists, find the power-series expansion to second-degree terms about the indicated point containing the given curve.

(a) $xp + yq = 0$; $(1,1,1)$; $x = 1$, $y = t^2$, $z = t^3$
(b) $yp - xq = 0$; $(1,0,1)$; $x = t$, $y = 0$, $z = t^2$
(c) $yp - xq = z(x^2 + y^2)$; $(1,0,0)$; $x = \cos t$, $y = \sin t$, $z = t$
(d) $yp - xq = (x^2 + y^2)$; $(0,1,0)$; $x = 0$, $y = t$, $z = 0$

23 With reference to Prob. 19, Eq. (1) in the case $n = 3$ becomes

$$P_1 u_x + P_2 u_y + P_3 u_z = R \tag{1}$$

where the P_i and R are functions of x, y, z, u. Let S be a given surface in xyz space defined parametrically by

$$x = f(s,t) \qquad y = g(s,t) \qquad z = h(s,t) \tag{2}$$

The Cauchy problem for (1) is to determine a solution of (1) such that at each point (x,y,z) on the surface S

$$u(x,y,z) = F(x,y,z) \tag{3}$$

where F is a given continuously differentiable function. A method of solution is as follows. Let w_1, w_2, w_3 be functionally independent integrals of the subsidiary equations (3) in Prob. 19. Eliminate the parameters s, t from the simultaneous equations

$$w_1\{f(s,t),g(s,t),h(s,t),F[f(s,t),(s,t),h(s,t)]\} = c_1$$

$$w_2 = c_2 \qquad w_3 = c_3$$

to obtain a functional relation $\Phi(c_1,c_2,c_3) = 0$. Then the solution is $\Phi(w_1,w_2,w_3) = 0$. Obtain a solution of each of the following Cauchy problems.

(a) $xu_x + yu_y + zu_z = nu$; $u = x + y + z$ on the surface $x = t$, $y = s$, $z = st$
(b) $u_x + xu_y + xyu_z = xyzu$; $u = x^2 + y^2$ on the surface $x = s$, $y = t$, $z = 0$

Sec. 1-2

1. Classify each of the following equations to the extent of the definitions given in section 1-2.

(a) $u_{xy} + x^2 u_{yy} + (\cos y) u_x = (\tan xy) u + x^2 y^2$

linear second order, two independent variables

(b) $\left(\frac{\partial \varphi}{\partial x}\right)^2 \frac{\partial^2 \varphi}{\partial x^2} + y^2 \frac{\partial^2 \varphi}{\partial y^2} + \varphi = 0$

non-linear, quasilinear, second order, two independent variables

(c) $x^2 r + y t - z p + (1 - z^3) q = x$

Non-linear, almost linear, second order, two independent variables

(d) $x^2 p - y z^2 q = 0$

not linear, quasilinear, first order, two independent variables

(f) $pq = z$

not linear, almost-linear, first order, two independent variables

(g) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ Laplace's equation

linear, second-order, three independent variables

(h) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ Cauchy Riemann equations

linear, first-order, two independent variables

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Sec. 1-3

3. Eliminate the arbitrary functions which appear and obtain a differentiable equation of lowest order

(a) $z = x f(y)$

$$x p - z = 0$$

(b) $u = e^{-x} F(x-2y)$

$$2u_x + u_y = -2u$$

(c) $G(x^2+y^2+z^2, z) = 0$

$$xq - yp = 0$$

(d) $u = x^n F(\frac{y}{x}, \frac{z}{x})$ n is a positive integer

$$x u_x + y u_y + z u_z = nu$$

(e) $z e^{-x^2} + f(x^2+y^2) = 0$

$$yp - xq = 2xyz$$

(f) $z = f(y)\cos x + g(y)\sin x$

$$z + z_{xx} = 0$$

(g) $u = f(x-ct) + g(x+ct)$

$$u_{tt} - c^2 u_{xx} = 0$$

(h) $u = f_1(y-mx) + x f_2(y-mx) + x^2 f_3(y-mx)$

$$u_{xxx} + m^3 u_{yyy} + 2m^2 u_{yxy} + m u_{yxx} + 2m u_{xxy} + m^2 u_{xyy} = 0$$

6. Eliminate arbitrary constants $a, b, c, \ldots$ which appear and obtain a partial differential equation of lowest order

(a) $z = e^{ax+by}$

$$xp + yq = z \log z$$

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(b) $a(x^2+y^2)+bz^2=1$

$$p^2-q^2+z(r-t)=0 \qquad b\neq 0$$

(c) $z=bx^a y^{1-a}$

$$xp+yq=z$$

(d) $z=ax^2+2bxy+cy^2$

$$z_{xxx}=0 \text{ or } z_{yyy}=0$$

(e) $u=Ae^{ax}\cos(ay)$

$$u_{xx}+u_{yy}=0$$

(f) $u=A\cos(ax)\cos(at)$

$$u_{xx}-u_{tt}=0$$

(g) $u=Ae^{-a^2t}\cos(ax)$

$$u_t=u_{xx}$$

(h) $az+b=a^2x+y$

$$z_{xx}=2 \text{ or } z_{yy}=0 \qquad a\neq 0$$

(i) $z=ax+by+a^2+b^2$

$$z=xp+yq+p^2+q^2$$

---

7. Find a partial differential equations of lowest order satisfied by each surface in the given family surfaces.

(a) All planes through the point $(1,0,0)$ not perpendicular to the xy-plane

$$(x-1)p+yq=z$$

(b) All spheres of the unite radius

$$z_x=-x/z \text{ or } z_y=-y/z$$

(c) The family of all tangent planes to the surface $z=xy$

$$xp + yq - pq = z$$

Sec. 1-4

8. Hold one independent variable constant and integrate with respect to the remaining variable so as to obtain a solution involving an arbitrary function. Verify that your answer is correct by substituting into the differential equation.

(a) $q = x^2 + y^2$

$$z = x^2 y + \frac{y^3}{3} + f(x)$$

(b) $p = \sin \frac{x}{y}$

$$z = -y \cos \frac{x}{y} + f(y)$$

(c) $z_x + xz = x^3 + 3xy$

$$z = f(y)\, e^{-x^2/2} + x^2 - 2 + 3y$$

---

9. Let $Ap + Bq = 0$ be a first-order equation; where A, B are constants. Assume a solution of the form $z = f(ax + by)$, f arbitrary and a, b constants. Substitute into the differential equation to determine suitable values of a, b

$$Aa + B \cdot b = 0$$

---

10. Obtain the general solution

(a) $3p - 4q = x^2$

$$z = f(4x + 3y) + x^3/9$$

(b) $p - 3q = \sin x + \cos y$

$$z = f(3x + y) - \cos x - \frac{1}{3} \sin y$$

(c) $5p + 4q + z = x^3 + 1 + 2e^{3y}$

$$z = e^{-x/5} f(4x - 5y) + x^3 - 15x^2 + 150x - 149 + \frac{2e^{3y}}{13}$$

(d) $p + 2q - 5z = \cos x + y^3 + 1$

$$z = e^{5x} f(2x - y) + \frac{1}{26} \sin x - \frac{5}{26} \cos x - \frac{1}{5} y^3 - \frac{6}{25} y^2 - \frac{24}{125} y - \frac{173}{625}$$

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(e) $p - aq = e^{mx}\cos by$ a, m, b constants

$$z = f(ax+y) + e^{mx}\frac{m\cos by - ab\sin by}{m^2 + a^2b^2}$$

---

11. Make the change of independent variables $\xi = \log x$, $\eta = \log y$ and reduce the differential equation to one with constant coefficients. Obtain the general solution.

(a) $4xp - 2yq = 0$

$$z = f(x^2y^4)$$

(b) $2xp + 3yq = \log x$

$$z = f(x^3y^2) + \frac{1}{4}\log^2 x$$

(c) $xp - 7yq = x^2y$

$$z = f(x^7y) - \frac{x^2y}{5}$$

(d) $8xp - 5yq + 4z = x^2\cos x$

(e) $axz_x + byz_y + cz = x^2 + y^2$

$$z = x^{-c/a}F(x^by^{-a}) + \frac{x^2}{2a+c} + \frac{y^2}{2b+c}$$

---

12. Follow the method of the text in Example 1-8 and obtain the general solution in the form (1-37)

(a) $xyp - x^2q + yz = 0$

$$z = \frac{f(x^2+y^2)}{x}$$

(b) $yp - xq = 0$

$$z = f(x^2+y^2)$$

(c) $(x+a)p + (y+b)q + cz = 0$ a, b, c constants

$$z = (x+a)^{-c} f\left(\frac{x+a}{y+b}\right)$$

---

13. Let $Lu = Au_x + Bu_y + Cu_z + Du = G$ be a linear first-order equation in three independent variables x, y, z and dependent variable u, where A, B, C, D are constants and $G(x,y,z)$ is a given function. Assume $A \neq 0$. To obtain a solution of the homogeneous equation $Lu = 0$ which involves an arbitrary function assume a solution of the form $u = e^{-Dx/A} f(ax+by+cz)$, substitute into $Lu = 0$, and show that the constants a, b, c must satisfy $Aa + Bb + Cc = 0$ if the assumed form satisfies $Lu = 0$ for arbitrary choice of f. Conversely, if a, b, c are chosen so that the preceding equation holds, then $u = e^{-Dx/A} f(ax+by+cz)$ is a solution for arbitrary (differentiable) f. If $A = 0$ but $B \neq 0$, one can proceed similarly with $u = e^{-Dx/B} f(ax+by+cz)$, etc. Find a solution involving an arbitrary function for each of the following equations.

(a) $2u_x - u_y + 4u_z + u = 0$

$$u = e^{-x/2} f(x - 2y - z)$$

(b) $u_x - 4u_y + 7u_z - u = x + y + z + 1$

$$u = e^{x} f(x + 2y + z) + \frac{x^2 + y^2 + z^2}{2} + z$$

Sec. 1-5

14. Obtain the general solution

(a) $p + xq = z$

$$z = F(2y - x^2, z e^{-x})$$

(b) $xp + yq = nz$ n constant

$$F\left(\frac{y}{x}, \frac{z}{nx}\right) = 0$$

(c) $(x+z)p + (y+z)q = 0$

$$F\left(z, \frac{x+z}{y+z}\right) = 0$$

(d) $(x+y)p + (y-x)q = z$

$$F\left(\tan^{-1}\frac{y}{x} + \frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) + \ln x, \frac{z}{x+y}\right) = 0$$

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(e) $zp + yq = x$

$$F\left(x^2 - z^2, \frac{x+z}{y}\right) = 0$$

(f) $(x+y)(p-q) = z$

$$F\left(x+y, \frac{x+y}{z}\right) = 0$$

(g) $yp - xq = x^3 y + x y^3$

$$z = \frac{x^2(x^2+y^2)}{2} + f(x^2+y^2)$$

(h) $(mz - ny)p + (nx - lz)q = ly - mx$ $\quad$ l, m, n constants

$$F(lx + my + nz, x^2 + y^2 + z^2) = 0$$

(i) $x^2 p + y^2 q = axy$ $\quad$ $a \neq 0$ constant

$$z = \frac{axy \log(x/y)}{x-y} + f\left(\frac{x-y}{xy}\right)$$

(j) $(y^3 x - 2x^4)p + (2y^4 - x^3 y)q = 9z(x^3 - y^3)$

(K) $x(y-z)p + y(z-x)q = z(x-y)$

$$xyz = f(x+y+z)$$

(l) $yp - xq + z + x^2 + y^2 - 1 = 0$

(m) $p - 2q = 3x^2 \sin(y+2x)$

$$z = x^3 \sin(y+2x) + f(y+2x)$$

(n) $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

(o) $\cos y \dfrac{\partial z}{\partial x} + \cos x \dfrac{\partial z}{\partial y} = \cos x \cos y$

$$z = \sin y + f(\sin x - \sin y)$$

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(p) $(z+e^{x})p+(z+e^{y})q=z^2-e^{x+y}$

(q) $xp+yq=2xy(a^2-z^2)^{\frac{1}{2}}$

$$\sin^{-1}(z/a)=2xy+f(y/x)$$

---

15. Refer to Prob. 2b and find the general form of the equation of all surfaces orthogonal to the given family of surfaces.

(a) $x^2+y^2+z^2=2ax$

$$z=yf\left(\frac{x^2+y^2+z^2}{y}\right)$$

(b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+z^2=c^2$

$$z=x^{a}f\left(\frac{x^{a^2}}{y^{b^2}}\right)$$

16.

(b) The velocity potential Φ of a stationary velocity field of fluid flow is $\Phi(x,y,z)=xy+xyz^2$. The velocity field is $V=\nabla\Phi$. The trajectories of the field are the curves C such that the tangent to the curve at each point has the direction of V at the point, i.e., the field lines. Find the trajectories.

(c) The potential Φ of an electrostatic field is $\Phi(x,y,z)=1/x+1/y+1/z$. The electric field is $E=-\nabla\Phi$. Find the field lines of the E field.

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17. Find the general form of the equation of all surfaces such that the tangent plane to each point on the surface passes through the fixed point (0,0,a).

$$z=a+xf(x/y)$$

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18. Find the general solution of each of the following.

(a) $P_1 u_x + P_2 u_y + P_3 u_z + Ru = 0$ P_1, P_2, P_3, R constant.

$u = e^{-Rx/P_1} f(P_2 x - P_1 y, P_3 x - P_1 z)$, $P_1 \neq 0$

(b) $(y+z)u_x + (z+x)u_y + (x+y)u_z = 0$

$u = f(x^2 - y^2 + 2xz - 2yz, x^2 - z^2 + 2xy - 2yz)$

(c) $3u_x + 5u_y - u_z = \cos y - 2e^{-z}$

$u = f(5x - 3y, 5z + y) + \frac{1}{5}\sin y - 2e^{-z}$

(d) $u_x - u_y + 7u_z + u = Ae^{ax+by+cz}$ A, a, b, c constants

(e) $(x+z)u_x + yu_y - 2u_z = 4e^z$

$u = f[z + 2\log y, z + 2\log(x+z-2)] - 2e^z$

(f) $xzu_x + yzu_y - (x^2+y^2)u_z = zy^2 \sin y$

(g) $(z-y)u_x + yu_y - zu_z = y(x+z) - y^2$

$u = f(yz, x+y+z) + xy$

(h) $(\tan x)u_x + (\tan y)u_y + u_z = \sin z$

$u = f(\sin x/\sin y, e^{-z}\sin x) - \sin z$

(i) $xu_x + yu_y + zu_z = nu$ n constant

$u = x^n f(\frac{x}{y}, \frac{z}{x})$

(j) $\sqrt{x}\, u_x + \sqrt{y}\, u_y + \sqrt{z}\, u_z + u = x + y$

(k) $\frac{\partial u}{\partial x_1} + x_1 \frac{\partial u}{\partial x_2} + x_1 x_2 \frac{\partial u}{\partial x_3} + x_1 x_2 x_3 \frac{\partial u}{\partial x_4} = 0$

$u = f(x_1^2 - 2x_2, x_2^2 - 2x_3, x_3^2 - 2x_4)$

(l) $x_1 \frac{\partial u}{\partial x_1} + x_1 x_2 \frac{\partial u}{\partial x_2} + x_1 x_2 x_3 \frac{\partial u}{\partial x_3} + x_1 x_2 x_3 x_4 \frac{\partial u}{\partial x_4} + u = x_1^2 x_2^2$

(g) $2xzp + 2yzq = z^2 - x^2 - y^2$; $x + y + z = 0$, $x^2 + y^2 + z^2 = a^2$

$$(x^2+y^2+z^2)^2 = 2a^2(x^2+y^2+xy)$$

(h) $(y-z)p + (z-x)q = x-y$; $x = t$, $y = 2t$, $z = 0$

$$x^2+y^2+z^2 = \frac{5}{9}(x+y+z)^2$$

(i) $x(x^2+y^2)p + 2y^2(xp+yq-z) = 0$; $x^2+y^2 = a^2$, $z = h$

$$(x^2+y^2)(a^2z^2 - h^2y^2) = a^2 b x^2 z$$

(j) $(x^2+y^2)p + 2xyq = xz$; $x = a$, $y^2+z^2 = a^2$

(K) $x(y-z)p + y(z-x)q = z(x-y)$; $x = y = z = t$

$$(x+y+z)^3 = 27xyz$$

(l) $z(x+z)p - y(y+z)q = z(x-y)$; $x = 1$, $y = t$, $z = \sqrt{t}$

(m) $yp - xq = 2xyz$; $x = t$, $y = t$, $z = t$

$$z = \left(\frac{x^2+y^2}{2}\right)^{\frac{1}{2}} \exp\left(\frac{x^2-y^2}{2}\right)$$

(n) $(y^2 - x^2 + 2xz)p + 2y(z-x)q = 0$; $x = 0$, $y^2 + 4z^2 = 4a^2$

(o) $p\sec x + aq = \cot y$; $z(a,y) = \sin y$

$$a\sin(y - a\sin x) - \log[\sin(y - a\sin x)] = az - \log\sin y$$

---

21. Find a surface orthogonal to the sphere $x^2+y^2+z^2-2ax = 0$, $a > 0$, and passing through the line $y = x$, $z = h$, $0 < h < a$

$$-x^2+y^2+z^2 = h^2$$

---

22. If a unique solution exists, find the power series expansion to second degree term about the indicated point containing the given curve.

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(a) $xp + yq = 0$; $(1,1,1)$; $x = 1$, $y = t^2$, $z = t^3$

$z = 1 - \frac{3(x-1)}{2} + \frac{3(y-1)}{2} + \frac{15(x-1)^2/4 - 9(x-1)(y-1)/2 + 3(y-1)^2/4}{2} + \dots$

(b) $yp - xq = 0$; $(1,0,1)$; $x = t$, $y = 0$, $z = t^2$

(c) $yp - xq = z(x^2+y^2)$; $(1,0,0)$, $x = \cos t$, $y = \sin t$, $z = t$

No solution exists

(d) $yp - xq = (x^2+y^2)$; $(0,1,0)$; $x = 0$, $y = t$, $z = 0$

---

23. Obtain a solution to each of the following Cauchy Problems

(a) $xu_x + yu_y + zu_z = nu$; $u = x+y+z$ on the surface $x = t$, $y = s$, $z = st$

$u = (x+y+z)\left(\frac{xy}{z}\right)^{n-1}$

(b) $u_x + xu_y + xyu_z = xyzu$; $u = x^2+y^2$ on the surface $x = s$, $y = t$, $z = 0$

condition that $M = L$ is that

$$E_{ij}(\mathbf{x}) = A_{ij}(\mathbf{x}) \qquad F_i(\mathbf{x}) = B_i(\mathbf{x}) \qquad G(\mathbf{x}) = C(\mathbf{x}) \qquad \begin{array}{l}\text{on } \mathscr{R} \\ i, j = 1, \ldots, n\end{array}$$

that is, the coefficients are identical. Clearly the foregoing is sufficient to ensure that $M = L$. Conversely, suppose $M = L$. Then $Mu = Lu$ holds for every polynomial in the variables $x_1, \ldots, x_n$. Choose $u = 1$. Then $G(\mathbf{x}) = C(\mathbf{x})$. Choose $u = x_1$. Then $G = C$ and $Mu = Lu$ implies $F_1(\mathbf{x}) = B_1(\mathbf{x})$. In this manner the identity of the coefficients in M and L can be established.

The operator L in Eq. (2-72) is called *self-adjoint* if $L^* = L$. It is clear from the foregoing that a necessary and sufficient condition for L to be self-adjoint is that the equations

$$\sum_{j=1}^{n} \frac{\partial A_{ij}}{\partial x_j} = B_i \qquad i = 1, \ldots, n \tag{2-79}$$

hold on $\mathscr{R}$. Accordingly every self-adjoint linear second-order operator has the form

$$L = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij} \frac{\partial}{\partial x_j}\right) + C \tag{2-80}$$

If L is self-adjoint, then Green's formula states that for each pair of functions u, v with the continuity properties assumed above,

$$\int_{\mathscr{R}} (vLu - uLv)\, d\tau = \int_{b(\mathscr{R})} \left(\sum_{i=1}^{n} P_i \gamma_i\right) d\sigma \tag{2-81}$$

where

$$P_i(x) = \sum_{j=1}^{n} A_{ij}\left(v \frac{\partial u}{\partial x_j} - u \frac{\partial v}{\partial x_j}\right) \qquad i = 1, \ldots, n \tag{2-82}$$

In the case where the coefficients in the operator L are constants, it follows from Eq. (2-79) that L is self-adjoint if, and only if, $B_i = 0$, $i = 1, \ldots, n$. Some classical examples of self-adjoint operators when $n = 3$ appear in the equations of Examples 2-14 to 2-16.

PROBLEMS

Sec. 2-2

1 **(a)** Show that the general solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is

$$u = f(x - ct) + g(x + ct)$$

19. Obtain the general solution of the following.

(a) $u_x + x u_y + xy u_z = xyzu$

$f(x^2-2y,\ y^2-2z,\ ue^{-z^2/2}) = 0$

(b) $x u_x + (z+u) u_y + (y+u) u_z = y+z$

(c) $x u_x + y u_y + z u_z = u + \frac{xy}{z}$

$u = x f\left(\frac{y}{x}, \frac{z}{x}\right) + \frac{xy \log x}{z}$

(d) $(s-x)u_x + (s-y)u_y + (s-z)u_z = s-u$, $s = x+y+z+u$

Sec. 1-6

20. Determine the integral surface which passes through the given curve.

(a) $p+q=z$; $z=\cos x$, $y=0$

$z = e^{y}\cos(x-y)$

(b) $z(p-q) = y-x$; $x=1$, $z=y^2$

$z = (y^2+4xy)^{\frac{1}{2}}$

(c) $xp - yq = 0$; $x=y=z=t$

$z^2 = xy$

(d) $(x+z)p + (y+z)q = 0$; $x=1-t$, $y=1+t$, $z=t$

$z = \frac{y-x}{2x+2z}$

(e) $x^2p + y^2q = z^2$; $x=t$, $y=2t$, $z=1$

$z = \frac{xy}{2x-y+xy}$

(f) $xzp + yzq + xy = 0$; $xy=a^2$, $z=h$

1 (b) Find the general solution of the inhomogeneous wave equation $u_{tt} - c^2u_{xx} = x^2 + xt - \sin \omega t$, $\omega > 0$ a constant.

2 Find the general solution of the equation of spherical waves

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$$

Hint: Make the change of dependent variable $v = ru$.

3 Symbolic methods of obtaining particular solutions of Eq. (2-1) exist for the constant-coefficient case. Let L be the linear operator in Eq. (2-3), where the coefficients are real constants. Then $L = P(D_x,D_y)$ is a polynomial in the operators D_x, D_y. If $f(x,y)$ is a function, it is understood in the following that the symbolic equation

$$\frac{f(x,y)}{P(D_x,D_y)} = \varphi(x,y)$$

means φ is a function such that

$$P(D_x,D_y)[\varphi(x,y)] = f(x,y)$$

Derive each of the following symbolic equations.

(a) $e^{ax+by}/P(D_x,D_y) = e^{ax+by}/P(a,b)$ provided the constants a, b are such that $P(a,b) \neq 0$.

(b) $\sin (ax + by)/P(D_x^2,D_y^2) = \sin (ax + by)/P(-a^2,-b^2)$ provided $P(-a^2,-b^2) \neq 0$. A similar equation holds for $\cos (ax + by)$.

(c) $\cos (ax + by)/P(D_x,D_y) = \text{Re}\,[e^{i(ax+by)}/P(ia,ib)]$ where $i = \sqrt{-1}$ and Re [ ] means "the real part of [ ]," provided a, b are real constants such that $P(ia,ib) \neq 0$.

(d) $\sin (ax + by)/P(D_x,D_y) = \text{Im}\,[e^{i(ax+by)}/P(ia,ib)]$ where $i = \sqrt{-1}$ and Im [ ] means "the imaginary part of [ ]," provided a, b are as stated in **c**.

(e) $e^{mx} \cos (ax + by)/P(D_x,D_y) = \text{Re}\,[e^{mx+i(ax+by)}/P(m + ia, ib)]$ provided m is a real constant and a, b are as stated in **c**.

(f) $x^n/(D_x + a)^m = (a^{-m} - ma^{-(m+1)}D_x + [m(m + 1)/2!]a^{-(m+2)}D_x^2 - \cdots)x^n$ provided $a \neq 0$ and n, m are positive integers. A similar result holds for $y^n/(D_y + a)^m$.

4 Obtain the general solution.

(a) $r - 10s + 9t = 0$

(b) $4z_{xy} + z_{yy} = \cos y + 1$

(c) $z_{xx} + z_x + x + y + 1 = 1$

(d) $r = xy$

(e) $z_{yy} + z = e^{x+y}$

(f) $r - t + p + q + x + y + 1 = 0$

(g) $2s + 3t - q = 6 \cos (2x - 3y) - 30 \sin (2x - 3y)$

(h) $s + ap + bq + abz = e^{mx+ny}$ a, b, m, n constants

(i) $r - 4t = 12x^2 + \cos y + 4$

(j) $r - t - 3p + 3q = xy + e^{x+2y}$

(k) $r - 2s + t = 4e^{3y} + \cos x$

5 If the operator L with constant coefficients is factorable as in Eq. (2-6), a particular solution of Eq. (2-1) may sometimes be obtained by the following procedure. With reference to Eq. (2-6) let $L_2z = v$. Now obtain a particular solution v_p of the first-order linear equation $L_1v = G$. In turn derive a particular solution z_p of the first-order equation $L_2z = v_p$. Then $Lz_p = L_1(L_2z_p) = L_1v_p = G$. Use this method to obtain a particular solution of each of the following equations. Also find the general solution.

(a) $r + 5s + 6t = \log(y - 2x)$ (b) $r - s - 2t = (y - 1)e^x$

(c) $r - 4t = \dfrac{4x}{y^2} - \dfrac{y}{x^2}$

6 Use simple integrations, reduction to an ordinary differential equation, etc., to obtain a solution involving two arbitrary functions.

(a) $t = x^3 + y^3$ (b) $z_{xy} = \dfrac{x}{y} + a$ $a = \text{const}$

(c) $t - xq = x^2$ (d) $xr = c^2p + x^2y^2$ $c = \text{const}$

(e) $yz_{xy} + z_x = \cos(x + y) - y \sin(x + y)$

(f) $y^2t + 2yq = 1$

(g) $(x - y)z_{xy} - z_x + z_y = 0$ Euler-Poisson-Darboux equation

Hint: Let $u = (x - y)z$.

7 Show that the equation

$$ax^2z_{xx} + 2bxyz_{xy} + cy^2z_{yy} + dxz_x + eyz_y + fz = G(x,y)$$

where a, b, c, d, e, f are constants, is transformed into an equation with constant coefficients under the transformation of independent variables $\xi = \log x$, $\eta = \log y$.

8 Use the method of Prob. 7 to obtain the general solution of each of the following equations.

(a) $x^2r - y^2t - 2xp + 2yq = 0$ (b) $x^2r - xys - xp = 1$

(c) $xyz_{xy} - y^2z_{yy} - 2xz_x + 2yz_y - 2z = 0$

(d) $x^2r - 2xys - 3y^2t + xp - 3yq = 2 \log xy + 4x$

(e) $x^2r - y^2t = xy$

(f) $x^2z_{xx} + 2xyz_{xy} + y^2z_{yy} - nxz_x - nyz_y + nz = x + y$ $n = \text{const} \neq 0$

(g) $xp + yq - xys = z$

(h) $(axD_x + byD_y)^2z + \lambda^2z = 0$ a, b, λ constants

9 **(a)** Let L be the linear operator in Eq. (2-3). Define a functionally invariant pair of Eq. (2-5) as done in the text for Eq. (2-12). Refer to the method in the text and show that ψ, φ is a functionally invariant pair of Eq. (2-5) if, and only if, ψ, φ satisfy the system of partial differential equations

$$A\varphi_x^2 + 2B\varphi_x\varphi_y + C\varphi_y^2 = 0 \qquad (1)$$

$$2[A\psi_x\varphi_x + B(\psi_x\varphi_y + \psi_y\varphi_x) + C\psi_y\varphi_y] + \psi(A\varphi_{xx} + 2B\varphi_{xy} + C\varphi_{yy} + D\varphi_x + E\varphi_y) = 0 \qquad (2)$$

$$L\psi = 0 \qquad (3)$$

(b) Let the coefficients in L be real constants. Then the characteristic equation (1) has real-valued solutions if, and only if, $\Delta = B^2 - AC \geq 0$. Assume $\Delta > 0$. If $A \neq 0$, let r_1, r_2 be the distinct roots of $Ar^2 + 2Br + C$, and let $\varphi_1 = r_1x + y$, $\varphi_2 = r_2x + y$. If $A = 0$, let $\varphi_1 = x$, $\varphi_2 = Cx - 2By$. Then φ_1, φ_2 are functionally independent solutions of (1). If $\varphi = r_1x + y$, show that (2) becomes

$$(Ar_1 + B)\psi_x + (Br_1 + C)\psi_y + \frac{Dr_1 + E}{2}\psi = 0 \qquad (4)$$

Let $\alpha = Ar_1 + B$, $\beta = Br_1 + C$, $\gamma = (Dr + E)/2$. Then the general solution of (4) is $\psi = e^{-\gamma x/\alpha}h(\beta x - \alpha y)$. Let $t = \beta x - \alpha y$, substitute this form of ψ into (3), and show that $h(t)$ must satisfy a first-order equation

$$ah'(t) + bh(t) = 0 \tag{5}$$

and so

$$h(t) = e^{-bt/a}$$

(c) Let the coefficients in L be real constants, $\Delta > 0$,

$$AE^2 - 2BDE + CD^2 + 4\Delta F = 0 \tag{6}$$

Show that Eq. (2-5) has the general solution

$$z = e^{(cx+dy)/2\Delta}[f(r_1x + y) + g(r_2x + y)]$$

$c = CD - BE$, $d = AE - BD$, r_1, r_2 as described in **b** above.

10 With reference to Prob. **9a**, a functionally independent pair of Eq. (2-5) when the coefficients are not constants can often be constructed as follows. Factor (1) as

$$A(\varphi_x - m_1\varphi_y)(\varphi_x - m_2\varphi_y) = 0$$

and obtain particular solutions of the linear first-order equations $\varphi_x - m_1\varphi_y = 0$, $i = 1, 2$. Now choose a simple but nontrivial solution of $L\psi = 0$ such that Eq. (2) in Prob. **9a** holds. Obtain the general solution of each of the following equations.

(a) $e^{2y}(r - p) = e^{2x}(t - q)$ **(b)** $(y^2 - x^2)(r - t) + 4(xp + yq - z) = 0$

(c) $xr + (y - x)s - yt = q - p$

Sec. 2-3

11 Obtain the general solution.

(a) $u_{xx} + 2u_{xy} + u_{yy} - 2u_{yz} - 2u_{xz} + u_{zz} = x \cos x + y \cos y + z \cos z$

(b) $2u_{xx} + u_{xy} + 2u_{xz} - u_{yy} - u_{yz} = x^2 + y^2 + z^2$

(c) $u_{xx} + 2u_{xy} + u_{yy} - u_{zz} - 2u_z - u = e^{2x} + 2 \cos y + z + 2$

(d) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} + 2yzu_{yz} + 2xzu_{xz} + z^2u_{zz} = \ln xyz$

(e) $u_{tt} = Au_{xx} + 2Bu_{xy} + Cu_{yy}$, A, B, C positive constants such that $B^2 - AC = 0$

12 Construct exponential-type solutions. Also find a particular solution if the equation is inhomogeneous.

(a) $z_{xx} - 2z_{xy} + z_y - z = 0$ **(b)** $z_{xx} + 4z_{xy} + z_{yy} + z_x + z_y = 3e^{2x} + xy$

(c) $z_{xx} - 2z_x - z_y = x + y$ **(d)** $z_{xx} + z_{yy} = x + ye^y$

(e) $u_{xx} + u_{yy} = 0$ **(f)** $u_{xx} + u_{yy} + u_{zz} = 0$

(g) $u_{xx} + u_{yy} + k^2u = 0$ $k = \text{const}$ **(h)** $u_{xx} + u_{yy} = ku_t$ $k = \text{const}$

(i) $c^2(u_{xx} + u_{yy}) = u_{tt}$ $c = \text{const}$

13 Find the equation which the constants l, m, n must satisfy if the function u defined by

$$u(x,y,t) = \int_\alpha^\beta f(lx + my + bt, \xi)\, d\xi$$

satisfies the two-dimensional wave equation in Example 2-3. Assume that it is legitimate to differentiate within the integral sign.

14 Assume that it is legitimate to differentiate within the integral sign. Verify that the function u defined by

$$u(x,y,z,t) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x \sin \xi \cos \eta + y \sin \xi \sin \eta + z \cos \xi + ct, \xi, \eta)\, d\xi\, d\eta$$

(Whittaker's solution) satisfies the three-dimensional wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$$

15 Assume that it is legitimate to differentiate within the integral sign. Verify that the function u defined by

$$u(x,y,z) = \int_0^{2\pi} f(x \cos \xi + y \sin \xi + iz, \xi)\, d\xi \qquad i = \sqrt{-1}$$

satisfies Laplace's equation in Example 2-5.

Sec. 2-4

16 Show that Eq. (2-30) holds.

17 Let L be the linear operator (2-3) in which the coefficients may be variable. Let Eq. (2-27) define a transformation of the independent variables where the functions have continuous second derivatives. Carry out the calculation of the derivatives of z, substitute into Eq. (2-1), and show that the transformed equation is

$$Q(\xi)z_{\xi\xi} + 2Q(\xi,\eta)z_{\xi\eta} + Q(\eta)z_{\eta\eta} + (L_1\xi + D\xi_x + E\xi_y)z_\xi$$
$$+ (L_1\eta + D\eta_x + E\eta_y)z_\eta + Fz = G$$

where $Q(\xi)$, $Q(\xi,\eta)$, and $Q(\eta)$ are given in the text [following Eq. (2-28)] and

$$L_1\varphi = A\varphi_{xx} + 2B\varphi_{xy} + C\varphi_{yy}$$

Hence show that the normal forms of Eq. (2-1) in the three types are

$$z_{\xi\eta} + \alpha z_\xi + \beta z_\eta + \gamma z = G'(\xi,\eta) \qquad \text{hyperbolic}$$

$$z_{\eta\eta} + \alpha z_\xi + \beta z_\eta + \gamma z = G'(\xi,\eta) \qquad \text{parabolic}$$

$$z_{\xi\xi} + z_{\eta\eta} + \alpha z_\xi + \beta z_\eta + \gamma z = G'(\xi,\eta) \qquad \text{elliptic}$$

where α, β, γ are (in general) functions of ξ, η.

18 **(a)** In the hyperbolic equation

$$z_{xy} + Dz_x + Ez_y + Fz = G$$

suppose that $\partial D/\partial x = \partial E/\partial y$ holds. Then $E\,dx + D\,dy$ is an exact differential. Choose a function $a(x,y)$ such that $da = E\,dx + D\,dy$. Then $E = \partial a/\partial x$, $D = \partial a/\partial y$. The equation now takes the form

$$z_{xy} + a_y z_x + a_x z_y + Fz = G$$

Make the change of dependent variable $z = ue^{-a(x,y)}$ and show that the equation is transformed into

$$u_{xy} + bu = Ge^{a}$$

$$b(x,y) = F - a_x a_y - a_{xy} = F - DE - \frac{\partial D}{\partial x}$$

In particular the transformation is possible whenever $D = D(y)$, $E = E(x)$.

(b) Consider the parabolic equation

$$z_{xx} + Dz_x + Ez_y + Fz = G$$

Define the function a by

$$a(x,y) = \int D(x,y)\,dx$$

y held fast. Make the change of dependent variable $z = ue^{-a/2}$ and show that the equation is transformed into

$$u_{xx} + Eu_y + bu = Ge^{a/2}$$

$$b(x,y) = F - \tfrac{1}{4}a_x^2 - \tfrac{1}{2}a_{xx} - \tfrac{1}{2}Ea_y$$

(c) In the elliptic equation

$$z_{xx} + z_{yy} + Dz_x + Ez_y + Fz = G$$

suppose that $\partial D/\partial y = \partial E/\partial x$ holds. Then $D\,dx + E\,dy$ is an exact differential. Choose a function $a(x,y)$ such that $da = D\,dx + E\,dy$. Then $D = \partial a/\partial x$, $E = \partial a/\partial y$. The equation now takes the form

$$z_{xx} + z_{yy} + a_x z_x + a_y z_y + Fz = G$$

Make the change of dependent variable $z = ue^{-a/2}$ and show that the equation is transformed into

$$u_{xx} + u_{yy} + bu = Ge^{a/2}$$

$$b(x,y) = F - \tfrac{1}{4}a_x^2 - \tfrac{1}{4}a_y^2 - \tfrac{1}{2}a_{xx} - \tfrac{1}{2}a_{yy}$$

In particular the transformation is possible whenever

$$D = D(x) \qquad E = E(y)$$

19 In the hyperbolic equation

$$z_{xy} + Dz_x + Ez_y + Fz = 0$$

the functions $h = \partial D/\partial x + DE - F$, $k = \partial E/\partial y + DE - F$ are called *invariants* of the differential equation. For transformations of the independent variables of the type $\xi = \xi(x)$, $\eta = \eta(y)$, or a transformation of the dependent variable $z = \psi(x,y)u$, the differential equation is transformed into one of the same form with invariants

$$h_1 = \mu h \qquad k_1 = \mu k$$

Show that if either $h = 0$ or $k = 0$, the general solution can be obtained by quadratures.

Hint: Suppose $h = 0$. Then

$$\frac{\partial^2 z}{\partial x\,\partial y} + D\frac{\partial z}{\partial x} + E\frac{\partial z}{\partial y} + Fz = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y} + Dz\right) + E\left(\frac{\partial z}{\partial y} + Dz\right)$$

Let $u = \partial z/\partial y + Dz$ and solve the first-order linear equation $u_x + Eu = G$ to obtain an intermediate integral.

20 Obtain the general solution:

(a) $z_{xy} + z_x + z_y + z = 0$

(b) $z_{xy} + xyz_x + yz = 0$

(c) $z_{xy} - \dfrac{z_x}{x-y} + \dfrac{z_y}{x-y} = \dfrac{1}{x-y}$

(d) $z_{xy} - \dfrac{2z_x}{x-y} + \dfrac{3z_y}{x-y} - \dfrac{3z}{(x-y)^2} = 0$

21 Let $L = AD_x^2 + 2BD_xD_y + CD_y^2 + DD_x + ED_y + F$ be a second-order linear operator with real constant coefficients.

(a) Assume L is hyperbolic, $A \neq 0$. Show that the characteristics are the two families of straight lines

$$\xi = y + m_1 x = c_1 \qquad \eta = y + m_2 x = c_2$$

where $m_{1,2}$ are defined by Eq. (2-33) and Δ is defined by Eq. (2-26). Use the equations of the characteristics to obtain a transformation of independent variables such that the equation $Lz = G$ is transformed into the normal form

$$z_{\xi\eta} + dz_\xi + pz_\eta + fz = H(\xi,\eta)$$

where d, p, f are constants. Thus in the special case where $D = E = F = 0$ show that the general solution of the homogeneous equation $Lz = 0$ is $z = f(y + m_1x) + g(y + m_2x)$. If one of the coefficients D, E, F is different from zero, show that the change of dependent variable $z = e^{-(d\eta+p\xi)}u$ further transforms the equation into

$$u_{\xi\eta} + \beta u = H(\xi,\eta)e^{(d\eta+p\xi)}$$

where $\beta = f - pd$. Discuss fully the case when $A = 0$.

(b) Assume that L is parabolic, $A \neq 0$. Show that the characteristic curves are the straight lines $\xi = Bx - Ay = C_1$. Choose as a second independent function $\eta = x$ and show directly that the equation $Lz = G$ is transformed into the normal form

$$z_{\eta\eta} + dz_\xi + pz_\eta + fz = H(\xi,\eta)$$

where d, p, f are constants. Thus in the special case where $D = E = F = 0$ show that the general solution of the homogeneous equation $Lz = 0$ is $z = xf(Bx - Ay) + g(Bx - Ay)$. If one of the coefficients D, E, F is different from zero, show that the change of dependent variable $z = ue^{-p\eta/2}$ further transforms the equation into $u_{\eta\eta} + du_\xi + \beta u = H(\xi,\eta)e^{p\eta/2}$, where $\beta = f - \frac{1}{4}p^2$. Discuss fully the case when $A = 0$.

(c) Assume L is elliptic, and show that this implies $A \neq 0$. Show that the characteristic differential equations are

$$y' = -m_1 \qquad y' = -\overline{m}_1$$

where $m_1 = (-B + i\sqrt{-\Delta})/A$, $i = \sqrt{-1}$, and $\overline{m}_1$ denotes the complex conjugate of m_1. The general solutions of these are $\varphi(x,y) = y + m_1x = c_1$ and $\psi(x,y) = y + \overline{m}_1x = c_2$ [so that $\psi(x,y) = \overline{\varphi(x,y)}$]. Write $\varphi = \xi + i\eta$ and determine the expressions for $\xi(x,y)$, $\eta(x,y)$ (these are real-valued functions). Under the one-to-one transformation $\xi = \xi(x,y)$,

$\eta = \eta(x,y)$ show directly that the equation $Lz = G$ is transformed into the normal form

$$z_{\xi\xi} + z_{\eta\eta} + dz_\xi + pz_\eta + fz = H(\xi,\eta)$$

Thus, in the special case where $D = E = F = 0$ show that the general solution of $Lz = 0$ is $z = f(y + m_1x) + g(y + \overline{m}_1x)$. If one of the coefficients D, E, F is different from zero, show that the change of dependent variable $z = ue^{-(d\xi+p\eta)/2}$ further transforms the equation into

$$u_{\xi\xi} + u_{\eta\eta} + \beta u = H(\xi,\eta)e^{(d\xi+p\eta)/2}$$

where $\beta = f - \frac{1}{4}d^2 - \frac{1}{4}p^2$.

22 Classify each equation. Reduce to normal form and obtain the general solution.

(a) $4z_{xx} - 8z_{xy} + 4z_{yy} = 1$ **(b)** $4z_{xx} - 4z_{xy} + 5z_{yy} = 0$

(c) $r - 2s + t + a(p - q) + cz = (x + 2y)^2$ a, c nonzero constants

(d) $r - t + p + q + x + y + 1 = 0$ **(e)** $x^2r + 2xys + y^2t = 4x^2$

(f) $xr - (x + y)s + yt = \dfrac{x + y}{x - y}(p - q)$

(g) $x^2r - y^2t = xy$ **(h)** $\dfrac{\partial}{\partial x}\left(x^2\dfrac{\partial z}{\partial x}\right) = x^2\dfrac{\partial^2 z}{\partial y^2}$

(i) $r - (2\sin x)s - (\cos^2 x)t - (\cos x)q = 0$

(j) $(\sec^4 y)r - t + (2\tan y)q = 0$

(k) $xy(t - r) + (x^2 - y^2)s = py - qx - 2(x^2 - y^2)$

(l) $y^2z_{xx} - 2yz_{xy} + z_{yy} = z_x + 6y$

(m) $\dfrac{\partial}{\partial x}\left[\left(1 - \dfrac{x}{h}\right)^2 z_x\right] = \dfrac{1}{a^2}\left(1 - \dfrac{x}{h}\right)^2\dfrac{\partial^2 z}{\partial t^2}$ a, h real positive constants

23 Classify each of the following equations.

(a) $c^2(u_{xx} + u_{yy} + u_{zz}) = u_{tt}$ $c > 0$ a real constant

(b) $\dfrac{\partial u}{\partial t} = k\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$ $k > 0$ a real constant

(c) $u_{xx} + u_{yy} + u_{zz} = 0$

(d) $u_{xx} + u_{yy} - u_{tt} = 0$

(e) $u_{xx} + u_{yy} + u_{zz} + \lambda u = 0$ λ a constant

(f) $u_{xx} + u_{yy} - u_{zz} - u_{tt} = 0$

Sec. 2-5

24 In each of the following Cauchy problems determine whether **(i)** there exists a unique solution, **(ii)** there exist infinitely many distinct solutions, **(iii)** no solution exists. If **i**, find the solution.

(a) $u_{xx} - u_{tt} = 0$; $u(x,0) = b$, b a constant, $u_t(x,0) = \sin x$

(b) $u_{xx} - u_{tt} = 0$; $u(x,0) = \sin x$, $u_t(x,0) = b$, b a constant

(c) $u_{xx} - u_{tt} = 0$; $u = 0$, $u_x = 1$ on C, C is the line $t = x$

(d) $u_{xx} - u_{tt} = 0$; $u = 0$, $u_x = x$ on C, C is the line $t = x$

(e) $u_{xx} - 10u_{xt} + 9u_{tt} = 0$; $u(x,0) = x^2$, $u_t(x,0) = 1$

(f) $z_{xy} = 0$; $z = \cos x$, $z_x = 1$ on C, C is the line $y = x$

(g) $z_{xy} = 0$; $z(x,0) = 1$, $z_y(x,0) = x^2$

(h) $z_{xy} = xy$; $z = \cos x$, $z_y = x^3$ on C, C is the line $y = x$

(i) $z_{xy} + z_x = x$; $z = x^2$, $z_x = 0$ on C, C is the line $y = x$

(j) $z_{xy} = 0$; $z(x,0) = x$, $z_y(x,0) = 1$

(k) $z_{xy} - 4z_{yy} = 16y + 4e^{-y}$; $z(x,0) = -1$, $z_y(x,0) = 4x + 1$

(l) $z_{xx} - 2z_{xy} + z_{yy} = 4e^{3y}$; $z(x,0) = x^2$, $z_y(x,0) = \frac{1}{3}$

(m) $z_{xx} = x^2 + y^2$; $z(0,y) = 1$, $z_x(0,y) = 0$

25 Solve each of the following Cauchy problems.

(a) $x^2u_{xx} - 2xyu_{xy} - 3y^2u_{yy} = 0$; $u(x,1) = x$, $u_y(x,1) = 1$

(b) $yu_{xx} + (x + y)u_{xy} + xu_{yy} = xy$; $u(0,y) = y$, $u_x(0,y) = 1$, $y > 0$

(c) $2u_{xx} - 2yu_{yy} - u_y = 0$; $u(x,1) = \dfrac{x^2}{2}$, $u_y(x,1) = 2$

(d) $u_{xx} + (2\cos x)u_{xy} - (\sin^2 x)u_{yy} - (\sin x)u_y = 0$; $u = x^2$, $u_y = 1$, on C, C is the curve $y = \sin x$

(e) $x^2u_{xx} - 2xu_{xy} + u_{yy} + u_y = 0$; $u(1,y) = y^2$, $u_x(1, y) = e^y$

Sec. 2-7

26 Let

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x\,\partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

where the coefficients are functions with continuous second derivatives in a region $\mathscr{R}$ of the xy plane.

(a) Write out in full the expression for the adjoint L^* of L.

(b) Show from Eq. (2-75) that for any pair of functions u, v with continuous second derivatives in $\mathscr{R}$ the identity

$$vLu - uL^*v = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

holds, where

$$Q = A\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + B\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(D - \frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}\right)uv$$

$$P = B\left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right) + C\left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}\right) + \left(\frac{\partial B}{\partial x} + \frac{\partial C}{\partial y} - E\right)uv$$

(c) Let Γ be a simple closed curve lying in $\mathscr{R}$, $\mathscr{R}_1$ the region enclosed by Γ. Use Green's formula (2-77) in the case $n = 2$ and show that

$$\int_{\mathscr{R}_1} (vLu - uL^*v)\,dx\,dy = \int_{\Gamma} P\,dx + Q\,dy$$

the line integral being taken in the positive sense around Γ.

(d) Show directly from the expression for the adjoint of L derived in **a** that the adjoint of the adjoint of L is L: $L^{**} = L$.

(e) Show that L is self-adjoint if, and only if, the equations

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} = D \qquad \frac{\partial B}{\partial x} + \frac{\partial C}{\partial y} = E$$

hold in $\mathscr{R}$. Thus every second-order self-adjoint operator, in the case $n = 2$, has the form

$$Lu = \frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(B\frac{\partial u}{\partial x} + C\frac{\partial u}{\partial y}\right) + Fu$$

(f) A special case is the laplacian operator

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Use Green's formula derived in **c** to obtain the relation

$$\int_{\mathscr{R}_1} (v\,\nabla^2 u - u\,\nabla^2 v)\,dx\,dy = \int_{\Gamma}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds$$

where $\partial u/\partial n$ denotes the directional derivative of u in the direction of the exterior normal $\mathbf{n}$ on Γ. The line integral is taken in the positive sense around Γ. This relation is called the symmetric form of Green's theorem in the plane.

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Sec. 2-2

1.

(b) Find the general solution of the inhomogeneous

$u_{tt} - c^2 u_{xx} = x^2 + xt - \sin \omega t \quad \omega > 0$ a constant

$u = f(x-ct) + g(x+ct) - \frac{1}{12c^2}x^4 + \frac{t}{6c^2}x^3 + \frac{1}{\omega^2}\sin\omega t$

2. Find the general solution of the equation of spherical waves $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$

$u = \frac{1}{r}\left[f(x-ct) + g(x+ct)\right]$

4. Obtain the general solution.

(a) $r - 10s + 9t = 0$

$z = f(x+y) + g(9x+y)$

(b) $4z_{xy} + z_{yy} = \cos y + 1$

$z = f(x-4y) - \cos y + \frac{1}{2}y^2$

(c) $z_{xx} + z_x + x + y + 1 = 0$

$z = f(y) + e^{-x}g(y) - \frac{x^2}{2} - xy$

(d) $r = xy$

$z = x f(y) + g(y) + \frac{x^3}{6}y$

(e) $z_{yy} + z = e^{x+y}$

$z = f(x)\cos y + g(x)\sin y + e^{x+y}/2$

(f) $r - t + p + q + x + y + 1 = 0$

$z = e^{-x}f(x+y) + g(x-y) - \frac{1}{2}x^2 + x - \frac{1}{2}y^2 - 2y$

(g) $2s + 3t - q = 6\cos(2x-3y) - 30\sin(2x-3y)$

$z = f(x) + e^{x/2}g(3x-2y) + \frac{4}{13}\cos(2x-3y) + 2\sin(2x-3y)$

(h) $s + ap + bq + abz = e^{mx+ny}$ a, b, m, n constants

$z = e^{-bx}f(y) + \frac{1-mn-ab}{am+bn}\exp(mx+ny)$

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(i) $r - 4t = 12x^2 + \cos y + 4$

$$z = f(2x+y) + g(2x-y) + x^4 + 2x^2 + \frac{\cos y}{4}$$

(j) $r - t - 3p + 3q = xy + e^{x+2y}$

(K) $r - 2s + t = 4e^{3y} + \cos x$

$$z = f(x+y) + xg(x+y) + \frac{4}{9}e^{3y} - \cos x$$

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5. Obtain a particular solution of each of the following equations. Also find the general solution.

(a) $r + 5s + 6t = \log(y - 2x)$

$$z = f(y-2x) + g(y-3x) + 7x^2 - 6xy + \frac{5y^2}{4} - \left(4x^2 - 3xy + \frac{y^2}{2}\right)\log(y-2x)$$

(b) $r - s - 2t = (y-1)e^x$

(c) $r - 4t = \frac{4x}{y^2} - \frac{y}{x^2}$

$$z = f(y-2x) + g(y+2x) + x\log y + y\log x$$

---

6. Use simple integrations, reduction to an ordinary differential equation, etc., to obtain a solution involving two arbitrary functions.

(a) $t = x^3 + y^3$

$$z = f(x) + y\,g(x) + \frac{x^3y^2}{2} + \frac{y^5}{20}$$

(b) $z_{xy} = \frac{x}{y} + a \qquad a = \text{constant}$

$$z = f(x) + g(y) + \frac{x^2}{2}\log y + axy$$

(c) $t - xq = x^2$

$$z = e^{xy}g(x) - \frac{f(x)}{x} - xy$$

(d) $xr = c^2p + x^2y^2 \qquad c = \text{const}$

$$z = \frac{(c^2+2)\,f(x,y)}{x} + \frac{x^3}{12}y^2 + g(y) + \frac{h(y)}{x} \;;\; f(x,y) = \int z\,dx$$

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(e) $y z_{xy} + z_x = \cos(x+y) - y\sin(x+y)$

$$z = f(y) + \frac{g(x)}{y} + \sin(x+y)$$

(f) $y^2 t + 2yq = 1$

$$z = f(x) + \frac{g(x)}{y} + \log y + 1$$

(g) $(x-y) z_{xy} - z_x + z_y = 0$ Euler-Poisson-Darboux equation

Hint: Let $u = (x-y)z$

$$z = \frac{f(x) + g(y)}{x-y}$$

---

8. Obtain the general solution of each of the following equations

(a) $x^2 r - y^2 t - 2xp + 2yq = 0$

$$z = f(xy) + x^3 g\left(\frac{y}{x}\right)$$

(b) $x^2 r - xys - xp = 1$

$$z = -\frac{1}{2}\log x + f(y)$$

(c) $xy z_{xy} - y^2 z_{yy} - 2x z_x + 2y z_y - 2z = 0$

$$z = y^2 f(x) + \frac{g(xy)}{x}$$

(d) $x^2 r - 2xys - 3y^2 t + xp - 3yq = 2\log xy + 4x$

(e) $x^2 r - y^2 t = xy$

$$z = xy\log x + f(xy) + x g\left(\frac{x}{y}\right)$$

(f) $x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} - nx z_x - ny z_y + nz = x + y$ $n = \text{const} \neq 0$

(g) $xp + yq - xys = z$

$$z = x f(y) + y g(x)$$

(h) $(ax D_x + by D_y)^2 z + \lambda^2 z = 0$ a, b, λ constants.

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10. obtain the general solution of each of the following equations.

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(a) $e^{2y}(r-p) = e^{2x}(t-q)$

$$z = f(e^x - e^y) + g(e^x + e^y)$$

(b) $(y^2-x^2)(r-t) + 4(xp+yq-z) = 0$

(c) $xr + (y-x)s - yt = q - p$

$$z = f(x+y) + g\left(\frac{y}{x}\right)$$

Sec. 2-3

11. Obtain the general solution.

(a) $u_{xx} + 2u_{xy} + u_{yy} - 2u_{yz} - 2u_{xz} + u_{zz} = x\cos x + y\cos y + z\cos z$

$$u = xf(x-y, x+z) + g(x-y, x+z) + 2\sin x - x\cos x + 2\sin y - y\cos y + 2\sin z - z\cos z$$

(b) $2u_{xx} + u_{xy} + 2u_{xz} - u_{yy} - u_{yz} = x^2 + y^2 + z^2$

$$u = f(x-y, x-z) + g(x+2y, 2z) + x^4/24 - y^4/12$$

(c) $u_{xx} + 2u_{xy} + u_{yy} - u_{zz} - 2u_z - u = e^{2x} + 2\cos y + z + 2$

$$u = e^x f(x+z, y+z) + e^{-x} g(x-z, y-z) + e^{2x}/3 - \cos y - z$$

(d) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + 2yz\, u_{yz} + 2xz\, u_{xz} + z^2 u_{zz} = \ln xyz$

(e) $u_{tt} = A u_{xx} + 2B u_{xy} + C u_{yy}$, A, B, C positive constants such that $B^2 - AC = 0$.

$$u = f(x + \sqrt{A}\,t, y + \sqrt{C}\,t) + g(x - \sqrt{A}\,t, y - \sqrt{C}\,t)$$

12. Construct exponential-type solutions. Also find a particular solution if the equation is inhomogeneous

(a) $z_{xx} - 2z_{xy} + z_y - z = 0$

$$z = \exp\left[hx + \frac{(1-h^2)y}{1-2h}\right]$$

(b) $z_{xx} + 4z_{xy} + z_{yy} + z_x + z_y = 3e^{2x} + xy$

$$z = \exp\left[hx + \frac{-4h - 1 \pm \sqrt{12h^2 + 4h + 1}}{2}y\right]$$

(c) $z_{xx} - 2z_x - z_y = x + y$

$$z = \exp[hx + (h^2 - 2h)y] - \frac{x^2 + x}{4} - \frac{y^2}{2}$$

(d) $z_{xx} + z_{yy} = x + ye^y$

$$z = \exp[h(x \pm iy)] + \frac{1}{6}x^3 + (y-2)e^y \quad ; \quad i = \sqrt{-1}$$

(e) $u_{xx} + u_{yy} = 0$

$$u = \exp[h(x \pm iy)] \quad ; \quad i = \sqrt{-1}$$

(f) $u_{xx} + u_{yy} + u_{zz} = 0$

$$u = \exp[\alpha x + \beta y \pm i\sqrt{\alpha^2 + \beta^2}\, z]$$

(g) $u_{xx} + u_{yy} + k^2 u = 0 \qquad k = \text{cons}$

$$u = \exp[hx \pm i(h^2 + k^2)^{\frac{1}{2}} y]$$

(h) $u_{xx} + u_{yy} = k u_t \qquad k = \text{cons}$

$$u = \exp[\alpha x + \beta y + \frac{1}{k}(\alpha^2 + \beta^2)]$$

(i) $c^2(u_{xx} + u_{yy}) = u_{tt} \qquad c = \text{cons}$

$$u = \exp[\alpha x + \beta y \pm c(\alpha^2 + \beta^2)^{\frac{1}{2}} t]$$

---

13. Find the equation which the constants l, m, n must satisfy if the function u defined by $u(x,y,t) = \int_\alpha^\beta f(lx + my + bt, \xi)\, d\xi$ satisfies the two-dimensional wave equation in Example 2-3. Assume that it is legitimate to differentiate with the integral sign

$$l^2 + m^2 = n^2/c^2$$

Sec. 2-4

22. Classify each equation. Reduce to normal form and obtain the general solution.

(a) $4z_{xx} - 8z_{xy} + 4z_{yy} = 1$

Parabolic; $z = f(x+y) + x\,g(x+y) + \frac{x^2}{8}$

(b) $4z_{xx} - 4z_{xy} + 5z_{yy} = 0$

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(c) $r - 2s + t + a(p - q) + cz = c(x+2y)^2$ a, c nonzero constants

Parabolic; $z = \exp[\lambda_1(x+2y)]f(x+y) + \exp[\lambda_2(x+2y)]g(x+y) + \frac{(x+2y+a/c)^2}{c} + \frac{a^2-2c}{c^3}$

(d) $r - t + p + q + x + y + 1 = 0$

(e) $x^2 r + 2xys + y^2 t = 4x^2$

Parabolic; $z = f\left(\frac{y}{x}\right) + y\,g\left(\frac{y}{x}\right) + 2x^2$

(f) $xr - (x+y)s + yt = \frac{x+y}{x-y}(p - q)$

(g) $x^2 r - y^2 t = xy$

Hyperbolic wherever $xy \neq 0$; $z = f(xy) + x\,g\left(\frac{x}{y}\right) + xy\log x$

(h) $\frac{\partial}{\partial x}\left(x^2 \frac{\partial z}{\partial x}\right) = x^2 \frac{\partial^2 z}{\partial y^2}$

(i) $r - (2\sin x)s - (\cos^2 x)t - (\cos x)q = 0$

Hyperbolic; $z = f(x + y - \cos x) + g(x - y + \cos x)$

(j) $(\sec^4 y)r - t + (2\tan y)q = 0$

(K) $xy(t - r) + (x^2 - y^2)s = py - qx - 2(x^2 - y^2)$

Hyperbolic wherever $x^2 + y^2 \neq 0$; $z = f(x^2 + y^2) + g\left(\frac{x}{y}\right) - xy$

(l) $y^2 z_{xx} - 2y z_{xy} + z_{yy} = z_x + 6y$

(m) $\frac{\partial}{\partial x}\left[\left(1 - \frac{x}{h}\right)^2 z_x\right] = \frac{1}{a^2}\left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 z}{\partial t^2}$ a, h real positive constants.

Hyperbolic wherever $x \neq h$; $z = \frac{f(x - at) + g(x + at)}{h - x}$

23. classify each of the following equations.

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(a) $c^2(u_{xx}+u_{yy}+u_{zz})=u_{tt}$ $c>0$ a real constant
Hyperbolic

(b) $\frac{\partial u}{\partial t}=k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$ $k>0$ a real constant

(c) $u_{xx}+u_{yy}+u_{zz}=0$
Elliptic

(d) $u_{xx}+u_{yy}-u_{tt}=0$

(e) $u_{xx}+u_{yy}+u_{zz}+\lambda u=0$ λ a constant
Elliptic

(f) $u_{xx}+u_{yy}-u_{zz}-u_{tt}=0$

Sec. 2.5

24. In each of the following Cauchy problems determine whether (i) there exists a unique solution, (ii) there exist infinitely many solutions, (iii) no solution exists. If i, find the solution.

(a) $u_{xx}-u_{tt}=0$; $u(x,0)=b$, b a constant, $u_t(x,0)=\sin x$
unique solution; $u=b+\sin(x)\sin(t)$

(b) $u_{xx}-u_{tt}=0$; $u(x,0)=\sin x$, $u_t(x,0)=b$, b a constant

(c) $u_{xx}-u_{tt}=0$; $u=0$, $u_x=1$ on C, C is the line $t=x$
If $f(\xi)$ is an arbitrary twice diff function s.t $f'(0)=1 \Rightarrow u=f(x-t)-f(0)$

(d) $u_{xx}-u_{tt}=0$; $u=0$, $u_x=x$ on C, C is the line $t=x$
If $f(\xi)$ is an arbitrary twice diff. function s.t $f'(0)=x \Rightarrow u=f(x-t)-f(0)$

(e) $u_{xx}-10u_{xt}+9u_{tt}=0$; $u(x,0)=x^2$, $u_t(x,0)=1$

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unique solution; $u = x^2 + t - t^2/9$

(f) $z_{xy} = 0$; $z = \cos x$, $z_x = 1$ on C, C is the line $y = x$

$z = x - y + \cos y$

(g) $z_{xy} = 0$; $z(x,0) = 1$, $z_y(x,0) = x^2$

No solution

(h) $z_{xy} = xy$; $z = \cos x$, $z_y = x^3$ on C, C is the line $y = x$

$z = \cos x - \frac{3}{8}x^4 + \frac{1}{8}y^4 + x^2y^2/4$

(i) $z_{xy} + z_x = x$; $z = x^2$, $z_x = 0$ on C, C is the line $y = x$

unique solution ; $z = \frac{x^2 + y^2}{2} + y - 1 + e^{x-y}(1 - x)$

(j) $z_{xy} = 0$; $z(x,0) = x$, $z_y(x,0) = 1$

If $f(\xi)$ is arbitrary twice diff function s.t $f'(0) = 1 \Rightarrow z = f(y) + x - f(0)$

(k) $z_{xy} - 4z_{yy} = 16y + 4e^{-y}$; $z(x,0) = -1$, $z_y(x,0) = 4x + 1$

unique solution ; $z = 4xy + y^2/2 - 2y^3/3 - e^{-y}$

(l) $z_{xx} - 2z_{xy} + z_{yy} = 4e^{3y}$; $z(x,0) = x^2$, $z_y(x,0) = \frac{1}{3}$

(m) $z_{xx} = x^2 + y^2$; $z(0,y) = 1$, $z_x(0,y) = 0$

unique solution ; $z = x^4/12 + x^2y^2/2 + 1$

---

25. Solve each of the following Cauchy Problems.

(a) $x^2 u_{xx} - 2xy\,u_{xy} - 3y^2 u_{yy} = 0$; $u(x,1) = x$, $u_y(x,1) = 1$

$u = x + y - 1$

(b) $y\,u_{xx} + (x+y)\,u_{xy} + x\,u_{yy} = xy$; $u(0,y) = y$, $u_x(0,y) = 1$, $y > 0$

(c) $2u_{xx} - 2y\,u_{yy} - u_y = 0$; $u(x,1) = \frac{x^2}{2}$, $u_y(x,1) = 2$

$u = x^2/2 + 2y - 2$

(d) $u_{xx} + (2\cos x)u_{xy} - (\sin^2 x)u_{yy} - (\sin x)u_y = 0$; $u = x^2$, $u_y = 1$ on C, C is the curve $y = \sin x$

(e) $x^2 u_{xx} - 2x u_{xy} + u_{yy} + u_y = 0$; $u(1, y) = y^2$, $u_x(1, y) = e^y$

$u = (\log x + y - x)^2 + 2(\log x + y) + x^2(e^y - 1) - x e^y$

where

$$A_{nk} = 2\frac{\displaystyle\int_0^a\int_0^{2\pi} r'f(r',\theta')J_n(\xi_{nk}r'/a)\cos n\theta'\,dr'\,d\theta'}{\pi a^2 J_{n+1}^2(\xi_{nk})}$$

$$B_{nk} = 2\frac{\displaystyle\int_0^a\int_0^{2\pi} r'f(r',\theta')J_n(\xi_{nk}r'/a)\sin n\theta'\,dr'\,d\theta'}{\pi a^2 J_{n+1}^2(\xi_{nk})}$$

$n = 1, 2, \ldots; \qquad k = 1, 2, \ldots$

and

$$A_{0k} = \frac{\displaystyle\int_0^a\int_0^{2\pi} r'f(r',\theta)J_0(\xi_{0k}r'/a)\,dr'\,d\theta'}{\pi a^2 J_1^2(\xi_{0k})}$$

PROBLEMS

Sec. 3-2

1 Determine conditions under which the following functions are harmonic in three dimensions.

(a) $u = ax^2 + by^2 + cz^2$ $\qquad a, b, c, d$ constants

(b) $u = f(ax + by + cz + d)$ $\qquad a, b, c, d$ constants

(c) $u = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz$

where the a_{ij} are constants.

(d) $u = Ae^{ax+by+cz}$ $\qquad A, a, b, c$ constants

(e) $u = A \sin ax \cosh by + B \cos ax \sinh by$

2 (a) Let φ be a harmonic function, not identically a constant. When is $u = f(\varphi)$ a harmonic function?

(b) Let φ, ψ be harmonic functions. When is $u = \varphi\psi$ harmonic? Interpret the condition geometrically.

3 A harmonic function in three-dimensional space which depends only on the distance from a fixed point is called a purely radially dependent potential. Let $r = (x^2 + y^2 + z^2)^{1/2}$. Determine all twice differentiable functions f such that $u = f(r)$ is a potential. Do this also in two dimensions.

4 Let u be a harmonic function in a region. Show that any derivative of u is also harmonic in the region (assume the property that a harmonic function has derivatives of all orders in the region).

5 Let $i = \sqrt{-1}$. It is known that each function f of the complex variable $z = x + iy$, x, y real variables, can be written

$f(z) = u(x,y) + iv(x,y)$

where u, v are real-valued functions. It is shown in the theory of functions of a complex variable that if f is an analytic function of z, then u and v are harmonic functions. For each of the following cases find the functions u, v and verify that they are harmonic.

(a) $f(z) = z^2$

(b) $f(z) = 4z^3 - 2z^2 + 1$

(c) $f(z) = e^z$

(d) $f(z) = \cos z$, where $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

(e) $f(z) = \dfrac{1}{z} \qquad z \neq 0$

6 (a) A linear transformation

$$x = a_{11}\xi + a_{12}\eta \qquad y = a_{21}\xi + a_{22}\eta$$

is called an orthogonal transformation if the coefficients (assumed real) satisfy

$$a_{11}{}^2 + a_{21}{}^2 = a_{12}{}^2 + a_{22}{}^2 = 1 \qquad a_{11}a_{21} + a_{12}a_{22} = 0$$

In this case the transformation represents a rotation of axes. Let $u(x,y)$ be a twice differentiable function of the variables x, y, and let $v(\xi,\eta) = u[x(\xi,\eta),y(\xi,\eta)]$. Show that

$$v_{\xi\xi} + v_{\eta\eta} = u_{xx} + u_{yy}$$

(b) Let the xy coordinate system be related to the coordinate system by the translation $x = \xi + a$, $y = \eta + b$. Let u, v be as described in **a**. Show that the same relation holds. It follows (by combining the results of **a** and **b**) that the laplacian is invariant under the group of rigid motions in the plane; that is, u is a scalar invariant. In particular, if u is a harmonic function, so is v.

7 Let u, v be harmonic functions in a region $\mathscr{R}$. Let S_1 be a closed surface such that (1) S_1 and its interior lie within $\mathscr{R}$, (2) the divergence theorem is applicable to S_1 and the region bounded by S_1. If $\mathbf{n}$ is the outward normal on S_1, show that

$$\int_{S_1} v\frac{\partial u}{\partial n}\,dS = \int_{S_1} u\frac{\partial v}{\partial n}\,dS$$

8 (a) Let $\overline{\mathscr{R}}$ be the set of all points in the plane such that $x^2 + y^2 \leq 1$. Let $u = x^3 - 3xy^2$. Find the maximum and minimum values of u on $\overline{\mathscr{R}}$.

(b) Let C be the circle $x^2 + y^2 = a^2$ about the origin. Verify Gauss' mean-value theorem in the plane for the function u of part **a**.

9 Let $\mathscr{R}$ be a bounded region with boundary S. Let u be harmonic in $\mathscr{R}$, continuous on $\overline{\mathscr{R}}$, and such that $u > 0$ on S. Prove that $u > 0$ in $\mathscr{R}$.

10 Let u, v, w be harmonic in $\mathscr{R}$, continuous on $\overline{\mathscr{R}}$, and such that $v(Q) \leq u(Q) \leq w(Q)$ holds for all points Q on S. Prove that this inequality must hold everywhere in $\mathscr{R}$.

11 Let u be harmonic in a region $\mathscr{R}$ of xyz space. Let P be a point in $\mathscr{R}$, S_p a sphere of radius r_0 with P as center such that S_p and its interior lie within $\mathscr{R}$. Prove the average-value property

$$u(P) = \frac{1}{V}\int_{\mathscr{R}_p} u\,d\tau$$

where $V = 4\pi r_0{}^3/3$ is the volume of the spherical region $\mathscr{R}_p$ whose boundary is S_p, and

$\bar{\mathscr{R}}_p = \mathscr{R}_p + S_p$. *Hint:* Multiply both sides of Eq. (3-12) by $4\pi r^2\,dr$ and integrate from $r = 0$ to $r = r_0$.

12 **(a)** Let $\mathscr{R}$ be a region in space with boundary S such that the divergence theorem

$$\int_{\mathscr{R}} \nabla \cdot \mathbf{A}\, d\tau = \int_S \mathbf{A} \cdot \mathbf{n}\, dS$$

is applicable to $\bar{\mathscr{R}} = \mathscr{R} + S$. Set $\mathbf{A} = v\,\nabla u$, and obtain the equation

$$\int_{\mathscr{R}} v\,\Delta u\, d\tau + \int_{\mathscr{R}} \nabla u \cdot \nabla v\, d\tau = \int_S v\frac{\partial u}{\partial n}\, dS$$

where $\mathbf{n}$ is the exterior normal on S (note that this assumes u has continuous second derivatives on $\bar{\mathscr{R}}$, v has continuous first derivatives on $\bar{\mathscr{R}}$). If u has continuous second derivatives on $\bar{\mathscr{R}}$, show that

$$\int_{\mathscr{R}} u\,\Delta u\, d\tau + \int_{\mathscr{R}} |\nabla u|^2\, d\tau = \int_S u\frac{\partial u}{\partial n}\, dS$$

(b) Prove that if u is harmonic in $\mathscr{R}$ and has continuous second derivatives on $\bar{\mathscr{R}}$ and $\partial u/\partial n = 0$ on S, then u is constant on $\bar{\mathscr{R}}$.

(c) Prove that any pair of solutions of the Neumann problem $\Delta u = f$ in $\mathscr{R}$, $\partial u/\partial n = g$ on S having continuous second derivatives on $\bar{\mathscr{R}}$, differ by a constant.

13 Prove that there is at most one solution having continuous second derivatives on $\bar{\mathscr{R}}$ of the mixed problem $\Delta u = f$ in $\mathscr{R}$, $\partial u/\partial n + hu = g$ on S, where h is a continuous non-negative function, not identically zero, on S.

14 **(a)** Derive Eq. (3-17).

(b) Derive Eq. (3-19).

(c) State and give a complete proof of the maximum-minimum principle for plane harmonic functions.

(d) Let $\mathscr{R}$ be a region in the plane, and let u be harmonic in $\mathscr{R}$. Let P be a point in $\mathscr{R}$ and C a circle of radius r_0 such that C, and its interior, lies within $\mathscr{R}$. Prove that

$$u(P) = \frac{1}{\pi r_0{}^2}\int_{\bar{\mathscr{R}}} u\, dx\, dy$$

where $\bar{\mathscr{R}}_1$ is the circular disk bounded by C.

15 **(a)** Let S be a simple closed surface, $\mathscr{R}$ the unbounded region exterior to S. Let u be harmonic in $\mathscr{R}$, continuous on $\mathscr{R} + S$, and vanishing uniformly at infinity. Suppose u is not identically zero and $M = \max |u(P)|$ for P on S. Prove that $|u(P)| < M$ holds in $\mathscr{R}$.

(b) Suppose that u is harmonic everywhere and vanishes uniformly at infinity. Prove that u must be identically zero. *Hints:* If $u = 0$ on S, then u is identically zero everywhere (see proof of Theorem 3-6). Hence assume $M \neq 0$. Let P be a point of $\mathscr{R}$. Choose a sphere S_r of radius r about the origin sufficiently large such that (1) P lies within S_r, (2) $|u(Q)| < M/2$ holds for all points Q on and outside S_r. What does the maximum-minimum principle imply for the region bounded by S and S_r? In **b** let $\epsilon > 0$, and choose a sphere S_r about 0 sufficiently large such that (1) P lies within S_r, and (2) $|u(Q)| < \epsilon$ holds for all points Q on and outside S_r.

Sec. 3-3

16 Laplace's equation in rectangular coordinates in two dimensions is $u_{xx} + u_{yy} = 0$. Assume a separable solution $u = X(x)Y(y)$. Substitute into the differential equation, and follow the method of the text used for the polar-coordinate case to show that

$$\frac{X''}{X} = -\lambda = \frac{Y''}{Y}$$

where λ is a separation constant (real or complex), and

$$X'' = \frac{d^2X}{dx^2} \qquad Y'' = \frac{d^2Y}{dy^2}$$

Let $\mu = \sqrt{\lambda}$, and show that separable solutions must be of the form

$$u = (ax + b)(cy + d) + (Ae^{i\mu x} + Be^{-i\mu x})(Ce^{\mu y} + De^{-\mu y})$$

17 **(a)** A thin rectangular homogeneous thermally conducting plate lies in the xy plane and occupies the rectangle $0 \le x \le a$, $0 \le y \le b$. The faces of the plate are insulated, and no internal sources or sinks are present. The edge $y = 0$ is held at 100°, while the remaining edges are held at 0°. Find the steady temperature $u(x,y)$ in the plate.
(b) Find $u(x,y)$ if the edge $y = 0$ is held at temperature $T \sin(\pi x/a)$, $0 \le x \le a$, where T is a constant.
(c) Find $u(x,y)$ if the edge $y = 0$ is held at temperature $Tx(x - a)$, where T is a constant.
Hint: Since no heat sources are present in the plate, the steady-state temperature u must satisfy

$$\Delta u = 0 \qquad 0 < x < a;\ 0 < y < b$$

The boundary conditions in **a** are

$$u(0,y) = u(a,y) = 0 \qquad 0 < y < b \qquad\qquad u(x,b) = 0 \qquad 0 \le x \le a$$
$$u(x,0) = 100 \qquad 0 \le x \le a$$

As in Prob. 16, assume a separable solution $u = XY$ of Laplace's equation in rectangular coordinates, and obtain the ordinary differential equations

$$X'' + \lambda X = 0 \qquad Y'' - \lambda Y = 0$$

Now the boundary conditions $u(0,y) = u(a,y) = 0$, $0 < y < b$, are satisfied by the separable solution if the factor X satisfies

$$X(0) = X(a) = 0$$

Show that this leads to the values $\lambda_n = n^2\pi^2/a^2$, $n = 1, 2, \ldots$, as the only possible values for the separation constant. The corresponding eigenfunctions are $X_n = \sin(n\pi x/a)$, $n = 1, 2, \ldots$. Thus the corresponding solutions for Y are

$$Y_n(y) = c_n \cosh\frac{n\pi y}{a} + d_n \sinh\frac{n\pi y}{a}$$

The separable solution $u_n = X_n(x)Y_n(y)$ will satisfy the boundary condition $u_n(x,b) = 0$ if

$$c_n \cosh\frac{n\pi b}{a} + d_n \sinh\frac{n\pi b}{a} = 0$$

Thus

$$Y_n(y) = c_n\left[\cosh\frac{n\pi y}{a} - \frac{\cosh(n\pi b/a)\sinh(n\pi y/a)}{\sinh(n\pi b/a)}\right]$$

$$= c_n'\sinh\frac{n\pi(b-y)}{a}$$

The separable solutions

$$u_n(x,y) = \sin\frac{n\pi x}{a}\sinh\frac{n\pi(b-y)}{a} \qquad n = 1, 2, \ldots$$

satisfy all the boundary conditions save one. Consider the superposition

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi(b-y)}{a}$$

In order to satisfy the condition along $y = 0$ it is necessary that

$$\sum_{n=1}^{\infty} B_n \sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a} = 100 \qquad 0 \le x \le a$$

This will be so if the coefficients

$$B_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a 100\sin\frac{n\pi x}{a}\,dx \qquad n = 1, 2, \ldots$$

18 Find the steady temperature in the plate of Prob. 17 if the edge $x = a$ is held at temperature 100° while the remaining edges are held at 0°.

19 Derive the steady temperature in the plate of Prob. 17 if the temperature is held at T_1 along the edge $y = 0$, at T_2 along the edge $x = a$, at T_3 along the edge $y = b$, and at T_4 along the edge $x = 0$, where T_1, T_2, T_3, T_4 are constants.

20 Let the prescribed conditions on the edges of the plate in Prob. 17 be as follows:

$$u(x,0) = T_1\sin\frac{\pi x}{2a} \qquad u(x,b) = 0;\ 0 \le x \le a$$

$$u(0,y) = T_2 y(b-y) \qquad u_x(a,y) = 0;\ 0 \le y \le b$$

where T_1 and T_2 are constants. Derive the expression for the temperature in the plate. *Hint:* Construct the superposition $u = v + w$ where v and w are harmonic functions in the rectangle which satisfy the boundary conditions

$$v(x,0) = T_1\sin\frac{\pi x}{2a} \qquad v(x,b) = 0$$

$$v(0,y) = 0 \qquad v_x(a,y) = 0$$

$$w(x,0) = 0 \qquad w(x,b) = 0$$

$$w(0,y) = T_2 y(b-y) \qquad w_x(a,y) = 0$$

21 (a) A thin homogeneous plate occupies the region

$$0 \le x \le a \qquad y \ge 0$$

in the xy plane. There are no heat sources in the plate, and the faces are insulated. The temperature is held at 0° along the edges $x = 0$, $x = a$, while $u = T$ along the edge $y = 0$ (T constant). Also $\lim_{y \to \infty} u(x,y) = 0$ uniformly for $0 \le x \le a$. Derive the series representation for the temperature in the plate. Obtain a closed-form expression for the temperature with the aid of the equation

$$\sum_{k=1}^{\infty} \frac{\sin [(2k-1)x] e^{-(2k-1)y}}{2k-1} = \frac{\tan^{-1} (\sin x / \sinh y)}{2}$$

(b) Derive the series expression for the temperature in the plate described in **a** if instead the boundary condition along the edge $x = a$ is

$$\frac{\partial u}{\partial x} + hu = 0$$

where h is a positive constant, the remaining boundary conditions being the same.

22 In the problem of the torsion of a beam in the theory of elasticity there occurs the *stress function* Ψ. If the stress function is known, the tangential stresses and the torsion moment can be determined. It is shown in the theory that if the axis of the beam coincides with the z axis and the beam is of uniform cross section, then Ψ must satisfy Poisson's equation $\Delta\Psi = -2$ in $\mathscr{R}$, where $\mathscr{R}$ is the generating cross section in the xy plane, and the boundary condition $\Psi = 0$ on C, where C is the simple closed curve which bounds $\mathscr{R}$. Show that if the beam has the rectangular cross section $0 \le x \le a$, $0 \le y \le b$, the stress function is

$$\Psi(x,y) = ax - x^2 - \frac{8a^3}{\pi^3} \sum_{k=1}^{\infty} \frac{\sin [(2k-1)\pi x/a]}{(2k-1)^3} \left\{ \frac{\sinh [(2k-1)\pi(b-y)/a] + \sinh [(2k-1)\pi y/a]}{\sinh [(2k-1)\pi b/a]} \right\}$$

Hint: Recall the discussion in Sec. 3-1. The function $v(x,y) = ax - x^2$ satisfies $\Delta v = -2$ in $\mathscr{R}$ and is zero along $x = 0$ and along $x = a$. Now construct a function w which is harmonic in $\mathscr{R}$ and such that

$$w(0,y) = w(a,y) = 0 \qquad 0 \le y \le b$$
$$w(x,0) = w(x,b) = x^2 - ax \qquad 0 \le x \le a$$

23 Solve the boundary-value problem

$$\Delta u = cx + dy \qquad \text{in } \mathscr{R} \qquad u = 0 \qquad \text{on the boundary}$$

where c and d are constants and $\mathscr{R}$ denotes the interior of the rectangle $0 \le x \le a$, $0 \le y \le b$.

24 In rectangular coordinates in three dimensions Laplace's equation is

$$\Delta u = u_{xx} + u_{yy} + u_{zz}$$

Assume a separable solution $u = X(x)Y(y)Z(z)$. Obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

Argue as before that

$$\frac{X''}{X} = -\alpha^2 = -\left(\frac{Y''}{Y} + \frac{Z''}{Z}\right)$$

where α is a separation constant. Further

$$\frac{Y''}{Y} = \alpha^2 - \frac{Z''}{Z} = -\beta^2$$

Hence the factors must satisfy

$$X'' + \alpha^2 X = 0 \qquad Y'' + \beta^2 Y = 0 \qquad Z'' - (\alpha^2 + \beta^2)Z = 0$$

Accordingly separable solutions are of the form

$$u = e^{\pm i\alpha x}e^{\pm i\beta y}e^{\pm \gamma z}$$

where $\alpha \neq 0$, $\beta \neq 0$, and $\gamma^2 = \alpha^2 + \beta^2$. If $\alpha = 0$, $\beta \neq 0$, separable solutions are

$$u = (ax + b)e^{\pm i\beta y}e^{\pm \beta z}$$

If $\alpha = \beta = 0$, separable solutions are

$$u = (ax + b)(cy + d)(ez + f)$$

where a, b, c, d, e, f are constants. It is clear from symmetry that other forms of solution can be written down from the above forms. For example,

$$u = e^{\pm \alpha x}e^{\pm i\beta y}e^{\pm i\gamma z}$$

where $\alpha^2 = \beta^2 + \gamma^2$ and $\beta \neq 0$, $\gamma \neq 0$, is a separable solution in rectangular coordinates.

25 A homogeneous solid bar occupies the region $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$. There are no heat sources within the bar. The base $z = 0$ is held at constant temperature T, while the remaining sides are held at 0°. Show that the steady-temperature distribution in the bar is given by

$$u(x,y,z) = \frac{16T}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin[(2n-1)\pi x/a]\sin[(2m-1)\pi y/b]}{(2n-1)(2m-1)\sinh \omega_{nm}c}\sinh[\omega_{nm}(c-z)]$$

where

$$\omega_{nm} = \left[\frac{(2n-1)^2\pi^2}{a^2} + \frac{(2m-1)^2\pi^2}{b^2}\right]^{1/2} \qquad n, m = 1, 2, \ldots$$

What is the temperature at the center of the bar? *Hint:* The steady temperature u must satisfy Laplace's equation inside the bar. The boundary conditions are

$$u(x,0,z) = u(x,b,z) = u(0,y,z) = u(a,y,z) = 0$$

and

$$u(x,y,0) = T \qquad u(x,y,c) = 0$$

As in Prob. 24 assume a separable solution $u = XYZ$. Show that X must satisfy

$$X'' + \alpha^2 X = 0 \qquad 0 \leq x \leq a;\ X(0) = X(a) = 0$$

and hence the separation constant $\alpha = \alpha_n = n\pi/a$, $n = 1, 2, \ldots$. Similarly show that

$$Y'' + \beta^2 Y = 0 \qquad 0 \leq y \leq b;\ Y(0) = Y(b) = 0$$

implies $\beta = \beta_m = m\pi/b$, $m = 1, 2, \ldots$. The z-dependent factor must satisfy

$$Z'' - ({\alpha_n}^2 + {\beta_m}^2)Z = 0 \qquad Z(c) = 0$$

Let $\gamma_{nm} = (\alpha_n^2 + \beta_m^2)^{1/2}$, and derive the ∞^2 separable solutions

$$u_{nm}(x,y,z) = \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sinh [\gamma_{nm}(c - z)] \qquad n = 1, 2, \ldots ; m = 1, 2, \ldots$$

which satisfy all the boundary conditions save one. Determine the coefficients B_{nm} in the superposition

$$u(x,y,z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sinh [\gamma_{nm}(c - z)]$$

so as to satisfy the remaining condition. Use the orthogonality properties

$$\int_0^b \int_0^a \varphi_{pq}(x,y)\varphi_{nm}(x,y)\, dx\, dy = 0 \qquad (p,q) \neq (n,m)$$

where

$$\varphi_{jk}(x,y) = \sin \frac{j\pi x}{a} \sin \frac{k\pi y}{b} \qquad j = 1, 2, \ldots ; k = 1, 2, \ldots$$

26 Solve Prob. 25 if instead the faces $x = 0$, $x = a$, $y = 0$, $y = b$ of the solid are insulated, while conditions on the top and bottom remain the same as stated there. An insulated face means that the derivative of the temperature u in the direction of the normal to the face is zero.

27 An infinitely long bar of homogeneous material occupies the region $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z < \infty$. There are no heat sources within the bar. The base $z = 0$ is held at the temperature $Txy(x - a)(y - b)$, where T is a constant, while the sides are held at 0°. Also the temperature satisfies the condition $\lim_{z\to\infty} u(x,y,z) = 0$ uniformly in x, y for $0 \leq x \leq a$, $0 \leq y \leq b$. Find the steady temperature in the solid.

28 **(a)** A thin thermally conducting homogeneous disk with insulated faces occupies the region $0 \leq r \leq a$ in the xy plane (where r, θ are polar coordinates). The rim is held at 100°. What is the steady temperature in the disk?

(b) Solve the preceding problem if instead the temperature on the rim is held at $100\theta(1 - \theta/2\pi)$, $0 \leq \theta \leq 2\pi$. What is the temperature at the center of the disk?

29 The disk of Prob. 28**a** has the following prescribed temperature on the rim $r = a$: $u = c$, c a constant, $0 < \theta < \alpha$, $u = 0$, $\alpha < \theta < 2\pi$, where α is a given angle, $0 < \alpha < 2\pi$. Find the series expression for temperature at interior points of the disk. In particular consider the case where $c = 100$ and $\alpha = \pi/2$.

For this case use Poisson's integral (3-35) to derive a closed-form expression for the temperature inside the disk. Use the closed-form expression to show that

$$\lim_{r\to a} u(r,\theta) = 100 \qquad 0 < \theta < \frac{\pi}{2}$$

$$\lim_{r\to a} u(r,\theta) = 0 \qquad \frac{\pi}{2} < \theta < 2\pi$$

What is the temperature at the center of the disk?

30 A thin homogeneous metal sheet with insulated faces occupies the region $0 \leq r \leq a$, $0 \leq \theta \leq \alpha$, in the xy plane. Here r, θ are polar coordinates and α is given angle, $0 < \alpha < 2\pi$. The temperature along the edges $\theta = 0$, $\theta = \alpha$ is held at zero. On $r = a$, $0 < \theta < \alpha$, the temperature is given by $f(\theta)$, where f is a continuous function. Derive the series representation

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n \left(\frac{r}{a}\right)^{n\pi/\alpha} \sin \frac{n\pi\theta}{\alpha}$$

for the temperature in the sheet, where

$$B_n = \frac{2}{\alpha}\int_0^{\alpha} f(\theta) \sin \frac{n\pi\theta}{\alpha}\, d\theta \qquad n = 1, 2, \ldots$$

Consider the particular case where $f(\theta) = 100$ and $\alpha = \pi/2$. Use the formula for the sum of the series given in Prob. 21 to derive the closed-form expression

$$u(r,\theta) = \frac{200 \tan^{-1} [2a^2r^2 \sin 2\theta/(a^4 - r^4)]}{\pi}$$

for the temperature. Observe that $u(r,0) = u(r,\pi/2) = 0$, and verify that

$$\lim_{\substack{r\to a\\ r<a}} u(r,\theta) = 100 \qquad 0 < \theta < \frac{\pi}{2}$$

31 Find a function u harmonic in the region $0 \le r < a$, $0 < \theta < \pi/2$ such that $u = 1$ on the edge $\theta = 0$, $0 < r < a$, $u = 0$ on the edge $\theta = \pi/2$, $0 < r < a$, and $u = 0$ on the rim $r = a$, $0 < \theta < \pi/2$. *Hint:* Consider that the harmonic function $v = 1 - 2\theta/\pi$ satisfies the boundary conditions along $\theta = 0$, $\theta = \pi/2$. Now construct a harmonic function w in the region such that the superposition $u = v + w$ satisfies the conditions of the problem.

32 A thin annulus occupies the region $0 < a \le r \le b$, $0 \le \theta \le 2\pi$, where $b > a$. The faces are insulated, and along the inner edge the temperature is maintained at 0°, while along the outer edge the temperature is held at 100°. Show that the temperature in the annulus is given by

$$u = 100 \frac{\log (r/a)}{\log (b/a)}$$

33 Determine the temperature distribution in the annulus of Prob. 32 if instead the temperature on the outer rim $r = b$ is held at $u = T \cos (\theta/2)$, $0 < \theta < 2\pi$, where T is a constant.

34 After division by r, and replacement of R by y and λ by $-\lambda$, show that Eq. (3-23) can be rewritten as

$$Ly + \lambda\rho y = D(rDy) + \frac{\lambda y}{r} = 0 \qquad 0 < r$$

Here $D = d/dr$, and the differential operator is self-adjoint (see Sec. 1, Appendix 2). Let $0 < a < b$, a, b fixed, and consider the self-adjoint Sturm-Liouville problem

$$Ly + \lambda\rho y = 0 \qquad a \le r \le b;\ y(a) = 0;\ y(b) = 0$$

(a) Review Sec. 1 of Appendix 2, and show, directly from the differential equation and the boundary conditions, that eigenfunctions y_n, y_m corresponding to distinct eigenvalues λ_n, λ_m, respectively, are orthogonal on $[a,b]$ with weight function $\rho = 1/r$:

$$\int_b^a \frac{y_n y_m}{r}\, dr = 0 \qquad n \ne m$$

and also that the eigenvalues are real and positive.

(b) Find the linearly independent solutions

$$y = \exp(\pm i\omega \log r) \qquad \omega = \sqrt{\lambda};\ i = \sqrt{-1}$$

and hence the general solution

$$y = A\cos(\omega \log r) + B\sin(\omega \log r)$$

(c) Show that the eigenvalues are $\lambda_n = n^2\pi^2/[\log(b/a)]^2$, $n = 1, 2, \ldots$, and that corresponding real-valued eigenfunctions are

$$y_n = \sin\left(\omega_n \log \frac{r}{a}\right) \qquad \omega_n = \sqrt{\lambda_n};\ n = 1, 2, \ldots$$

35 A thin thermally conducting sheet occupies the region $0 < a \le r \le b$, $0 \le \theta \le \alpha$, in the xy plane, where r, θ are polar coordinates, a and b are given numbers such that $a < b$, and α is a given angle, $0 < \alpha < 2\pi$. The edges $r = a$, $r = b$ are held at 0°, as is also the edge $\theta = 0$. The edge $\theta = \alpha$ is held at 100°. Find the steady temperature in the sheet.

36 Let C be the circle $r = a$ in the xy plane and $\mathscr{R}$ the region interior to C. Derive the solution of the boundary-value problem

$$\Delta u = x^2 - y^2 \quad \text{in } \mathscr{R} \qquad u = 0 \quad \text{on } C$$

in the form

$$u(r,\theta) = \frac{(r^4 - a^2r^2)\cos 2\theta}{12}$$

Hint: Examine the form of the function on the right in the differential equation, and derive the particular solution

$$v = \frac{x^4 - y^4}{12} = \frac{r^4 \cos 2\theta}{12}$$

of the Poisson equation. Now derive the solution of the Dirichlet problem

$$\Delta w = 0 \quad \text{in } \mathscr{R} \qquad w(a,\theta) = -v(a,\theta) \quad \text{on } C$$

37 **(a)** A homogeneous thermally conducting cylinder occupies the region $0 \le r \le a$, $0 \le \theta \le 2\pi$, $0 \le z \le h$, where r, θ, z are cylindrical coordinates. There are no sources of heat within the cylinder. The top $z = h$ and the lateral surface $r = a$ are held at 0°, while the base $z = 0$ is held at 100°. Find the steady-temperature distribution within the cylinder.

(b) Solve the problem in **a** if the top is held at 100° instead of 0°, the remaining conditions being the same.

38 The cylinder of Prob. 37 has its base held at 100°. The lateral surface and the top radiate into an infinite medium, which is at temperature 0°. Thus there are the boundary conditions

$$\left(\frac{\partial u}{\partial r} + \gamma u\right)\bigg|_{r=a} = 0 \qquad \left(\frac{\partial u}{\partial z} + \gamma u\right)\bigg|_{z=h} = 0$$

where $\gamma > 0$ is a constant.

Derive the solution

$$u = 400\gamma a \sum_{n=1}^{\infty} \frac{J_0(\xi_n r/a)}{(\xi_n^{\,2} + \gamma^2 a^2) J_0(\xi_n)} \left[\xi_n \cosh \frac{\xi_n(h-z)}{a} + \gamma a \sinh \frac{\mu_n(h-z)}{a} \right]$$

where $0 < \xi_1 < \xi_2 < \cdots$ are the positive roots of the equation

$$\xi J_0'(\xi) + \gamma a J_0(\xi) = 0$$

39 A wedge-shaped solid occupies the region described by the inequalities $0 \leq r \leq a$, $0 \leq \theta \leq \beta$, $0 \leq z \leq h$, where β is a given angle, $0 < \beta < 2\pi$. The top $z = h$, the lateral surface $r = a$, and the faces $\theta = 0$ and $\theta = \beta$ are insulated. The base $z = 0$ is held at temperature $f(r,\theta)$. Derive the expression for the steady temperature in the solid if there are no sources of heat within. Consider the special case $f(r,\theta) = 100$.

Sec. 3-4

40 In potential theory it is shown that the gravitational potential ψ due to matter distributed in space satisfies Laplace's equation in regions free of matter, and in a region containing matter of density p satisfies Poisson's equation

$$\Delta\psi = -4\pi\rho$$

If S is a simple closed surface which bounds a region $\mathscr{R}$ of space containing matter of density ρ, and if the region exterior to S is free of matter, the potential and its first partial derivatives are continuous across S. Also, the potential in this case must vanish uniformly at infinity. Let S be a sphere of radius a, and suppose the interior of S contains matter of constant density ρ. Derive the expression for the potential **(a)** at points inside S, and **(b)** at points outside S. *Hint:* Use spherical coordinates with origin at the center of the sphere. Then ψ is spherically symmetric and satisfies

$$\psi_{rr} + \frac{2\psi_r}{r} = \begin{cases} -4\pi\rho & 0 \leq r \leq a \\ 0 & r > a \end{cases}$$

Integrate the equations directly and impose the requirements which the potential must satisfy.

41 Let $0 < a < b$, a, b fixed numbers. The spherical annulus in space which is bounded by the spheres $r = a$, $r = b$ is filled with matter. The density varies according to the formula $\rho = 1/r$, where r is the distance from the origin. Derive the expressions in spherical coordinates for the potential in the regions $0 \leq r \leq a$, $0 \leq r \leq b$, $r \geq b$.

42 Determine the electrostatic potential φ in the annular region bounded by the concentric spheres $r = a$, $r = b$, $0 < a < b$, if the inner sphere $r = a$ is held at constant potential V_0 and the outer sphere $r = b$ is held at constant potential V_1, $V_1 \neq V_0$.

43 A homogeneous thermally conducting solid is bounded by the concentric spheres $r = a$, $r = b$, $0 < a < b$. There are no heat sources within the solid. The inner surface $r = a$ is held at constant temperature u_1, and at the outer surface there is radiation into the medium $r > b$, which is at constant temperature u_2. Determine the steady temperature u in the solid. *Hint:* The steady temperature is spherically symmetric and satisfies Laplace's equation in regions where there are no sources. At $r = b$ the boundary condition is

$$\frac{\partial u}{\partial r} + h(u - u_2) = 0 \qquad h \text{ a positive constant}$$

44 Heat is generated at a constant rate Q within a homogeneous solid ball of radius a. The surface $r = a$ is held at the constant temperature T. Show that the steady temperature inside the ball is given by

$$u = \frac{Q(a^2 - r^2)}{6K} + T$$

What is the net flux of heat out through the surface $r = a$? *Hint:* The temperature must be finite, spherically symmetric, and satisfy Poisson's equation

$$\Delta u = -\frac{Q}{K}$$

inside the sphere.

45 Solve Prob. 44 if instead there is radiation of heat out into the region $r > a$ and the external medium has constant temperature zero. In this case the boundary condition at $r = a$ is

$$\frac{\partial u}{\partial r} + hu = 0$$

46 **(a)** Let (r,θ,φ) be spherical coordinates, as shown in Fig. 3-3. A homogeneous solid ball of radius a contains no heat sources. The portion of the surface defined by $r = a$, $0 \leq \theta < \pi/2$, is held at a constant temperature T, while the remainder is at temperature zero. Show that the temperature inside the ball is given by

$$u = \frac{T}{2} + \frac{T}{2}\sum_{k=0}^{\infty}\left(\frac{r}{a}\right)^{2k+1}[P_{2k}(0) - P_{2k+2}(0)]P_{2k+1}(\cos\theta)$$

(b) Solve the problem in **a** if instead the bottom hemisphere is held at temperature $-T$. *Hint:* In **b** let u denote the solution of the original problem in **a**. What properties do the functions $v = 2u$, $w = -T$ possess?

47 Determine the temperature in the ball of Prob. 46 if instead the surface temperature is $u = T(1 + 2\sin^2\theta)$, $0 \leq \theta \leq \pi$, where T is a constant.

48 The solid described in Prob. 43 has the inner surface $r = a$ held at the temperature $u = f_1(\theta)$, and the outer surface $r = b$ is held at the temperature $u = f_2(\theta)$, where f_1, f_2 are given functions of θ. Show that the steady temperature in the solid is given by

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(C_n r^n + \frac{D_n}{r^{n+1}}\right)P_n(\cos\theta)$$

where

$$C_n = \frac{a^{n+1}A_n - b^{n+1}B_n}{a^{2n+1} - b^{2n+1}} \qquad D_n = \frac{a^{-n}A_n + b^{-n}B_n}{a^{-(2n+1)} - b^{-(2n+1)}}$$

$$A_n = \frac{2n+1}{2}\int_0^{\pi} f_1(\theta)P_n(\cos\theta)\sin\theta\,d\theta$$

$$B_n = \frac{2n+1}{2}\int_0^{\pi} f_2(\theta)P_n(\cos\theta)\sin\theta\,d\theta$$

$n = 0, 1, 2, \ldots$. Consider the particular case where $f_1(\theta) = T_1$, $f_2(\theta) = T_2(1 - \cos\theta)$,

T_1, T_2 constants. *Hint:* The temperature is axially symmetric. Assume a solution

$$u = \sum_{n=0}^{\infty} \left(C_n r^n + \frac{D_n}{r^{n+1}} \right) P_n(\cos\theta)$$

The boundary conditions imply

$$\sum_{n=0}^{\infty} \left(C_n a^n + \frac{D_n}{a^{n+1}} \right) P_n(\cos\theta) = f_1(\theta)$$

$$\sum_{n=0}^{\infty} \left(C_n b^n + \frac{D_n}{b^{n+1}} \right) P_n(\cos\theta) = f_2(\theta)$$

Let m be a fixed nonnegative integer. Multiply both sides of each equation by $P_m(\cos\theta)\sin\theta$ and integrate over $0 \le \theta \le \pi$. Use the orthogonality properties of the Legendre polynomials to obtain two equations in the unknowns C_m, D_m.

49 A homogeneous conducting solid hemisphere is bounded by the xy plane and the surface $r = a$, $0 \le \theta \le \pi/2$. The curved surface is held at the temperature $u = T(1 - \cos\theta)$, T a constant. The base is insulated. Find the steady temperature in the solid. *Hint:* Insulated base means $\left.\frac{1}{r}\frac{\partial u}{\partial \theta}\right|_{\theta=\pi/2} = 0$.

Sec. 3-5

50 Let $\bar{\mathscr{R}}$ be the closed rectangle $0 \le x \le a$, $0 \le y \le b$. Show that the eigenfunction expansion of the solution of the boundary-value problem

$$\Delta\Psi = -2 \quad \text{in } \mathscr{R} \qquad \Psi = 0 \quad \text{on } C$$

where C is the boundary of the rectangle and $\mathscr{R}$ is its interior, is

$$\Psi(x,y) = \frac{32}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varphi_{nm}(x,y)$$

where

$$\varphi_{nm} = \frac{\sin[(2n-1)\pi x/a] \sin[(2m-1)\pi y/b]}{(2n-1)(2m-1)[(2n-1)^2/a^2 + (2m-1)^2/b^2]}$$

51 Find the eigenfunction expansion of the solution of the boundary-value problem in Prob. 23.

52 Let $\bar{\mathscr{R}}$ be the closed rectangle $0 \le x \le a$, $0 \le y \le b$. Find the eigenfunction expansion of the solution of the boundary-value problem

$$D\,\Delta\varphi - c\varphi = F \quad \text{in } \mathscr{R} \qquad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on } C$$

where D, c, and F are positive constants, C is the boundary of the rectangle, and $\mathbf{n}$ is the exterior normal on C.

53 **(a)** Let Δ be the three-dimensional laplacian. Let $\bar{\mathscr{R}}$ be the closed parallelepiped in xyz space defined by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$, $\mathscr{R}$ the interior of $\bar{\mathscr{R}}$, and S the bounding surface of the parallelepiped. Derive the complete set of eigenfunctions of the eigenvalue problem

$$\Delta\varphi + \lambda\varphi = 0 \quad \text{in } \mathscr{R} \qquad \varphi = 0 \quad \text{on } S$$

$$\varphi_{nmp} = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{p\pi z}{c} \qquad n = 1, 2, \ldots; m = 1, 2, \ldots; p = 1, 2, \ldots$$

and show that the eigenvalues are

$$\lambda_{nmp} = \pi^2\left(\frac{n_2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}\right)$$

(b) Let $\mathscr{R}$ and S be as described in **a** above. Find the eigenfunction expansion of the solution of the boundary-value problem

$$\Delta u = xyz \quad \text{in } \mathscr{R} \qquad u = 0 \quad \text{on } S$$

54 Find the eigenfunction expansion of the solution of the boundary-value problem

$$\Delta u = a^2 - r^2 \quad \text{in } \mathscr{R} \qquad u = 0 \quad \text{on } C$$

where C is the circle of radius a about the origin, $\mathscr{R}$ denotes the region interior to C, and r and θ are polar coordinates.

55 Solve the boundary-value problem in Prob. 50 by means of an eigenfunction expansion if the region $\mathscr{R}$ is the region interior to the circle $r = a$ in the xy plane.

56 Let $\mathscr{R}$ and C be as described in Prob. 54. Show that the eigenfunction expansion of the solution of the boundary-value problem

$$\Delta\varphi = x^2 - y^2 \quad \text{in } \mathscr{R} \qquad \varphi = 0 \quad \text{on } C$$

is

$$\varphi(r,\theta) = -\left[2a^4 \sum_{m=1}^{\infty} \frac{J_2(\xi_{2m}r/a)}{\xi_{2m}{}^3 J_3(\xi_{2m})}\right] \cos 2\theta$$

57 In the theory of elasticity it is shown that if a thin elastic plate (of uniform thickness) lies with its midplane in the xy plane and a surface force density f (force/unit area) acts in the vertical z direction, the vertical deflection $w(x,y)$ satisfies the fourth-order elliptic-type partial differential equation

$$\Delta\Delta w = \frac{f(x,y)}{N} \qquad \Delta\Delta w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}$$

where N is a material constant.

(a) The homogeneous equation $\Delta\Delta w = 0$ is called the *biharmonic equation*. Show that if u is a harmonic function in a region $\mathscr{R}$, then u is a solution of the biharmonic equation in $\mathscr{R}$.

(b) Show that if u, v are harmonic in $\mathscr{R}$, then $w = xu + v$ is biharmonic in $\mathscr{R}$. Hence the general solution of the biharmonic equation is

$$w = F(x + iy) + xF(x + iy) + G(x - iy) + xG(x - iy)$$

where $i = \sqrt{-1}$ and F, G are arbitrary functions. *Hint:* Consider the vector identity

$$\Delta(\varphi\psi) = \varphi\,\Delta\psi + \psi\,\Delta\varphi + 2\nabla\varphi \cdot \nabla\psi$$

(c) Let $\bar{\mathscr{R}}$ be the closed rectangle $0 \le x \le a$, $0 \le y \le b$ and let C be its boundary. Consider the eigenvalue problem

$$\Delta\Delta u + \lambda u = 0 \qquad \text{in } \mathscr{R} \qquad\qquad u = \Delta u = 0 \qquad \text{on } C$$

Show that all the eigenvalues are negative. *Hint:* The reality of the eigenvalues can be shown by an argument similar to that given for the self-adjoint problem discussed in the text. To show the eigenvalues are positive, let λ be an eigenvalue and u a corresponding eigenfunction. Then

$$\int_{\mathscr{R}} u\, \Delta\Delta u\, dx\, dy = -\lambda \int_{\mathscr{R}} u^2\, dx\, dy$$

Now in the Green's formula for the operator Δ replace v by u and $\nabla^2 u$ by $\Delta\Delta u$. Then

$$\int_{\mathscr{R}} (u\, \Delta\Delta u - |\Delta u|^2)\, dx\, dy = \int_C \left[u \frac{\partial}{\partial n} (\Delta u) - \Delta u \frac{\partial u}{\partial n} \right] ds = 0$$

since $u = \Delta u = 0$ on C. Hence

$$\lambda \int_{\mathscr{R}} u^2\, dx\, dy = - \int_{\mathscr{R}} |\Delta u|^2\, dx\, dy \le 0$$

and $\lambda = 0$ if, and only if, $\Delta u = 0$ in $\mathscr{R}$. But if $\Delta u = 0$ in $\mathscr{R}$, then $u = 0$ everywhere on $\bar{\mathscr{R}}$.

(d) Since $\lambda < 0$, let $\lambda = -\omega^2$, ω real and positive. Consider that

$$(\Delta - \omega)(\Delta + \omega) = (\Delta + \omega)(\Delta - \omega) = \Delta^2 - \omega^2 = \Delta\Delta - \omega^2$$

Thus if u satisfies

$$\Delta u + \omega u = 0 \qquad \text{in } \mathscr{R} \qquad\qquad u = \Delta u = 0 \qquad \text{on } C$$

then u is an eigenfunction of the eigenvalue problem in **c** corresponding to eigenvalue $-\omega^2$. But by Example 3-5 the eigenfunctions of this problem are $\varphi_{nm}(x,y) = \sin (n\pi x/a) \times \sin (m\pi y/b)$ with corresponding eigenvalues

$$\omega_{nm} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \qquad n = 1, 2, \ldots ; m = 1, 2, \ldots$$

Hence show that the values $\lambda_{nm} = -\omega_{nm}{}^2$ are eigenvalues of the problem in **c**, with φ_{nm} corresponding eigenfunctions; that is,

$$\Delta\Delta\varphi_{nm} - \omega_{nm}{}^2\varphi_{nm} = 0 \qquad \varphi_{nm} = \Delta\varphi_{nm} = 0 \qquad \text{on } C$$

The completeness of the orthogonal sequence $\{\varphi_{nm}\}$ implies that these are all the eigenvalues.

(e) If there are no vertical deflections and no bending moments along the edge of the rectangular plate, the deflection w is the solution of the boundary-value problem

$$\Delta\Delta w = \frac{f(x,y)}{N} \qquad \text{in } \mathscr{R} \qquad\qquad w = \Delta w = 0 \qquad \text{on } C$$

(these boundary conditions are often termed the *Navier conditions*). Assume the solution is

$$w(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm}\varphi_{nm}(x,y)$$

Then $w = \Delta w = 0$ on C. Since w is a solution of the differential equation, it follows that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm}\omega_{nm}{}^2\varphi_{nm}(x,y) = \frac{f(x,y)}{N}$$

Let p, q be a fixed pair of positive integers. Follow the procedure of Example 3-5, and use the orthogonality properties of the sequence $\{\varphi_{nm}\}$ to show that

$$c_{pq} = \frac{A_{pq}}{N\omega_{pq}{}^2} = \frac{4}{N\omega_{pq}{}^2 ab} \int_0^a \int_0^b f(x,y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}\, dx\, dy$$

Hence the solution

$$w(x,y) = \frac{1}{Nab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_{nm}}{\omega_{nm}{}^2} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

(f) Show that if the plate is uniformly loaded so that $f(x,y) = f_0$, a constant, the deflection is

$$w(x,y) = \frac{16 f_0}{\pi^6 N} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin [(2n-1)\pi x/a] \sin [(2m-1)\pi y/b]}{(2n-1)(2m-1)\beta_{nm}}$$

where

$$\beta_{nm} = \left[\frac{(2n-1)^2}{a^2} + \frac{(2m-1)^2}{b^2}\right]^2$$

If the plate is square, show that the deflection at the center is approximately $4 f_0 a^4/\pi^6 N$.

where the functions $C_0(t)$, $C_q(t)$, and $C_{nmq}(t)$ are to be determined by the method set forth in the text. One obtains

$$C_0(t) = \int_0^t d\xi \int_0^\xi F_0(\eta)\,d\eta \qquad c_q(t) = \frac{1}{\omega_q}\int_0^t F_q(\xi)\sin\,[\omega_q(t-\xi)]\,d\xi$$

$$C_{nmq}(t) = \frac{1}{\omega_{nmq}}\int_0^t F_{nmq}(\xi)\sin\,[\omega_{nmq}(t-\xi)]\,d\xi$$

where

$$F_0(t) = \frac{4}{\pi h(b^2-a^2)}\int_a^b\int_0^{\pi/2}\int_0^h F(r,\theta,z,t)r\,dr\,d\theta\,dz$$

$$F_q(t) = \frac{8}{\pi h(b^2-a^2)}\int_a^b\int_0^{\pi/2}\int_0^h F(r,\theta,z,t)\cos\frac{q\pi z}{h}\,r\,dr\,d\theta\,dz \qquad q = 1, 2, \ldots$$

$$F_{nmq}(t) = \frac{1}{\|\psi_{nmq}\|^2}\int_a^b\int_0^{\pi/2}\int_0^h F(r,\theta,z,t)\psi_{nmq}(r,\theta,z)r\,dr\,d\theta\,dz$$

$$n = 0, 1, \ldots;\ m = 1, 2, \ldots;\ q = 0, 1, \ldots$$

Now the superposition

$$u = v + w$$

yields the solution of the original problem.

PROBLEMS

Sec. 4-2

1 (a) A solution of the homogeneous wave equation (4-5) of the form

$$u(x,y) = \psi(x)e^{\pm i\omega t} \qquad i = \sqrt{-1};\ \omega \text{ real and positive}$$

is called *harmonic time-dependent*. Substitute into Eq. (4-5), and show that the *amplitude factor* ψ must satisfy

$$\psi'' + k^2\psi = 0 \qquad k^2 = \frac{\omega^2}{c^2}$$

where primes denote derivatives with respect to x. Hence show that harmonic time-dependent wave functions are of the form

$$u(x,y) = Ae^{i(kw\pm\omega t)} \qquad k = \pm\frac{\omega}{c};\ A = \text{const}$$

(b) Assume a solution of Eq. (4-5) of the form

$$u(x,t) = \psi(x)e^{\pm\omega t} \qquad \omega \text{ real and positive}$$

and derive wave functions of the form

$$u(x,t) = Ae^{kx\pm\omega t} \qquad k = \pm\frac{\omega}{c};\ A = \text{const}$$

(c) Let f be a twice continuously differentiable function. Show that

$$u(x,t) = f(x-ct) - f(-x-ct)$$

is a wave function which satisfies the boundary condition

$u(0,t) = 0$

If, in addition, f is an even function of its argument, $f(-\zeta) = f(\zeta)$, then u satisfies the initial condition

$u(x,0) = 0$

(d) Construct a nontrivial real-valued wave function which satisfies the stated condition.

(i) u is harmonic time-dependent and $u(0,t) = 0$ all t.

(ii) u is harmonic time-dependent, $u(0,t) = 0$ all t, and $u(x,0) = 0$ all x.

(iii) $u(0,t) = 0, \quad \lim_{x \to +\infty} u(x,t) = 0, \quad \lim_{x \to -\infty} u(x,t) = 0.$

(iv) $u_x(0,t) = 0$ all t and $u_t(x,0) = 0$ all x.

(v) $(u_x + \alpha u)|_{x=0} = 0$ all t, α a real constant.

2 The one-dimensional homogeneous wave equation with damping is

$$u_{tt} + 2\gamma u_t - c^2 u_{xx} = 0$$

where γ is a real positive constant [recall Eq. (4-3)]. Solutions of this equation may represent waves whose amplitudes are *damped out* with increasing time or waves which are *attenuated* as they travel.

(a) Assume a solution of the form $u = e^{-\gamma t} v(x,t)$. Show that v must satisfy

$$v_{tt} - \gamma^2 v - c^2 v_{xx} = 0$$

To obtain particular solutions of this equation assume

$$v = \psi(x) e^{\pm i\omega t} \qquad i = \sqrt{-1};\ \omega \text{ real}$$

and show that ψ must satisfy

$$\psi'' + k^2 \psi = 0 \qquad k^2 = \frac{\gamma^2 + \omega^2}{c^2}$$

Hence derive particular solutions of the homogeneous damped wave equation of the form

$$u = A e^{-\gamma t} e^{\pm i(kx - \omega t)} \qquad k = \pm \frac{(\omega^2 + \gamma^2)^{1/2}}{c}$$

where A is a constant. These represent traveling waves with amplitude $A' = Ae^{-\gamma t}$ and with speed

$$c' = \frac{\omega}{k} = \frac{c\omega}{(\omega^2 + \gamma^2)^{1/2}} < c$$

(b) Assume a solution of the form $u = \psi(x)e^{-i\omega t}$, ω a real and positive constant. Show that ψ must satisfy

$$\psi'' - \alpha^2 \psi = 0 \qquad \alpha^2 = -\frac{\omega^2 + 2i\omega\gamma}{c^2}$$

Write $\alpha = a + ib$, a, b real. Calculate α^2, equate real and imaginary parts, and obtain

$$a^2 - b^2 = -\frac{\omega^2}{c^2} \qquad ab = -\frac{\omega\gamma}{c^2}$$

Show that $b = \mp k(\omega)$ and $a = \pm \omega\gamma/c^2k(\omega)$, where

$$k(\omega) = \frac{\sqrt{2}\,\omega\gamma}{c[\omega(\omega^2 + 4\gamma^2)^{1/2} - \omega^2]^{1/2}} > 0$$

Hence derive the particular solutions

$$u = Ae^{-h(\omega)x}e^{i(kx-\omega t)} \qquad h(\omega) = \frac{[\omega(\omega^2 + 4\gamma^2)^{1/2} - \omega^2]^{1/2}}{\sqrt{2}c}$$

The amplitude $A' = Ae^{-h(\omega)x} \to 0$ as $x \to +\infty$. Thus as the wave progresses to the right, the wave is attenuated. The speed

$$c' = \frac{\omega}{k} = \frac{c^2h(\omega)}{\gamma}$$

depends on the frequency ω. In a superposition of such waves with different frequencies each component has a different speed.

(c) Derive the solutions

$$u = Ae^{-\gamma t}e^{kx-\omega t} \qquad k = \pm\frac{(\omega^2 - \gamma^2)^{1/2}}{c}\ ;\ \omega^2 > \gamma^2$$

$$u = Ae^{ikx}e^{-(\gamma+\omega)t} \qquad k = \pm\frac{(\gamma^2 - \omega^2)^{1/2}}{c}\ ;\ \gamma^2 > \omega^2$$

3 Use D'Alembert's solution (4-15) to construct the solution of the initial-value problem (4-17) with the given initial data. Note that f, g may not satisfy the differentiability conditions assumed in the text at every point. Nevertheless verify that the function obtained by applying D'Alembert's formula satisfies the initial conditions. Show also that the wave equation is satisfied except possibly at points along the characteristic curves of Eq. (4-5). Sketch the solution at times $t = 0$, $t = 1/c$, and $t = 2/c$.

(a) $f(x) = e^{-x}, g(x) = 0, -\infty < x < \infty$
(b) $f(x) = 1/(1 + x^2), g(x) = 0, -\infty < x < \infty$
(c) $f(x) = A \sin \omega x, g(x) = B \cos \mu x, -\infty < x < \infty$
(d) $f(x) = 0, g(x) = A \sinh ax, -\infty < x < \infty$
(e) $f(x) = 1, |x| \le 1, f(x) = 0, |x| > 1; g(x) = 0, -\infty < x < \infty$
(f) $f(x) = 1 - |x|, |x| \le 1, f(x) = 0, |x| > 1; g(x) = 0, -\infty < x < \infty$
(g) $f(x) = \cos x, |x| < \pi/2, f(x) = 0, |x| \ge \pi/2; g(x) = 0, -\infty < x < \infty$
(h) $f(x) = 0, -\infty < x < \infty; g(x) = 1, |x| < \epsilon, g(x) = 0, |x| \ge \epsilon, \epsilon > 0$ a constant

4 Recall the rule for differentiation of an integral with respect to a parameter. Show that if f is twice continuously differentiable and g is continuously differentiable, D'Alembert's solution (4-15) satisfies all the conditions of the initial-value problem (4-17).

5 Construct the solution of the initial-value problem (4-11) if the data are as described.

(a) $F(x,t) = 1, f(x) = \sin \omega x, g(x) = 0$
(b) $F(x,t) = xt, f(x) = g(x) = 0$
(c) $F(x,t) = 4x + t, f(x) = 0, g(x) = \cosh bx$
(d) $F(x,t) = A \sin \omega x \sin \mu t, f(x) = g(x) = 0$
(e) $F(x,t) = A \sin (kx - \omega t), f(x) = g(x) = 0, k = \omega c$

6 Verify that the function w defined by Eq. (4-16) is a solution of problem (4-18).

7 Consider the boundary- and initial-value problem

$$u_{tt} = c^2 u_{xx} \qquad x > 0;\ t > 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad x \geq 0$$

$$u(0,t) = 0 \qquad t \geq 0$$

Note that in order for the data to agree at (0,0) it is necessary that $f(0) = 0$, $g(0) = 0$ (see Fig. P4-1). The given functions f, g are defined only for $x \geq 0$; however, assume for the moment that f, g are defined in some manner on $(-\infty,\infty)$, and apply D'Alembert's formula (4-15) to the problem. Impose the condition along $x = 0$, and obtain

$$\tfrac{1}{2}[f(-ct) + f(ct)] + \frac{1}{2c}\int_{-ct}^{ct} g(\xi)\, d\xi = 0 \qquad t \geq 0$$

This equation will hold provided f, g are defined such that

$$f(ct) = -f(-ct) \qquad \int_{-t}^{t} g(\xi)\, d\xi = 0 \qquad t > 0$$

Hence, the appropriate procedure is to extend f and g as *odd functions* on $(-\infty,\infty)$:

$$f(x) = -f(-x) \qquad g(x) = -g(-x) \qquad x < 0$$

Now D'Alembert's formula defines the solution of the problem

$$u(x,t) = \begin{cases} \tfrac{1}{2}[f(x - ct) + f(x + ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(\xi)\, d\xi & 0 \leq ct \leq x \\ \tfrac{1}{2}[-f(ct - x) + f(x + ct)] + \dfrac{1}{2c}\displaystyle\int_{ct-x}^{x+ct} g(\xi)\, d\xi & 0 \leq x \leq ct \end{cases}$$

Verify that u satisfies the boundary condition along $x = 0$ as well as the initial conditions. Note that u satisfies the wave equation except possibly along the characteristics $x = \pm ct$

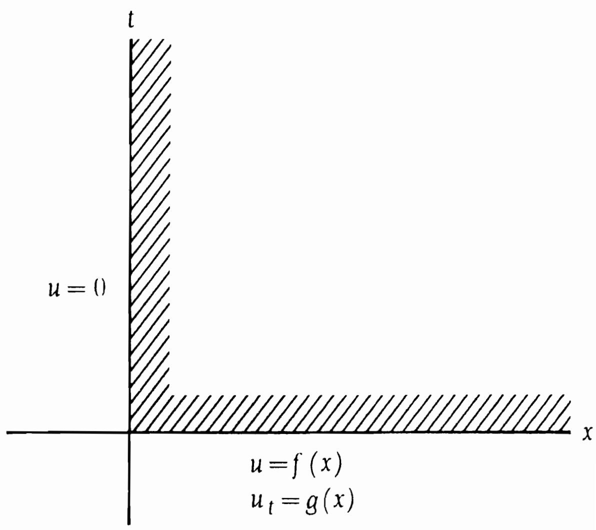

Figure P4-1

in the xt plane. Use this formula to construct the solution of the boundary- and initial-value problem corresponding to the following prescribed data. Sketch the solution at $t = 0$, $t = 1/c$, $t = 3/2c$, $t = 2/c$.

(a) $f(x) = 1$; $g(x) = -\cos x$, $x \geq 0$

(b) $f(x) = x^2$; $g(x) = 0$, $x \geq 0$

(c) $f(x) = 1$, $0 \leq x < 1$, $f(x) = 0$, $x \geq 1$; $g(x) = 0$, $x \geq 0$

(d) $f(x) = 0$, $0 \leq x < 1$, $f(x) = 2(x - 1)$, $1 \leq x \leq \frac{3}{2}$
$f(x) = 2(2 - x)$, $\frac{3}{2} \leq x \leq 2$, $f(x) = 0$, $x > 2$; $g(x) = 0$, $x \geq 0$

8 Consider the following boundary- and initial-value problem

$$u_{tt} = c^2 u_{xx} \qquad x > 0;\ t > 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad x \geq 0$$

$$u(0,t) = h(t) \qquad t \geq 0$$

Assume f, h are twice continuously differentiable and g is continuously differentiable, on $[0,+\infty)$. Note that if the boundary and initial values agree at the corner (0,0) in the xt plane, the necessary conditions $h(0) = f(0)$, $h'(0) = g(0)$ are implied in the data. Let v denote the function constructed as the solution in Prob. 7. Suppose w is a solution of the wave equation (except possibly along the characteristic $x = ct$) such that

$$w(x,0) = 0 \qquad w_t(x,0) = 0 \qquad x \geq 0 \qquad\qquad w(0,t) = h(t) \qquad t \geq 0$$

Then $u = v + w$ satisfies the conditions of the boundary- and initial-value problem first proposed, except that the wave equation may not be satisfied along the characteristic. To construct w assume $w = \varphi(x - ct)$, $x \neq ct$. Impose the boundary condition at $x = 0$; then

$$\varphi(-ct) = h(t) \qquad t \geq 0$$

Let $\xi = -ct$, so that $\varphi(\xi) = h(-\xi/c)$ and

$$\varphi(x - ct) = h\left(t - \frac{x}{c}\right)$$

Define w as

$$w(x,t) = 0 \qquad 0 \leq ct \leq x \qquad\qquad w(x,t) = h\left(t - \frac{x}{c}\right) \qquad 0 \leq x \leq ct$$

Verify that w has the desired properties at $t = 0$ and satisfies the wave equation, except possibly along $x = ct$, $t \geq 0$. Use the results of these considerations to construct the solution of the boundary- and initial-value problem for each of the following cases.

(a) $f(x) = e^{-x}$, $g(x) = 1$, $h(t) = 1$

(b) $f(x) = x$, $g(x) = 0$, $h(t) = \sin t$

(c) $f(x) = g(x) = 0$, $h(t) = 1$, $0 \leq t \leq T$, $h(t) = 0$, $t > T$, where T is a given constant

(d) $f(x) = g(x) = 0$, $h(t) = A \sin \omega t$

9 Consider the problem

$$u_{tt} - c^2 u_{xx} = 0 \qquad u = f(x) \qquad u_t = g(x) \qquad \text{on } C$$

where C is the characteristic $t = x/c$ in the xt plane. Attempt to fit the general solution (4-6) of the wave equation to the prescribed data. Show that no solution exists unless the given functions f, g satisfy the relation

$$g(x) = \frac{cf'(x)}{2} + a$$

for some constant a. Show that if f, g satisfy such a relation, there are infinitely many distinct solutions of the form

$$u = \varphi(x - ct) + f\left(\frac{x + ct}{2}\right) - \varphi(0)$$

where φ is any twice continuously differentiable function such that $\varphi'(0) = -a/c$.

10 Let k be a fixed real number such that $k > 1/c$. Consider the following problem involving the homogeneous wave equation:

$$u_{tt} = c^2 u_{xx} \qquad x > 0;\ 0 < t < kx$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad x \geq 0$$

$$u(x,kx) = h(x) \qquad x \geq 0$$

Here the curve in the xt plane on which the data are prescribed consists of the nonnegative x axis and the half line $t = kx$, $x \geq 0$ (see Fig. P4-2). Let v denote the function constructed as the solution in Prob. 7. Then v satisfies the conditions prescribed on $t = 0$. Also

$$v(x,kx) = G(x) = \tfrac{1}{2}\{-f[(kc - 1)x] + f[(kc + 1)x]\} + \frac{1}{2c}\int_{(kc-1)x}^{(kc+1)x} g(\xi)\, d\xi$$

It is desired to construct the solution of the present problem by superposition:

$$u = v + w$$

Evidently w must satisfy the homogeneous wave equation and the initial conditions

$$w(x,0) = w_t(x,0) = 0 \qquad w(x,kx) = h(x) - G(x) \qquad x \geq 0$$

To construct w assume

$$w(x,t) = 0 \qquad 0 \leq ct \leq x \qquad\qquad w(x,t) = \varphi(x - ct) \qquad 0 \leq x \leq ct$$

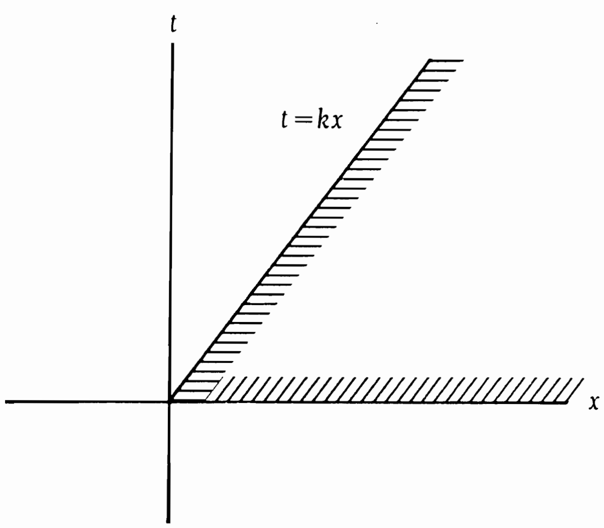

Figure P4-2

where φ is a function to be determined. Then w satisfies the necessary conditions along $t = 0$. To satisfy the remaining condition it is necessary that

$$\varphi(x) = h(x) - G(x) \qquad x \geq 0$$

Hence

$$\varphi(\xi) = h\left(\frac{\xi}{1 - kc}\right) - G\left(\frac{\xi}{1 - kc}\right)$$

and

$$\varphi(x - ct) = h\left(\frac{x - ct}{1 - kc}\right) - G\left(\frac{x - ct}{1 - kc}\right)$$

Complete the remaining steps of the derivation and show that the solution of the problem is

$$u(x,t) = \begin{cases} v(x,t) & 0 \leq ct \leq x \\ h\left(\dfrac{x - ct}{1 - kc}\right) + \tfrac{1}{2}\{f[\beta(x,t)] - f[\alpha(x,t)]\} + \dfrac{1}{2c}\displaystyle\int_\alpha^\beta g(\xi)\, d\xi & 0 \leq x \leq ct \end{cases}$$

where

$$\alpha(x,t) = \frac{1 + kc}{(1 - kc)(x - ct)} \qquad \beta(x,t) = x + ct$$

Verify that u satisfies the conditions of the problem. Note that u is a solution of the homogeneous wave equation, except possibly along the characteristic $x = ct$. Write down the solution of the Cauchy problem if

$$f(x) = g(x) = 0 \qquad h(x) = A \sin \omega x \qquad x \geq 0$$

where $A > 0$ is a constant. Discuss the behavior of u along the characteristic $x = ct$ for this case.

11 Recall the discussion in Sec. 2-3. The general homogeneous linear second-order hyperbolic equation with constant coefficients in two independent variables x, t is

$$Lu = Au_{xx} + 2Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = 0$$

where $\Delta = B^2 - AC > 0$. Let a be a fixed real positive constant, and suppose that for arbitrary choice of function f

$$u = f(x - at)$$

is a solution of the differential equation. Utilize the arbitrariness of f to show that in this case the relations

(i) $F = 0$ **(ii)** $D - aE = 0$ **(iii)** $Ca^2 - 2Ba + A = 0$

must hold. Conversely suppose $F = 0$ and the coefficients are such that

(iv) $AE^2 - 2BDE + CD^2 = 0$

Show that plane-wave solutions of arbitrary shape and with common speed $a = D/E$ exist. On the other hand, if $F \neq 0$, or if $F = 0$, $D \neq 0$, $E \neq 0$, and **iv** does not hold, or if $F = D = 0$, $E \neq 0$ (or $F = E = 0$, $D \neq 0$), the equation does not admit plane-wave solutions of arbitrary profile having a common speed. Show that if $C > 0$ and

$$F = D = E = 0 \qquad a_{1,2} = \frac{B \pm \sqrt{\Delta}}{C}$$

plane-wave solutions of arbitrary profile having common speed a_1 (or a_2) exist.

12 **(a)** The system of first-order linear partial differential equations

$$\frac{\partial v}{\partial x} + L\frac{\partial i}{\partial t} + Ri = 0 \qquad \frac{\partial i}{\partial x} + C\frac{\partial v}{\partial t} + Gv = 0$$

occurs in the theory of electric transmission lines; they are called the *transmission-line equations*. Here $v(x,t)$ denotes the voltage and $i(x,t)$ the current, at distance x along the line at time t. The constants L, C, R, G are real and nonnegative. They denote inductance, capacitance, resistance, and conductance (per unit length). Differentiate the equations with respect to x and t in an appropriate fashion, and deduce that if v, i are solutions of the transmission-line equations, then v must satisfy the *telegrapher's equation*

$$LCv_{tt} + (RC + LG)v_t + RGv - v_{xx} = 0$$

[see Eq. (4-4)]. Show that i satisfies this equation also. If $L > 0$ and $C > 0$, the equation is hyperbolic. If $L = 0$, or if $C = 0$, the equation is parabolic. Assume $L > 0$ and $C > 0$. If $R = 0$ and $G \neq 0$, show that the telegrapher's equation can be rewritten

$$v_{tt} + bv_t - c^2v_{xx} = 0$$

which is the damped-wave equation, where

$$c = \frac{1}{\sqrt{LC}}$$

If $R = G = 0$, then the wave equation results.

(b) Utilize the results of Prob. 11 and show that if $R \neq 0$, or if $G \neq 0$, the telegrapher's equation does not admit traveling-wave solutions

$$v = f(x - at)$$

of arbitrary profile having common speed a. In turn this implies the damped-wave equation does not admit traveling waves of arbitrary profile.

(c) A transmission line (assumed to be characterized by the telegrapher's equation) is called *distortionless* if the line admits traveling waves

$$v = e^{-\mu t}f(x - at)$$

of arbitrary profile having common speed a, for some real nonnegative constant μ. If $\mu > 0$, the waves are damped out exponentially with increasing time. Show that a necessary and sufficient condition for a distortionless line is

$$RC = LG$$

In this event show that the common speed is $a = 1/\sqrt{LC}$.

13 **(a)** Let L be the linear hyperbolic operator defined by

$$Lu = \frac{\partial^2 u}{\partial x\, \partial y} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u$$

Assume the coefficients are twice continuously differentiable in some region $\mathscr{R}$ of the xy plane. Verify from Eq. (2-74) that the adjoint of L is the linear hyperbolic operator L^*

defined by

$$L^*v = \frac{\partial^2 v}{\partial x\, \partial y} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial y}(bv) + cv$$

Utilize Lagrange's identity to show that if u, v are twice continuously differentiable, then

$$vLu - uL^*v = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

where

$$P = \tfrac{1}{2}(uv_x - vu_x) - buv \qquad Q = \tfrac{1}{2}(vu_y - uv_y) + auv$$

Let Γ be a simple closed curve in $\mathscr{R}$, and let $\mathscr{R}_1$ denote the region enclosed by Γ. Green's formula (2-77) states that if u, v are as described above, then

$$\iint_{\mathscr{R}_1} (vLu - uL^*v)\, dx\, dy = \int_\Gamma P\, dx + Q\, dy$$

where the line integral is taken in the positive sense around Γ.

(b) (*Riemann's method*) Let C be a given smooth curve lying in $\mathscr{R}$ and such that any line parallel to a coordinate axis intersects C in at most one point. Thus C is noncharacteristic. Consider the Cauchy problem

$$Lu = F(x,t) \qquad (x,y) \text{ in } \mathscr{R}$$

$$u = f(x) \qquad \frac{\partial u}{\partial n} = g(x) \qquad \text{on } C$$

where F is a given continuous function, f is twice continuously differentiable, and g is continuously differentiable. The symbol $\partial u/\partial n$ denotes the derivative of u in the direction of the normal $\mathbf{n}$ to C. Let $P_0(x_0,y_0)$ be a point not on C. Then Riemann's method of deriving the expression for the solution of the Cauchy problem at (x_0,y_0) is as follows. Construct the closed curve Γ as shown in Fig. P4-3. The segments BP_0, AP_0 are parallel

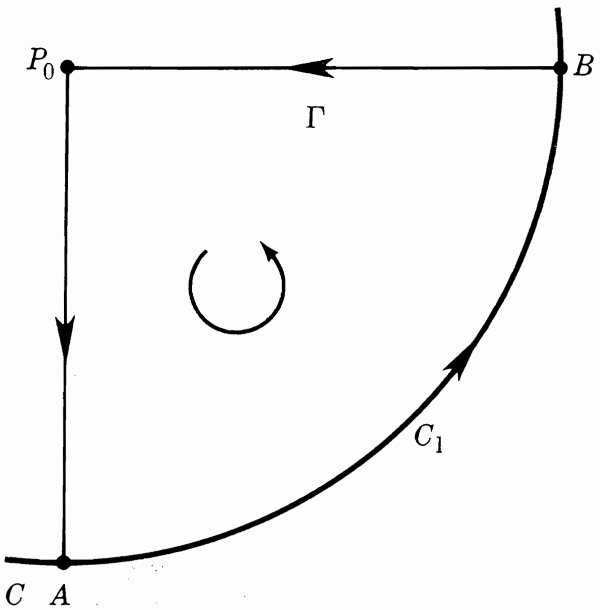

Figure P4-3

to the coordinate axes. Let $\mathscr{R}_1$ denote the region enclosed by Γ. If u is the solution of the Cauchy problem, then by Green's formula derived in **a** above

$$\iint_{\mathscr{R}_1} vF\,dx\,dy = \iint_{\mathscr{R}_1} uL^*v\,dx\,dy + \int_\Gamma P\,dx + Q\,dy$$

for any twice continuously differentiable function v. The appropriate choice for v is now derived. Rewrite P, Q as

$$P = -\frac{(uv)_x}{2} + u(v_x - bv) \qquad Q = \frac{(uv)_y}{2} - u(v_y - av)$$

The line integral around Γ is calculated as follows. Along the characteristic BP_0

$$\int_{BP_0} P\,dx + Q\,dy = \int_B^{P_0} P\,dx = \frac{(uv)|_B - (uv)|_{P_0}}{2} + \int_B^{P_0} u(v_x - bv)\,dx$$

Along the characteristic P_0A

$$\int_{P_0A} P\,dx + Q\,dy = \int_{P_0}^{A} Q\,dy = \frac{(uv)|_A - (uv)|_{P_0}}{2} - \int_{P_0}^{A} u(v_y - av)\,dy$$

Thus

$$\int_\Gamma P\,dx + Q\,dy = -(uv)|_{P_0} + \tfrac{1}{2}[(uv)|_B + (uv)|_A]$$
$$+\int_B^{P_0} u(v_x - bv)\,dx - \int_{P_0}^{A} u(v_y - av)\,dy + \int_{C_1} P\,dx + Q\,dy$$

where C_1 is the portion of C lying between the points A, B. Assume now that $v(x,y;x_0,y_0)$ is a function satisfying the following conditions:

$L^*v = 0 \qquad \text{in } \mathscr{R}_1$

$v_x = bv \qquad \text{on } y = y_0 \qquad\qquad v_y = av \qquad \text{on } x = x_0$

$v(x_0,y_0;x_0,y_0) = 1$

The function v is called the *Riemann Green's function* of the Cauchy problem. From the expressions derived above it follows that

$$u|_{P_0} = \tfrac{1}{2}[(uv)|_A + (uv)|_B] + \int_{C_1} P\,dx + Q\,dy - \iint_{\mathscr{R}_1} vF\,dx\,dy$$

Since the right-hand member involves only known functions, the value $u(x_0,y_0)$ is now determined.

14 **(a)** Let L be the hyperbolic operator defined by

$Lu = u_{xy}$

Show that L is self-adjoint: $L^* = L$. Show that the Riemann Green's function $v(x,y;x_0,y_0)$ associated with L must satisfy $Lv = 0$ and

$v_x = 0 \qquad \text{on } y = y_0 \qquad\qquad v_y = 0 \qquad \text{on } x = x_0$

$v(x_0,y_0;x_0,y_0) = 1$

Clearly the function $v(x,y;x_0,y_0) = 1$, all x, y, has the requisite properties.

(b) Let C be the line $y = x$ in the xy plane. Consider the Cauchy problem

$$u_{xy} = F(x,y) \qquad u|_C = f(x) \qquad \left.\frac{\partial u}{\partial n}\right|_C = g(x)$$

where F, f, g are given functions and $\partial u/\partial n$ denotes the derivative of u in the direction of the normal $\mathbf{n}$ to C. Thus $\partial u/\partial n = (u_y - u_x)/\sqrt{2}$. Apply the results of Prob. 13, and derive the solution of the Cauchy problem in the form

$$u(x_0,y_0) = \tfrac{1}{2}[f(x_0) + f(y_0)] + \frac{1}{\sqrt{2}}\int_{x_0}^{y_0} g(x)\,dx - \iint_{\mathscr{R}_1} F(x,y)\,dx\,dy$$

where $\mathscr{R}_1$ is the triangular region in the xy plane bounded by C and the lines $x = x_0$, $y = y_0$ through (x_0,y_0).

15 **(a)** In the initial-value problem (4-11) make the change of independent variables

$$\xi = x - ct \qquad \eta = x + ct$$

Show that the transformed problem is

$$u_{\xi\eta} = G(\xi,\eta) \qquad G(\xi,\eta) = \frac{-F[(\eta + \xi)/2, (\eta - \xi)/2c]}{4c^2}$$

$$u|_C = f(\xi) \qquad \left.\frac{\partial u}{\partial n}\right|_C = \frac{g(\xi)}{\sqrt{2}c}$$

where C is the line $\eta = \xi$ in the $\xi\eta$ plane and $\partial u/\partial n$ denotes the derivative of u in the direction of the normal $\mathbf{n}$ to C. Show that the transformation is such that the upper half plane $t \geq 0$ in the xt plane is mapped onto the half plane $\eta \geq \xi$ in the $\xi\eta$ plane. Is the transformed problem equivalent to problem (4-11)? Why?

(b) Use the results of Prob. 14 together with the result in **a** above to derive the solution of problem (4-11).

16 **(a)** Let L be the hyperbolic operator defined by

$$Lu = u_{xy} + au$$

where $a > 0$ is a constant. Show that L is self-adjoint: $L^* = L$. Show that the Riemann Green's function $v(x,y;x_0,y_0)$ associated with L must satisfy $Lv = 0$ and

$$v_x = 0 \quad \text{on } y = y_0 \qquad v_y = 0 \quad \text{on } x = x_0$$

$$v(x_0,y_0;x_0,y_0) = 1$$

(b) To determine the Riemann Green's function assume

$$v = \varphi[(x - x_0)(y - y_0)]$$

where φ is a function to be determined. Show this assumed form for v satisfies the required conditions on $y = y_0$ and on $x = x_0$. Substitute v into $Lv = 0$ and show that φ must satisfy

$$s\varphi''(s) + \varphi'(s) + a\varphi(s) = 0 \qquad s = (x - x_0)(y - y_0)$$

Make the change of variable $t = 2\sqrt{as}$, and show that the preceding differential equation becomes

$$t\ddot{\varphi} + \dot{\varphi} + t\varphi = 0$$

where dots denote differentiation with respect to t. This is Bessel's equation of order zero. Hence

$$\varphi(s) = J_0(2\sqrt{as})$$

and so

$$v = J_0[2\sqrt{a(x - x_0)(y - y_0)}]$$

is the Riemann Green's function of L.

(c) Let C be the line $y = x$ in the xy plane. Consider the Cauchy problem

$$u_{xy} + au = F(x,y) \qquad u|_C = f(x) \qquad \left.\frac{\partial u}{\partial n}\right|_C = g(x)$$

where F, f, g are given functions and $\partial u/\partial n$ denotes the derivative of u in the direction of the normal $\mathbf{n}$ to C. Thus $\partial u/\partial n = (u_y - u_x)/\sqrt{2}$. Apply the results of Prob. 13 and part **b** of the present problem, and derive the solution of the Cauchy problem in the form

$$u(x_0,y_0) = \tfrac{1}{2}[f(x_0) + f(y_0)] + \frac{1}{\sqrt{2}}\int_{x_0}^{y_0} J_0(\mu)g(x)\,dx$$
$$+ a(x_0 - y_0)\int_{x_0}^{y_0} \frac{J_0'(\mu)}{\mu} f(x)\,dx - \iint_{\mathscr{R}_1} J_0(2\sqrt{as})F(x,y)\,dx\,dy$$

where $s = (x - x_0)(y - y_0)$, $\mu = 2\sqrt{a(x - x_0)(x - y_0)}$, and $\mathscr{R}_1$ is the triangular region in the xy plane bounded by C and the lines $x = x_0$, $y = y_0$ through (x_0,y_0).

17 **(a)** The one-dimensional telegrapher's equation [recall Eq. (4-4)] is

$$u_{tt} + 2\gamma u_t + \beta u - c^2 u_{xx} = F(x,t)$$

where the constants γ, β, c are such that

$$\gamma > 0 \qquad \beta \geq 0 \qquad c > 0$$

Make the change of dependent variable

$$v(x,t) = e^{\gamma t}u(x,t)$$

and show that the resulting equation is

$$v_{tt} + (\beta - \gamma^2)v - c^2 v_{xx} = e^{\gamma t}F(x,t)$$

Note that if $\beta = \gamma^2$, the wave equation results. Now make the change of independent variables

$$\xi = x - ct \qquad \eta = x + ct$$

and show that the differential equation is transformed into

$$v_{\xi\eta} + av = H(\xi,\eta)$$

where

$$a = \frac{\gamma^2 - \beta}{4c^2} \qquad H(\xi,\eta) = -e^{\gamma(\eta-\xi)/2c}\frac{F[(\eta+\xi)/2, (\eta-\xi)/2c]}{4c^2}$$

(b) Consider the problem of solving the telegrapher's equation subject to the initial conditions

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad -\infty < x < \infty$$

where f, g are given functions. Show that this initial-value problem is equivalent to the Cauchy problem

$$v_{\xi\eta} + av = H(\xi,\eta)$$

$$v = f(\xi) \qquad \frac{\partial v}{\partial n} = \frac{\gamma f(\xi) + g(\xi)}{\sqrt{2}c} \qquad \text{on } C$$

where C is the line $\eta = \xi$ in the $\xi\eta$ plane and $\partial v/\partial n$ is the derivative of v in the direction of the normal **n** to C. The constant a and the function H are as described in **a** above.

(c) Assume $\gamma^2 > \beta$, so that $a > 0$. Apply the results of Prob. 16 and part **b** of the present problem, and derive the solution of the initial-value problem for the one-dimensional telegrapher's equation in the form

$$u(x,t) = e^{-\gamma t}v(x,t)$$

where

$$v(x,t) = \tfrac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} J_0(\mu)[\gamma f(s) + g(s)]\,ds - 2act$$
$$\times \int_{x-ct}^{x+ct} \frac{J_0'(\mu)}{\mu} f(s)\,ds + \frac{1}{2c}\iint_{\mathscr{R}_1} J_0\{2\sqrt{a[(s-x)^2 - c^2(\tau-t)^2]}\}e^{\gamma\tau}F(s,\tau)\,ds\,d\tau$$

where

$$a = \frac{\gamma^2 - \beta}{4c^2} \qquad \mu = 2\sqrt{a(x-s)^2 - c^2t^2}$$

and $\mathscr{R}_1$ is the domain of determinacy of the interval $[x - ct, x + ct]$.

Sec. 4-3

18 **(a)** The idealized string described in the text has fastened ends at $x = 0$, $x = b$. In the case where the string is released from rest and no external force acts, the subsequent displacement is given by Eq. (4-30) with $B_n = 0$, $n = 1, 2, \ldots$. Use the identity

$$\sin(a+b) + \sin(a-b) = 2\sin a\cos b$$

and show that the displacement is given by

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi(x-ct)}{b} + \frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi(x+ct)}{b}$$

Now let f_0 be the function which is the odd periodic extension of f to $(-\infty,\infty)$, with period $2b$. That is,

$$f_0(x) = f(x) \qquad 0 \le x \le b \qquad\qquad f_0(x) = -f(-x) \qquad -b \le x < 0$$
$$f_0(x + 2b) = f_0(x) \qquad -\infty < x < \infty$$

Assume f is continuous and piecewise smooth and

$$f(0) = f(b) = 0$$

State why it is true that the displacement of the freely vibrating string is given by

$$u(x,t) = \tfrac{1}{2}[f_0(x - ct) + f_0(x + ct)] \qquad 0 \le x \le b;\ t \ge 0$$

Interpret this result graphically in terms of a superposition of traveling waves. Relate to D'Alembert's solution (4-15).

(b) Consider the freely vibrating string which is released from its equilibrium position, so that $f(x) = 0$, $0 \le x \le b$, and with speed $g(x)$, $0 \le x \le b$. Assume g is not identically zero, continuous, and piecewise smooth and

$$g(0) = g(b) = 0$$

Let g_0 be the odd periodic extension of g to $(-\infty,\infty)$ with period $2b$. Show that

$$u_t(x,t) = \sum_{n=1}^{\infty} \omega_n B_n \sin\frac{n\pi x}{b}\cos\frac{n\pi ct}{b}$$

$$= \tfrac{1}{2}[g_0(x - ct) + g_0(x + ct)]$$

and hence

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(\xi)\, d\xi$$

Relate to D'Alembert's solution (4-15).

19 The string with fastened ends at $x = 0$, $x = b$ executes free vibrations after release from the initial displacement $f(x)$ and with initial speed $g(x)$. Determine the series expression for the subsequent displacement $u(x,t)$.

(a) $f(x) = 4hx(b - x)/b^2$, $g(x) = 0$, $0 \le x \le b$ ($h =$ const)

(b) $f(x) = 10 \sin(\pi x/b)$, $g(x) = 0$, $0 \le x \le b$

(c) $f(x) = A \sin \omega x$, $g(x) = 1$, $0 \le x \le b$ (ω is a constant and is not an integral multiple of π/b)

(d) $f(x) = 0$, $0 \le x \le b$; $g(x) = v_0$, $b/2 - \epsilon \le x \le b/2 + \epsilon$, $g(x) = 0$ elsewhere (ϵ is a small positive constant)

20 **(a)** The kinetic energy of an element ds of the vibrating string is

$$\frac{(dm)u_t^2}{2} \cong \frac{(\rho\, dx)u_t^2}{2}$$

where ρ is the (constant) linear density. Hence the total kinetic energy of motion is

$$K = \frac{\rho}{2}\int_0^b u_t^2\, dx$$

To obtain the expression for the potential energy consider an element dx of the string when the string lies in its equilibrium configuration along the x axis. Let ds be the length of the element at a subsequent time $t > 0$. The change in length is

$$ds - dx = (1 + u_x^2)^{1/2}\, dx - dx \cong u_x^2 \frac{dx}{2}$$

if higher-order terms in u_x^2 are neglected. The extension occurs in the presence of an elastic restoring force of magnitude T_0. Hence the increase in potential energy is

$$T_0 u_x^2 \frac{dx}{2}$$

The net potential energy is

$$V = \frac{T_0}{2} \int_0^b u_x^2 \, dx$$

The total energy is $E = K + V$.

(b) Show that in a traveling wave $u = f(x \pm ct)$, where $c = \sqrt{T_0/\rho}$, the kinetic energy equals the potential energy.

(c) The nth normal mode of vibration of the string with fastened ends is given by Eq. (4-27). Show that u_n can be rewritten

$$u_n(x,t) = C_n \sin \frac{n\pi x}{b} \cos \left(\frac{n\pi ct}{b} - \epsilon_n \right)$$

where the *amplitude* C_n and *phase* ϵ_n are given by

$$C_n = (A_n^2 + B_n^2)^{1/2} \qquad \tan \epsilon_n = \frac{B_n}{A_n}$$

Obtain the expressions

$$K_n(t) = \frac{T_0 \pi^2 n^2 C_n^2 \sin^2 (n\pi ct/b - \epsilon_n)}{4b}$$

$$V_n(t) = \frac{T_0 \pi^2 n^2 C_n^2 \cos^2 (n\pi ct/b - \epsilon_n)}{4b}$$

for the kinetic and potential energy of the nth mode. Hence the total energy is

$$E_n = K_n + V_n = \frac{T_0 \pi^2 n^2 C_n^2}{4b} = \frac{T_0 b \omega_n^2 C_n^2}{4c^2} = \frac{M \omega_n^2 C_n^2}{4}$$

where M is the total mass of the string. Observe that the energy of the nth mode is proportional to the square of the amplitude and proportional to the square of frequency.

(d) Subsequent to release from an initial displacement $f(x)$ with initial velocity $g(x)$ the string with fastened ends executes free vibrations. Use the series (4-30) and show that the energy is

$$E = \sum_{n=1}^{\infty} E_n^2 = \frac{M}{4} \sum_{n=1}^{\infty} \omega_n^2 C_n^2$$

that is, the sum of the individual energies of the normal modes.

21 **(a)** The string with fastened ends at $x = 0$, $x = b$ executes forced vibrations under the external force per unit mass

$$F(t) = F_0 \sin \omega t$$

where F_0, ω are given positive constants. The string is released from rest with zero displacement. Use Eq. (4-41) to obtain an expression for the displacement $u(x,t)$. Distinguish the cases **(i)** $\omega \neq \omega_k$, all k; **(ii)** $\omega = \omega_k$, some k. Case **ii** illustrates *resonance* for the vibrating string with fastened ends.

(b) Obtain the expression for $u(x,t)$ if

$$f(x) = 1 - \cos\frac{2m\pi x}{b} \qquad g(x) = 0;\, 0 \leq x \leq b$$

where m is a given positive integer, and the external driving force per unit mass is

$$F(x,t) = F_0 e^{-t} \sin\frac{\pi x}{b}$$

where F_0 is a positive constant.

(c) Obtain the expression for $u(x,t)$ if f, g are identically zero and the external force per unit mass is

$$F(x,t) = \begin{cases} F_0 & x_0 - \epsilon < x < x_0 + \epsilon;\ 0 < t < \delta \\ 0 & x \text{ not in } (x_0 - \epsilon,\, x_0 + \epsilon) \text{ or } t > \delta \end{cases}$$

Here F_0 is a constant, and ϵ, δ are small positive numbers. This represents a force of magnitude F_0 confined to the interval $(x_0 - \epsilon, x_0 + \epsilon)$ about the point x_0 and of duration δ. In the expression for the resulting displacement u let $\epsilon \to 0$, $\delta \to 0$, and obtain the motion due to a concentrated force applied to the point $x = x_0$ at time $t = 0$.

22 **(a)** Recall the properties of the unit impulse function given in Sec. 3-5. A concentrated impulsive force per unit mass and of unit magnitude applied at time $t = \tau$ to the point $x = \xi$ on the string can be represented

$$F(x,t) = \delta(x - \xi)\delta(t - \tau)$$

Thus

$$F(x,t) = 0 \qquad (x,t) \neq (\xi,\tau)$$

and

$$\int_0^t dt \int_0^b F(x,t)\, dx = 1 \qquad t > \tau$$

Let the ends of the string be fastened, and assume the initial conditions are zero. Use Eq. (4-41) and the properties of the δ function to obtain the formal series

$$G(x,t;\xi,\tau) = \frac{2}{\pi c} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{b} \sin\frac{n\pi\xi}{b} \sin\left[\frac{n\pi c}{b}(t - \tau)\right]$$

for the subsequent displacement. It can be shown that the series converges for all values of x, t, ξ, τ. Note that G, as a function of x, satisfies the boundary conditions

$$G(0,t;\xi,\tau) = 0 \qquad G(b,t;\xi,\tau) = 0 \qquad t \geq 0$$

Also

$$G(x,\tau;\xi,\tau) = 0 \qquad 0 \leq x \leq b$$

$$G(\xi,t;x,\tau) = G(x,t;\xi,\tau) \qquad \text{all } x,\ \xi$$

$$G(x,\tau;\xi,t) = -G(x,t;\xi,\tau) \qquad \text{all } t,\ \tau$$

The function G is called the *Green's function* of the boundary- and initial-value problem embodied in Eqs. (4-22) to (4-24). The function G is not a solution of the wave equation in the usual classical sense; however, it has many useful properties.

(b) Verify that the solution (4-41) of problem (4-35) can be written

$$u(x,t) = \int_0^t \int_0^b G(x,t;\xi,\tau)F(\xi,\tau)\, d\xi\, d\tau$$

by substitution of the series expression for the Green's function and a formal interchange of the operations of summation and integration so as to obtain the series solution (4-41). Interpret the integral in a physical way.

(c) The string with fastened ends is at rest in its equilibrium configuration along the x axis. At time $t = 0$ the string is struck in such a way that the point $x = \xi$ receives a velocity impulse of unit magnitude. Thus the initial conditions are

$$u(x,0) = 0 \qquad u_t(x,0) = \delta(x - \xi) \qquad 0 \le x \le b$$

Apply Eqs. (4-30), (4-33), and (4-34) in a formal manner, and use the properties of the δ function to show that the subsequent motion is given by

$$G(x,t;\xi,0) = \frac{2}{\pi c} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{b} \sin \frac{n\pi \xi}{b} \sin \frac{n\pi c t}{b}$$

(d) Substitute the series expression for $G(x,t;\xi,0)$, and formally interchange the operations of summation and integration to verify that the solution of problem (4-25) in the case where f is identically zero on $[0,b]$ is given by

$$u(x,t) = \int_0^b G(x,t;\xi,0)g(\xi)\, d\xi$$

where g is the initial velocity. Interpret this result in a physical way.

(e) Let $G(x,t;\xi,\tau)$ be the Green's function obtained in **a** above. Substitute the series expression and formally interchange the operations of summation and integration to verify that the series solution (4-30) of problem (4-25) is given by

$$u(x,t) = \int_0^b G_t(x,t;\xi,0)f(\xi)\, d\xi + \int_0^b G(x,t;\xi,0)g(\xi)\, d\xi$$

where f is the initial displacement and g is the initial velocity.

23 A pair of concentrated impulsive forces, of equal magnitude F_0 but oppositely directed, is applied transversely to the interior points of trisection of the string with fastened ends. Obtain the formal series expression for the subsequent displacement. Show that the midpoint of the string remains at rest.

24 **(a)** Consider the boundary- and initial-value problem embodied in Eqs. (4-22) to (4-24). Suppose the external force is independent of t.

$$F(x,t) = F_0(x)$$

Choose $v(x)$ such that

$$v''(x) = \frac{-F_0(x)}{c^2} \qquad v(0) = v(b) = 0$$

Let $w(x,t)$ be the solution of problem (4-25) with $f(x)$ replaced by $f(x) - v(x)$. Verify that the superposition

$$u = w + v$$

satisfies Eqs. (4-22) to (4-24).

(b) Solve the problem in Example 4-1 of the text by the method outlined in **a**.

(c) Use a slight modification of the method outlined in **a** to derive a formal solution of the boundary- and initial-value problem.

$$u_{tt} - c^2u_{xx} = F_0 \sin \omega x \qquad 0 \leq x \leq b;\ t \geq 0$$

$$u(0,t) = h_0 \qquad u(b,t) = h_1 \qquad t \geq 0$$

$$u(x,0) = x(b - x) \qquad u_t(x,0) = 0 \qquad 0 \leq x \leq b$$

where F_0, ω, h_0, h_1 are given positive real constants.

25 **(a)** Consider the boundary- and initial-value problem embodied in Eqs. (4-22), (4-23), and (4-24) when the driving function has the form

$$F(x,t) = F_1(x) \sin \omega t$$

where $F_1(x)$ is a given function and ω is a real positive constant. Assume a particular solution of Eq. (4-22) of the form

$$v(x,t) = X(x) \sin \omega t$$

Show that X must satisfy

$$X'' + \mu^2 X = \frac{-F_1(x)}{c^2} \qquad \mu = \frac{\omega}{c}$$

Conversely, if X is a solution of this ordinary differential equation such that

$$X(0) = X(b) = 0$$

then the function v is a solution of Eq. (4-22) which satisfies the boundary conditions (4-23). Suppose $w(x,t)$ is a solution of problem (4-25) with $g(x)$ replaced by $g(x) - \omega X(x)$; show that the superposition

$$u = v + w$$

satisfies Eqs. (4-22) to (4-24).

(b) It should be noted that the method outlined in **a** assumes a solution of the two-point boundary-value problem

$$X'' + \mu^2 X = \frac{-F_1(x)}{c^2} \qquad X(0) = 0 \qquad X(b) = 0 \qquad \mu = \frac{\omega}{c}$$

exists. However, a solution of the problem may or may not exist. Prove that **(i)** if ω is not a characteristic frequency ($\omega \neq \omega_k$, $k = 1, 2, \ldots$), a solution of the two-point problem exists; **(ii)** if $\omega = \omega_m = m\pi c/b$ for some positive integer m (that is, the case of *resonance*, ω coincides with a characteristic frequency), no solution of the two-point problem exists unless

$$\int_0^b F_1(\xi) \sin \frac{m\pi\xi}{b}\, d\xi = 0$$

In this event there are infinitely many distinct solutions.

(c) Solve Prob. 21**a** by the method outlined in **a** of the present problem.

(d) Solve the boundary- and initial-value problem

$$u_{tt} - c^2 u_{xx} = F_0 \sin \nu x \cos \omega t \qquad 0 \le x \le b;\ t \ge 0$$
$$u(0,t) = h_0 \qquad u(b,t) = h_1 \qquad t \ge 0$$
$$u(x,0) = x(b - x) \qquad u_t(x,0) = 0 \qquad 0 \le x \le b$$

where F_0, ν, ω, h_0, h_1 are given real positive constants. Distinguish the cases

(i) $\omega \ne n\pi c/b$, $n = 1, 2, \ldots$

(ii) $\omega = m\pi c/b$, m a positive integer.

26 If, instead of being fastened, the end $x = 0$ of the string is elastically constrained, the boundary condition is

$$u_x(0,t) - hu(0,t) = 0 \qquad h = \frac{K}{T_0} > 0$$

where K is the spring constant and T_0 is the horizontal component of the tension. Let the end $x = b$ be fixed. Assume the initial conditions are given by Eq. (4-24).

(a) Separate variables in the wave equation, and show the x-dependent factor X must satisfy

$$X'' + \lambda X = 0 \qquad X'(0) - hX(0) = 0 \qquad X(b) = 0$$

where λ is a separation constant. This is a Sturm-Liouville problem with unmixed boundary conditions. Review Sec. 1 of Appendix 2, and appeal to the appropriate theorems in order to verify that the eigenvalues form a real monotone discrete sequence $\{\lambda_n\}$ and that all the eigenvalues are positive. Now, by direct integration of the differential equation and imposition of the boundary conditions, show that $\lambda = 0$ is not an eigenvalue. Derive the eigenfunctions and eigenvalues

$$X_n = \sin [\mu_n(x - b)] \qquad \lambda_n = \mu_n^2 \qquad n = 1, 2, \ldots$$

where the μ_n are the real positive roots of the transcendental equation

$$h \tan b\mu + \mu = 0$$

arranged in increasing order. Sketch the graphs of the equations

$$y = \tan b\mu \qquad y = -\frac{\mu}{h}$$

on the same set of axes, and illustrate graphically the occurrence of the roots of the transcendental equation. Hence verify the discreteness and monotonicity of the sequence of eigenvalues $\{\lambda_n\}$. Note that

$$\frac{(2n - 1)\pi}{2b} < \mu_n < \frac{(2n + 1)\pi}{2b} \qquad n = 1, 2, \ldots$$

and for large n

$$\mu_n \sim \frac{(2n - 1)\pi}{2b}$$

By direct integration show that the eigenfunctions $\{X_n\}$ form an orthogonal sequence on $[0,b]$.

(b) Derive the normal modes

$$u_n(x,t) = \sin [\mu_n(x - b)](A_n \cos \omega_n t + B_n \sin \omega_n t) \qquad n = 1, 2, \ldots$$

where the characteristic frequencies $\omega_n = c\mu_n$, $n = 1, 2, \ldots$.

(c) Show that if the superposition of normal modes

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin [\mu_n(x - b)]$$

is suitably convergent and satisfies the initial conditions (4-24), the coefficients must be given by

$$A_n = \frac{1}{\alpha_n} \int_0^b f(x) \sin [\mu_n(x - b)]\, dx$$

$$B_n = \frac{1}{\alpha_n \omega_n} \int_0^b g(x) \sin [\mu_n(x - b)]\, dx$$

where

$$\alpha_n = \frac{hb + \cos^2 \mu_n b}{2h} \qquad n = 1, 2, \ldots$$

27 Derive the formal series solution of the problem

$$u_{tt} = c^2 u_{xx} \qquad 0 \le x \le b;\ t \ge 0$$

$$u_x(0,t) - h_1 u(0,t) = 0 \qquad u_x(b,t) + h_2 u(b,t) = 0 \qquad t \ge 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad 0 \le x \le b$$

where h_1 and h_2 are given positive constants.

28 Instead of being fastened, the end $x = 0$ of the string is forced (by some external means) to undergo a prescribed displacement. Then the boundary conditions are

$$u(0,t) = h(t) \qquad u(b,t) = 0 \qquad t \ge 0 \tag{1}$$

where h is a given function. Let the initial conditions be given by Eq. (4-24). Continuity of the spring implies

$$h(0) = f(0) \qquad \dot{h}(0) = g(0)$$

Choose a simple function v which satisfies the boundary conditions (1). Let w be a solution of Eq. (4-22) with F replaced by

$$F(x,t) - v_{tt} + c^2 v_{xx}$$

Assume also that w satisfies the homogeneous boundary conditions (4-23) and the initial conditions

$$w(x,0) = f(x) - v(x,0) \qquad w_t(x,0) = g(x) - v_t(x,0)$$

Verify that the superposition $u = v + w$ satisfies Eq. (4-22), the boundary conditions (1) and the initial conditions (4-24). An example of a suitable function v is

$$v(x,t) = \left(1 - \frac{x}{b}\right) h(t)$$

This function interpolates linearly between the value $h(t)$ at $x = 0$ and the value 0 at $x = b$.

29 **(a)** Solve the boundary- and initial-value problem

$$u_{tt} - c^2 u_{xx} = -g \qquad 0 \le x \le b;\ t \ge 0$$
$$u(0,t) = h_0 \sin \omega t \qquad u(b,t) = 0$$
$$u(x,0) = 0 \qquad u_t(x,0) = h_0\omega$$

where g, h_0, ω are given real positive constants and

$$\omega \ne \frac{n\pi c}{b} \qquad n = 1, 2, \ldots$$

(b) Solve the boundary- and initial-value problem

$$u_{tt} - c^2 u_{xx} = x(b - x) \sin \nu t$$
$$u(0,t) = h_0 \sin \omega t \qquad u(b,t) = h_1 \cos \omega t$$
$$u(x,0) = 0 \qquad u_t(x,0) = \frac{(b - x)h_0\omega}{b_0}$$

where ν, h_0, h_1, ω are given real constants and

$$\nu \ne \frac{n\pi c}{b} \qquad \omega \ne \frac{n\pi c}{b} \qquad n = 1, 2, \ldots$$

30 If the string vibrates in a medium, e.g., air, a frictional force may be present. In the case where the friction force is proportional to the speed the equation of motion of the idealized string is

$$u_{tt} + 2\gamma u_t - c^2 u_{tt} = F(x,t) \qquad 0 \le x \le b;\ t \ge 0 \tag{1}$$

Equation (1) is called the *damped-wave equation*. Here γ is a known positive constant and $c = \sqrt{T_0/\rho}$. It was observed in Prob. 2 that solutions of the homogeneous equation may represent traveling waves whose amplitudes are attenuated as they travel. Assume the ends of the string are fastened. Then the boundary conditions are given by Eq. (4-23). Let the initial conditions be given by Eq. (4-24).

(a) The *homogeneous damped-wave equation* is (1) with F identically zero. Separate variables in the homogeneous equation, and derive the *normal modes*

$$u_n(x,t) = e^{-\gamma t} \sin \frac{n\pi x}{b} (A_n \cos \omega_n t + B_n \sin \omega_n t) \qquad n = 1, 2, \ldots$$

where the *normal* (or characteristic) *frequencies*

$$\omega_n = \left(\frac{n^2\pi^2c^2}{b^2} - \gamma^2\right)^{1/2} \qquad n = 1, 2, \ldots$$

The normal modes are damped oscillatory provided the damping constant γ is sufficiently small. Henceforth assume this is so.

(b) Consider the homogeneous damped-wave equation subject to the boundary conditions (4-23) and initial conditions (4-24). Derive the formal series solution

$$u(x,t) = e^{-\gamma t} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{b} (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

where

$$A_n = \frac{2}{b} \int_0^b f(x) \sin \frac{n\pi x}{b}\, dx \qquad B_n = \frac{\gamma A_n}{\omega_n} + \frac{2}{b\omega_n} \int_0^b g(x) \sin \frac{n\pi x}{b}\, dx$$

(c) Consider the damped-wave equation (1) subject to the homogeneous boundary conditions (4-23) and the homogeneous initial conditions

$$u(x,0) = 0 \qquad u_t(x,0) = 0 \qquad 0 \le x \le b$$

Assume a solution of the form in Eq. (4-36). Show that the functions φ_k must satisfy

$$\ddot{\varphi}_k + 2\gamma\dot{\varphi}_k + \frac{k^2\pi^2c^2}{b^2} \varphi_k = F_k(t) \qquad \varphi_k(0) = \dot{\varphi}_k(0) = 0$$

$k = 1, 2, \ldots$, where the F_k are defined in Eq. (4-39). Thus derive the solution

$$u(x,t) = e^{-\gamma t} \sum_{n=1}^{\infty} \left\{ \frac{1}{\omega_n} \int_0^t e^{\gamma\xi} \sin [\omega_n(t - \xi)] F_n(\xi)\, d\xi \right\} \sin \frac{n\pi x}{b}$$

31 Derive the formal series solution.

$$u_{tt} + 2\gamma u_t - c^2 u_{xx} = -g \qquad 0 \le x \le b;\ t \ge 0$$
$$u(0,t) = 0 \qquad u(b,t) = 0 \qquad t \ge 0$$
$$u(x,0) = x(b - x) \qquad u_t(x,0) = 0 \qquad 0 \le x \le b$$

32 With regard to the string described in Prob. 30, a concentrated impulsive force

$$F(x,t) = \delta(x - \xi)\delta(t - \tau)$$

of unit magnitude is applied to the string at time $t = \tau$ and at the point $x = \xi$. The motion starts from rest with zero displacement. Show that the subsequent motion is given by

$$G(x,t;\xi,\tau) = \frac{2}{b} e^{-\gamma t} \sum_{n=1}^{\infty} \frac{1}{\omega_n} \sin \frac{n\pi x}{b} \sin \frac{n\pi\xi}{b} \sin [\omega_n(t - \tau)]$$

where the ω_n are the characteristic frequencies derived in Prob. 30. Use substitution and formal interchange of the operations of integration and summation to show that the solution of the problem of forced motion derived in Prob. 30**c** can be written

$$u(x,t) = \int_0^t \int_0^b G(x,t;\xi,\tau) F(\xi,\tau)\, d\xi\, d\tau$$

Use this expression and the *Green's function* together with superposition to derive the solution of Prob. 31. What relation exists between this Green's function and the Green's function obtained in Prob. 22?

33 Give a complete discussion and derive the formal series solution of the problem

$$u_{tt} + 2\gamma u_t - c^2 u_{xx} = F_1(x) \sin \omega t \qquad 0 \le x \le b;\ t \ge 0$$
$$u(0,t) = h_1 \qquad u(b,t) = h_2 \qquad t \ge 0$$
$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad 0 \le x \le b$$

Here γ, h_1, h_2, and ω are given real positive constants, and $F_1(x), f(x), g(x)$ are given real-valued functions.

34 In the determination of the voltage distribution $v(x,t)$ along a transmission line of length b the following boundary- and initial-value problem arises.

$$v_{xx} = LCv_{tt} + RCv_t \qquad 0 \le x \le b;\ t \ge 0$$
$$v(0,t) = 0 \qquad v_x(b,t) = 0 \qquad t \ge 0$$
$$v(x,0) = v_0 \qquad v_t(x,0) = 0 \qquad 0 \le x \le b$$

Here L, C, R, and v_0 are given real positive constants, and $L > R^2Cb^2/\pi^2$. Derive the formal series solution

$$v(x,t) = e^{-Rt/2L}\sum_{n=1}^{\infty} A_n \sin\frac{(2n-1)\pi x}{2b}\sin(\omega_n t + \epsilon_n)$$

where

$$A_n = \frac{4v_0}{\pi(2n-1)\sin\epsilon_n} \qquad \tan\epsilon_n = \frac{2L\omega_n}{R}$$

$$\omega_n = \frac{[L\pi^2(2n-1)^2 - R^2Cb^2]^{1/2}}{2bLC^{1/2}} \qquad n = 1, 2, \ldots$$

35 Derive the formal series solution

$$u_{tt} + 2\gamma u_t - c^2u_{xx} = F_0 \sin\nu t \qquad 0 \le x \le b;\ t \ge 0$$
$$u(0,t) = 0 \qquad u_x(b,t) = h_1 \sin\omega t \qquad t \ge 0$$
$$u(x,0) = 0 \qquad u_t(x,0) = 0 \qquad 0 \le x \le b$$

Here F_0, γ, ν, h_1, ω are given real positive constants.

36 The vibrating string illustrates *transverse waves*: the direction of motion of the individual particles is perpendicular to the direction of propagation of waves. The occurrence of elastic waves in a solid bar, e.g., metal, illustrates *longitudinal waves*: the direction of motion of individual particles is the same as the direction of propagation of waves. Consider a long, slender cylindrical homogeneous solid bar, of uniform cross section, which is at rest with its axis coincident with the x axis. The ends of the bar are at $x = 0, x = b > 0$. The rod is assumed to be perfectly elastic, so that if an elongation takes place as a result of the application of external forces at the ends of the bar, tensile forces, directed parallel to the x axis, are set up within the bar. If now the forces are removed, the bar vibrates longitudinally in accordance with the laws of elasticity. Let $\xi(x,t)$ denote the longitudinal displacement at time t of the point in the bar whose equilibrium position was x for $t < 0$. It can be shown that the function ξ must satisfy the homogeneous wave equation

$$\xi_{tt} - c^2\xi_{xx} = 0 \qquad 0 \le x \le b;\ t \ge 0$$

where $c = \sqrt{E/\rho}$, $E =$ modulus of elasticity, $\rho =$ density. Accordingly waves of longitudinal displacement occur. Assume the end $x = 0$ of the bar is held fixed, and the end $x = b$ is free. Then the boundary conditions are

$$\xi(0,t) = 0 \qquad \xi_x(b,t) = 0 \qquad t \ge 0$$

The second boundary condition is a consequence of the relation $T = EA\xi_x$, where T is the tensile force within the bar at the point x and A is the cross-sectional area of the bar. The initial conditions are

$$\xi(x,0) = \frac{(b_1 - b)x}{b} \qquad \xi_t(x,0) = 0 \qquad 0 \le x \le b$$

where b_1 is the length of bar at maximum extension. Derive a formal series solution of the boundary- and initial-value problem.

37 Derive a formal series solution of the problem

$$\xi_{tt} - c^2\xi_{xx} = 0 \qquad 0 \le x \le b;\ t \ge 0$$

$$\xi(0,t) = 0 \qquad M\xi_{tt}(b,t) - EA\xi_x(b,t) = 0 \qquad t \ge 0$$

$$\xi(x,0) = \frac{(b_1 - b)x}{b} \qquad \xi_t(x,0) = 0 \qquad 0 \le x \le b$$

where M, E, A, b_1 are given real positive constants.

38 Let L be the linear operator defined by

$$Lu = A(x)u_{xx} + C(t)u_{tt} + A_1(x)u_x + C_1(t)u_t + [A_0(x) + C_0(t)]u$$

Assume the coefficients are twice continuously differentiable, and $A(x)C(t) < 0$ in the region of the xt plane under consideration. Then L is a hyperbolic operator.

(a) Consider first the boundary- and initial-value problem

$$Lu = 0 \qquad 0 \le x \le b;\ t \ge 0$$

$$a_1u(0,t) + a_2u_x(0,t) = 0 \qquad b_1u(b,t) + b_2u_x(b,t) = 0 \qquad t \ge 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad 0 \le x \le b$$

where $[0,b]$ is a given fixed interval, a_1, a_2, b_1, b_2 are given real constants such that

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) \ne 0$$

and f, g are given real-valued twice continuously differentiable functions. Assume a separable solution

$$u(x,t) = X(x)T(t)$$

of $Lu = 0$, and show that the factors X, T must satisfy the linear second-order ordinary differential equations

$$AX'' + A_1X' + A_0X = \lambda X$$

$$C\ddot{T} + C_1\dot{T} + C_0T = -\lambda T$$

respectively, where λ is a separation constant. Here primes denote derivatives with respect to x, and dots denote derivatives with respect to t. Without loss of generality it can be assumed that $A(x) < 0$, $0 \le x \le b$. Multiply the differential equation satisfied by X by the function

$$p(x) = \frac{-\exp\left[\int (A_1/A)\,dx\right]}{A}$$

Show that the self-adjoint equation

$$(pX')' + qX + \lambda pX = 0$$

is obtained, where $p = -A\rho$ and $q = -A_0\rho$. Note that $p(x) > 0$ and $\rho(x) > 0$. The appropriate boundary conditions on X are

$$a_1X(0) + a_2X'(0) = 0 \qquad b_1X(b) + b_2X'(b) = 0$$

Review Sec. 1 of Appendix 2. The problem in X is a regular self-adjoint Sturm-Liouville problem. Accordingly the eigenvalues constitute a real sequence $\{\lambda_n\}$ such that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots \qquad \lim_{n\to\infty} \lambda_n = +\infty$$

There are at most a finite number of negative eigenvalues. If $A_0(x) \geq 0$, all eigenvalues are nonnegative. Corresponding to each eigenvalue λ_n is a real-valued eigenfunction φ_n. The sequence $\{\varphi_n\}$ of eigenfunctions forms an orthogonal sequence on $[0,b]$ with weight function ρ

$$\int_0^b \rho(x)\varphi_n(x)\varphi_m(x)\,dx = 0 \qquad n \neq m$$

(b) It can be assumed that $C(t) > 0$, $t \geq 0$. For each positive integer n there exists a fundamental set $T_n^{(1)}(t)$, $T_n^{(2)}(t)$ of the second-order linear equation

$$C\ddot{T} + C_1\dot{T} + (C_0 + \lambda_n)T = 0 \qquad t \geq 0$$

such that $T_n^{(1)}(0) = 1$, $\dot{T}_n^{(1)}(0) = 0$, $T_n^{(2)}(0) = 0$, and $\dot{T}_n^{(2)}(0) = 1$. Define the *normal modes* of the boundary- and initial-value problem

$$u_n(x,t) = \varphi_n(x)[A_nT_n^{(1)}(t) + B_nT_n^{(2)}(t)] \qquad n = 1, 2, \ldots$$

Each normal mode satisfies $Lu = 0$ as well as the homogeneous boundary conditions of the problem.

(c) Consider a superposition of normal modes

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)[A_nT_n^{(1)}(t) + B_nT_n^{(2)}(t)]$$

Show that if u is a solution of the boundary- and initial-value problem, the coefficients in the series must be given by

$$A_n = \frac{1}{\|\varphi_n\|^2}\int_0^b \rho(x)f(x)\varphi_n(x)\,dx \qquad B_n = \frac{1}{\|\varphi_n\|^2}\int_0^b \rho(x)g(x)\varphi_n(x)\,dx$$

where

$$\|\varphi_n\|^2 = \int_0^b \rho(x)\varphi_n^2(x)\,dx \qquad n = 1, 2, \ldots$$

(d) Consider now the problem

$$Lu = F(x,t) \qquad 0 \leq x \leq b;\ t \geq 0$$
$$a_1u(0,t) + a_2u_x(0,t) = 0 \qquad b_1u(b,t) + b_2u_x(b,t) = 0 \qquad t \geq 0$$
$$u(x,0) = 0 \qquad u_t(x,0) = 0 \qquad 0 \leq x \leq b$$

involving homogeneous boundary and initial conditions and the inhomogeneous partial differential equation. Here F is a given real-valued twice continuously differentiable function. Assume a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)\psi_n(t)$$

where the φ_n are the eigenfunctions obtained in **a** above and the functions ψ_n are to be determined. Then u satisfies the boundary conditions. It will satisfy the initial conditions if

$$\psi_n(0) = 0 \qquad \dot{\psi}_n(0) = 0 \qquad n = 1, 2, \ldots$$

Substitute u into the partial differential equation, and show that the ψ_n must satisfy

$$C\ddot{\psi}_n + C_1\dot{\psi}_n + (C_0 + \lambda_n)\psi_n = F_n(t) \qquad n = 1, 2, \ldots$$

where

$$F_n(t) = \frac{1}{\|\varphi_n\|^2}\int_0^b \rho(x)F(x,t)\varphi_n(x)\,dx \qquad n = 1, 2, \ldots$$

The initial conditions and the differential equation uniquely determine the ψ_n. With the ψ_n determined this way, the formal series solution of the problem is obtained. Now, using superposition, the formal series solution of the problem

$$Lu = F(x,t) \qquad 0 \le x \le b;\ t \ge 0$$
$$a_1u(0,t) + a_2u_x(0,t) = 0 \qquad b_1u(b,t) + b_2u_x(b,t) = 0 \qquad t \ge 0$$
$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad 0 \le x \le b$$

can be derived. If the boundary conditions are inhomogeneous

$$a_1u(0,t) + a_2u_x(0,t) = h_1(t) \qquad b_1u(b,t) + b_2u_x(b,t) = h_2(t) \qquad t \ge 0$$

where h_1, h_2 are given functions, application of the technique outlined in Prob. 28 together with superposition again leads to the formal solution.

39 Derive the formal series solution.

(a) $u_{tt} - c^2u_{xx} - \omega^2u = F_0 \sin \nu t \qquad 0 \le x \le b;\ t \ge 0$

$u(0,t) = 0 \qquad u(b,t) = 0 \qquad t \ge 0$

$u(x,0) = f(x) \qquad u_t(x,0) = g(x)$

where c, ω, F_0, ν are given real positive constants, $\omega \neq \nu$.

(b) $\dfrac{u_{tt}}{c^2} - u_{xx} - 2u_x = 0 \qquad 0 \le x \le b;\ t \ge 0$

$u(0,t) = 0 \qquad u(b,t) = 0 \qquad t \ge 0$

$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad 0 \le x \le b$

(c) $u_{tt} - c^2x^2u_{xx} = 0 \qquad 0 < a \le x \le b;\ t \ge 0$

$u(a,t) = 0 \qquad u(b,t) = 0 \qquad t \ge 0$

$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad a \le x \le b$

40 Derive the formal solution of the problem

$$u_{tt} - c^2[(b^2 - x^2)u_x]_x = 0 \qquad 0 < x < b;\ t > 0$$
$$u(0,t) = 0 \qquad \lim_{\substack{x\to b\\ x<b}} u(x,t) \text{ exists} \qquad t \ge 0$$
$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad 0 \ge x \ge b$$

in the form

$$u(x,t) = \sum_{n=1}^{\infty} P_{2n-1}\left(\frac{x}{b}\right)(A_n \cos \omega_n t + B_n \sin \omega_n t)$$

where the P_n are the Legendre polynomials, and the coefficients

$$A_n = \frac{4n-1}{b}\int_0^b f(x)P_{2n-1}\left(\frac{x}{b}\right)dx \qquad B_n = \frac{4n-1}{b\omega_n}\int_0^b g(x)P_{2n-1}\left(\frac{x}{b}\right)dx$$

The characteristic frequencies are

$$\omega_n = c[2n(2n-1)]^{1/2} \qquad n = 1, 2, \ldots$$

41 Derive the formal series solution of the problem

$$u_{tt} - c^2(xu_x)_x = 0 \qquad 0 < x < b;\ t > 0$$

$$\lim_{\substack{x\to 0\\ x>0}} u(x,t) \text{ exists} \qquad u(b,t) = 0 \qquad t \geq 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad 0 \leq x \leq b$$

42 **(a)** Let L be the linear operator defined in Prob. 38. Assume C, C_1 are constants, $C > 0$, $C_1 \geq 0$. Also assume $A(x) < 0$, $0 \leq x \leq b$. Then L is a hyperbolic operator. Suppose u is a solution of the homogeneous equation $Lu = 0$. Define the function

$$v(x,t) = u(x,t)\exp\left(\frac{C_1}{2C}t\right)$$

Show that v must satisfy the equation

$$A(x)v_{xx} + A_1(x)v_x + Cv_{tt} + \left(A_0 + C_0 - \frac{C_1^2}{4C}\right)v = 0$$

Conversely, if v satisfies this partial differential equation, then u is a solution of $Lu = 0$. Define the function

$$p(x) = \exp\left(\left(\int \frac{A_1}{A}\,dx\right)\right)$$

Show that multiplication by p transforms the differential equation in v into
$(pv_x)_x - rv_{tt} - qv = 0$
where

$$r(x) = \frac{-Cp(x)}{A(x)} \qquad q(x,t) = \frac{[C_1^2/4C - A_0(x) - C_0(t)]p(x)}{A(x)}$$

Since $p(x) > 0$, $r(x) > 0$, $0 \leq x \leq b$, it follows that the transformed equation is hyperbolic also.

(b) Consider the problem

$$Lu = A(x)u_{xx} + Cu_{tt} + A_1(x)u_x + C_1u_t + A_0(x)u = 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad 0 \leq x \leq b$$

with boundary conditions one of the following:

$$u(0,t) = u(b,t) = 0 \tag{1}$$

$$u(0,t) = u_x(b,t) = 0 \tag{2}$$

$$u_x(0,t) = u(b,t) = 0 \tag{3}$$

$$u_x(0,t) - h_1u(0,t) = 0 \qquad u_x(b,t) + h_1u(b,t) = 0 \qquad h_1 > 0;\ h_2 > 0 \tag{4}$$

$$u_x(0,t) = u_x(b,t) = 0 \tag{5}$$

Assume C, C_1 are constants, $C > 0$, $C_1 \geq 0$, and $A(x) < 0$, $0 \leq x \leq b$, as in part **a** above. In addition assume

$$A_0(x) \geq \frac{C_1^2}{4C} \qquad 0 \leq x \leq b$$

Then the problem can have at most one solution; i.e., if a solution exists, it is unique. The proof of this fact can be made with the results of **a** above. By linearity, the difference of two solutions of the given problem is a solution of the homogeneous problem, i.e., with f and g identically zero. If u denotes this difference and v is as defined in **a**, then v satisfies the homogeneous problem

$$(pv_x)_x - rv_{tt} - qv = 0 \qquad 0 \leq x \leq b;\ t \geq 0$$

$$v(x,0) = 0 \qquad v_t(x,0) = 0 \qquad 0 \leq x \leq b$$

$$B(v) = 0$$

where $B(v) = 0$ is symbolic of one of the boundary conditions (1) to (5), with u replaced by v. Now v can be shown to be the trivial solution as follows. Define the *energy integral*

$$E(t) = \frac{1}{2}\int_0^b (pv_t^2 + rv_t^2 + qv^2)\,dx$$

Since $p(x) > 0$, $r(x) > 0$, and $q(x) \geq 0$, it follows that $E(t) \geq 0$, $t \geq 0$. Differentiation yields

$$\frac{dE}{dt} = \int_0^b (pv_xv_{xt} + rv_tv_{tt} + qvv_t)\,dx$$

$$= pv_xv_t\Big|_0^b - \int_0^b (pv_x)_x - rv_{tt} - qv)v_t\,dx$$

$$= p(b)v_x(b,t)v_t(b,t) - p(0)v_x(0,t)v_t(0,t)$$

Show that if the boundary conditions are either (1), (2), (3), or (5), then dE/dt is zero for $t \geq 0$, and so E has a constant value. Apply the initial conditions and show $E = 0$. In turn prove that $E = 0$ implies v is identically zero. If the boundary conditions are (4), show that

$$\frac{dE}{dt} = -\frac{1}{2}\frac{d}{dt}[h_1p(0)v^2(0,t) + h_2p(b)v^2(b,t)]$$

and so

$$E(t) - E(0) = -\tfrac{1}{2}[h_1p(0)v^2(0,t) + h_2p(b)v^2(b,t)]$$

By the initial conditions, $E(0) = 0$. Thus $E(t) \leq 0$, $t \geq 0$. In turn this implies $E(t) = 0$, $t \geq 0$, and so v is identically zero.

43 **(a)** Prove the following existence theorem for problem (4-25). Let f have a continuous fourth derivative and g have a continuous third derivative on $[0,b]$ and such that

$$f(0) = f''(0) = f(b) = f''(b) = 0 \qquad g(0) = g''(0) = g(b) = g''(b) = 0$$

Then the series (4-30), where the coefficients are given by Eqs. (4-33) and (4-34), defines a function u which is twice continuously differentiable in x and t for $0 \leq x \leq b$, $t \geq 0$ and satisfies the homogeneous wave equation and the boundary and initial conditions of problem (4-25). *Hint:* Use integration by parts and show that

$$A_n = \frac{2b^3}{n^4\pi^4}\int_0^b f^{(4)}(x) \sin\frac{n\pi x}{b}\,dx \qquad B_n = -\frac{2b^3}{n^4\pi^4 c}\int_0^b g^{(3)}(x) \cos\frac{n\pi x}{b}\,dx$$

Hence there exists a constant $M > 0$ such that

$$|A_n| < \frac{M}{n^4} \qquad |B_n| < \frac{M}{n^4} \qquad n = 1, 2, \ldots$$

Now

$$\left|\sin\frac{n\pi x}{b}\left(A_n \cos\frac{n\pi ct}{b} + B_n \sin\frac{n\pi ct}{b}\right)\right| \leq |A_n| + |B_n| < \frac{2M}{n^4}$$

Let $T > 0$ be fixed. Use the Weierstrass test in conjunction with the convergent series of constants

$$\sum_{n=1}^{\infty} \frac{2M}{n^4}$$

to prove the series (4-30) converges uniformly for $0 \leq x \leq b$, $0 \leq t \leq T$ and so defines a continuous function u there. Since $T > 0$ is arbitrary, the function u defined by the series is continuous for $0 \leq x \leq b$, $t \geq 0$. Clearly u satisfies the boundary conditions of problem (4-25). Since the series for $u(x,0)$, $u_t(x,0)$ converge to f and g, respectively, it follows that u satisfies the initial conditions. It remains to show that u is a solution of the homogeneous wave equation. Since each normal mode u_n is a solution of the wave equation, one need only show that the series can be differentiated twice with respect to x or twice with respect to t. Show, for example, that the series

$$\sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial x^2} = \sum_{n=1}^{\infty} \frac{n^2\pi^2}{b^2} \sin\frac{n\pi x}{b}\,(A_n \cos\omega_n t + B_n \sin\omega_n t)$$

converges suitably. Do the same for the series obtained by differentiating termwise twice with respect to t.

(b) Prove the following existence theorem for problem (4-35). Let F have continuous third partial derivatives with respect to x and t such that

$$0 = F(0,t) = F(b,t) = F_{xx}(0,t) = F_{xx}(b,t) \qquad t \geq 0$$

Then the series (4-41), with the functions F_n defined as in Eq. (4-39), converges and defines a function u which is twice continuously differentiable for $0 \leq x \leq b$, $t \geq 0$ and which satisfies problem (4-35). *Hint:* From Eq. (4-37) and the form of the series (4-41) it follows that if the series is suitably convergent, the function u so defined satisfies the homogeneous boundary and initial conditions. Fix $T > 0$, and let

$$M(T) = \max\,[F_{xxx}(x,t)] \qquad 0 \leq x \leq b;\ 0 \leq t \leq T$$

Use integration by parts to show

$$F_n(t) \leq \frac{2b^3 M(T)}{n^3\pi^3} \qquad n = 1, 2, \ldots$$

and hence

$$|\varphi_n(t)| \le \frac{2b^3TM(T)}{n^4\pi^4c} \qquad n = 1, 2, \ldots$$

Use these inequalities to prove the series (4-41) converges uniformly on $0 \le x \le b$, $0 \le t \le T$ and so defines a continuous function there. In addition show that the series obtained by differentiating termwise twice with respect to x converges uniformly and

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} \left(-\frac{n^2\pi^2}{b^2}\right)\varphi_n(t) \sin\frac{n\pi x}{b} \qquad 0 \le x \le b;\ 0 \le t \le T$$

Since

$$\ddot{\varphi}_n = -\omega_n^2\varphi_n + F_n$$

show that

$$|\ddot{\varphi}_n(t)| \le \omega_n^2\,|\varphi_n(t)| + |F_n(t)| \le \frac{2b^2M(T)(cT + b)}{n^2\pi^2}$$

Thus the series obtained by differentiating termwise twice with respect to t converges uniformly, and

$$u_{tt} = \sum_{n=1}^{\infty} \ddot{\varphi}_n(t) \sin\frac{n\pi x}{b} = \sum_{n=1}^{\infty} [-\omega_n^2\varphi_n(t) + F_n(t)] \sin\frac{n\pi x}{b}$$

Hence

$$u_{tt} - c^2u_{xx} = \sum_{n=1}^{\infty} F_n(t) \sin\frac{n\pi x}{b} = F(x,t) \qquad 0 \le x \le b;\ 0 \le t \le T$$

44 Recall Prob. 12 and the telegrapher's equation, which results from the transmission-line equations. In Prob. 17 the initial-value problem for this equation was solved with the aid of the Riemann Green's function. If the transmission line is of finite length, boundary conditions on the voltage and current occur at the ends of the line, in addition to the prescribed initial conditions. In addition an external impressed voltage may be present. The resulting boundary- and initial-value problem in terms of the telegrapher's equation has the form

$$Lu = u_{tt} + 2\gamma u_t + \omega^2u - c^2u_{xx} = F(x,t) \qquad a \le x \le b;\ t \ge 0$$

$$a_1u(a,t) + a_2u_x(a,t) = h_1(t) \qquad b_1u(b,t) + b_2u_x(b,t) = h_2(t) \qquad t \ge 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad a \le x \le b$$

Here γ, ω are real nonnegative constants, and c is a real positive constant. If $\omega = 0$, $\gamma = 0$, the wave equation results. If $\omega = 0$, $\gamma > 0$, the damped-wave equation is obtained (see Probs. 30 to 35). Note that the operator L is a special case of the operator discussed in Prob. 38 and also is a special case of the operator considered in Prob. 42. Accordingly the results obtained in those problems apply to the present problem. In particular, if the corresponding homogeneous boundary conditions take one of the forms (1) to (5) stated in Prob. 42, there is at most one solution.

(a) As a special case consider the problem

$$Lu = F(x,t) \qquad a \le x \le b;\ t \ge 0$$

$$u(a,t) = h_1(t) \qquad u_x(b,t) = h_2(t) \qquad t \ge 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad a \le x \le b$$

where $\gamma > 0$, $\omega > 0$, and F, h_1, h_2, f, and g are given functions. First it is desired to remove the inhomogeneity in the boundary conditions. Consider the function

$$w(x,t) = \frac{x(x-2b)h_1(t)}{a(a-2b)} + \frac{ax(a-x)h_2(t)}{a(a-2b)}$$

This function satisfies the boundary conditions. If v satisfies

$$Lv = F(x,t) - Lw \qquad a \leq x \leq b;\ t \geq 0$$

$$v(a,t) = 0 \qquad v_x(b,t) = 0 \qquad t \geq 0$$

$$v(x,0) = f(x) \qquad v_t(x,0) = g(x) \qquad a \leq x \leq b$$

then $u = v + w$ is a formal solution of the problem.

(b) Assume now that the constants ω, γ are such that

$$\gamma^2 < \omega^2 + \left[\frac{\pi c}{2(b-a)}\right]^2$$

Consider the problem

$$Lu = 0 \qquad a \leq x \leq b;\ t \geq 0$$

$$u(a,t) = 0 \qquad u_x(b,t) = 0 \qquad t \geq 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad a \leq x \leq b$$

Separate variables, and derive the formal series solution

$$u(x,t) = e^{-\gamma t} \sum_{n=1}^{\infty} (A_n \cos \nu_n t + B_n \sin \nu_n t) \sin [\mu_n(x-a)]$$

where

$$A_n = \frac{2}{b-a} \int_a^b f(x) \sin [\mu_n(x-a)]\, dx$$

$$B_n = \frac{\gamma A_n}{\nu_n} + \frac{2}{\nu_n(b-a)} \int_a^b g(x) \sin [\mu_n(x-a)]\, dx$$

$$\mu_n = \frac{(2n-1)\pi}{2(b-a)} \qquad \nu_n = (\omega_n^2 + \omega^2 - \gamma^2)^{1/2}$$

$$\omega_n = c\mu_n \qquad n = 1, 2, \ldots$$

(c) Consider the problem

$$Lu = F(x,t) \qquad a \leq x \leq b;\ t \geq 0$$

$$u(a,t) = 0 \qquad u_x(b,t) = 0 \qquad t \geq 0$$

$$u(x,0) = 0 \qquad u_t(x,0) = 0 \qquad a \leq x \leq b$$

where F is a given function. Derive the formal series solution

$$u(x,t) = \sum_{n=1}^{\infty} \left\{\frac{1}{\nu_n} \int_0^t e^{-\gamma(t-\tau)} \sin [\nu_n(t-\tau)] F_n(\tau)\, d\tau\right\} \sin [\mu_n(x-a)]$$

where

$$F_n(t) = \frac{2}{b-a}\int_a^b F(x,t)\sin[\mu_n(x-a)]\,dx$$

(d) Choose $F(x,t) = \delta(x-\xi)\delta(t-\tau)$, and derive the formal series expression for the Green's function

$$G(x,t;\xi,\tau) = \frac{2}{b-a}e^{-\gamma(t-\tau)}\sum_{n=1}^{\infty}\frac{1}{\nu_n}\sin[\mu_n(x-a)]\sin[\mu_n(\xi-a)]\sin[\nu_n(t-\tau)]$$

Thus, if $H(x,t) = Lw$, where w is the function described in **a**, the formal solution of the problem posed in **a** is

$$u(x,t) = e^{-\gamma t}\sum_{n=1}^{\infty}(A_n\cos\nu_n t + B_n\sin\nu_n t)\sin[\mu_n(x-a)]$$
$$+\int_0^t\int_a^b G(x,t;\xi,\tau)[F(\xi,\tau) - H(\xi,\tau)]\,d\xi\,d\tau$$

Sec. 4-4

45 **(a)** A function of the form

$$u = \psi(x,y)e^{\pm i\omega t} \qquad i = \sqrt{-1};\ \omega \text{ real and positive}$$

is called *harmonic time-dependent*. Show that in order for u to be a solution of the homogeneous wave equation (4-45) it is necessary that the *amplitude factor* satisfy

$$\psi_{xx} + \psi_{yy} + k^2\psi = 0 \qquad k^2 = \frac{\omega^2}{c^2}$$

the *scalar Helmholtz equation* in two dimensions.

(b) Assume a solution of the Helmholtz equation of the form

$$\psi = e^{i(\alpha x+\beta y)} \qquad \alpha,\ \beta \text{ real constants}$$

and so derive the plane harmonic wave functions

$$u = e^{i(\alpha x\pm\beta y\pm\omega t)} \qquad \alpha^2 + \beta^2 = k^2 = \frac{\omega^2}{c^2}$$

These have sinusoidal variation in the direction of each space axis as well as time.

(c) Assume a solution of the Helmholtz equation of the form

$$\psi = e^{i\alpha x-\beta y} \qquad \alpha,\ \beta \text{ real constants}$$

and so derive the plane harmonic wave functions

$$u = e^{i(\alpha x\pm\omega t)-\beta y} \qquad \beta^2 = \alpha^2 - k^2 = \alpha^2 - \frac{\omega^2}{c^2}$$

If $\beta > 0$, such a wave function has its amplitude decaying exponentially in the positive y direction.

(d) Derive the wave functions

$$u = e^{\alpha x+\beta y\pm\omega t} \qquad \alpha,\ \beta \text{ real};\ \alpha^2 + \beta^2 = \frac{\omega^2}{c^2}$$

(e) Construct a nontrivial real-valued wave function such that

$$\lim_{x\to\infty} u(x,y,t) = 0 \qquad u(x,0,t) = 0 \qquad u(x,y,0) = 0$$

(f) Construct a nontrivial real-valued wave function such that

$$u(x,y,0) = 0 \quad \text{all } x, y \qquad u(0,y,t) = 0 \quad \text{all } y, t \qquad u(x,0,t) = 0 \quad \text{all } x, t$$

46 (a) Assume that for each choice of (suitably differentiable) function φ, the function u defined by Eq. (4-46) is a solution of the wave equation (4-45). Show that the functions η, ζ must satisfy the system (4-47).

(b) Suppose ζ is defined by Eq. (4-51). Show that the function η must satisfy the wave equation and

$$\mathbf{n}\cdot\nabla\eta \mp \frac{\eta_t}{c} = 0$$

where $\nabla\eta = \eta_x\mathbf{i} + \eta_y\mathbf{j}$ is the gradient of η. This is a linear first-order partial differential equation with constant coefficients, the independent variables being x, y, t. Apply the results of Prob. 18, Chap. 1, and show that η must be of the form

$$\eta = f(x \pm cn_x t, y \pm cn_y t)$$

Let $r = x \pm n_x ct$ and $s = y \pm cn_y t$. Substitute the function η into the wave equation, and show that

$$n_y^2 f_{rr} \pm 2n_x n_y f_{rs} + n_x^2 f_{ss} = 0$$

This is a factorable linear second-order equation. Apply the results of Sec. 2-2, and show that

$$f(r,s) = rF(n_x r \pm n_y s) + G(n_x r \pm n_y s)$$

is the form of f. Thus deduce the general form of solution given in Eq. (4-52).

47 (a) Assume a solution of the wave equation (4-45) of the form

$$u = \eta(x,y)e^{i\zeta(x,y,t)} \qquad i = \sqrt{-1}$$

where η, ζ are real-valued twice continuously differentiable functions. Show that η, ζ must satisfy the system

$$\Delta\eta + \eta\left(\frac{\zeta_t^2}{c^2} - |\nabla\zeta|^2\right) = 0 \qquad 2\nabla\eta\cdot\nabla\zeta + \eta\left(\Delta\zeta - \frac{\zeta_{tt}}{c^2}\right) = 0$$

where ∇ is gradient operator, Δ the laplacian, in two dimensions x, y.

(b) With reference to part **a** assume

$$\zeta = \mathbf{k}\cdot\mathbf{r} - \omega t$$

where $\mathbf{k} = k_x\mathbf{i} + k_y\mathbf{j}$ is an arbitrary real vector and ω is an arbitrary real positive constant. Show η must satisfy the scalar Helmholtz equation

$$\Delta\eta + \left(\frac{\omega^2}{c^2} - k^2\right)\eta = 0 \qquad k = |\mathbf{k}|$$

and also

$$\mathbf{k}\cdot\nabla\eta = 0$$

which is a linear first-order equation with constant coefficients. Call η the *amplitude* and ζ the *phase*. At a fixed time t the lines $\zeta = \text{const}$ in the xy plane define a family of curves of constant phase. The *propagation vector* is $\mathbf{k} = \nabla\zeta$, and $\mathbf{k}$ is normal to each line of constant phase. The curves $\eta = \text{const}$ define the family of curves of constant amplitude of the wave. Thus the curves of constant phase and the curves of constant amplitude are orthogonal families.

(c) From **b** and Sec. 1-4 show η must be of the form

$$\eta = f(k_y x - k_x y)$$

Let $s = k_y x - k_x y$. Substitute η into the Helmholtz equation and obtain

$$f''(s) - \mu^2 f(s) = 0 \qquad \mu^2 = 1 - \frac{\omega^2}{k^2c^2}$$

Thus

$$\eta = e^{\pm\mu(k_y x - k_x y)}$$

(d) Choose a set of values k_x, k_y and a value $\omega > 0$ such that $\omega < kc$. Then (taking the positive root) $0 < \mu < 1$. Now verify that the function

$$u = Ae^{\pm\mu(k_y x - k_x y)}e^{\pm i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad A = \text{const}$$

is a solution of Eq. (4-45). Such a wave function represents a plane harmonic wave with amplitude

$$\eta = Ae^{\pm\mu(k_y x - k_x y)}$$

a function of x and y. The speed of the wave is

$$c' = \frac{\omega}{k} < c$$

For example, if $0 < \omega < c$, the function

$$u = e^{-\mu y}\cos(x - \omega t) \qquad \mu = \frac{(c^2 - \omega^2)^{1/2}}{c}$$

is a wave function and represents a two-dimensional sinusoidal wave profile which travels in a direction parallel to the x axis with speed $c' = \omega < c$. The amplitude $\eta = e^{-\mu y}$ decreases exponentially with increasing y. These examples illustrate *dispersion*: the speed of the wave is a function of the frequency ω.

48 Apply the Poisson-Parseval formula (4-57), and construct a solution of the initial-value problem (4-56) given the functions f and g.

(a) $f(x,y) = 1 \qquad g(x,y) = 0$ **(b)** $f(x,y) = 0 \qquad g(x,y) = 1$

49 In the initial-value problem (4-56) let the given functions f and g be independent of y. Then it is easy to verify directly that D'Alembert's formula (4-15) furnishes the unique solution. Show that the Poisson-Parseval formula (4-57) reduces to (4-15) in this case. *Hint:* Let $\xi = x + r\cos\theta$ and $\eta = y + r\sin\theta$ in the double integrals in (4-57). Then the domain of integration becomes the disk bounded by the circle

$$(\xi - x)^2 + (\eta - y)^2 = c^2t^2$$

Suppose f is independent of y. Then

$$\int_0^{ct}\int_0^{2\pi}\frac{f(x+r\cos\theta)}{\sqrt{c^2t^2-r^2}}\,r\,dr\,d\theta=\int_{x-ct}^{x+ct}f(\xi)\,d\xi\int_{-\mu+y}^{\mu+y}\frac{d\eta}{\sqrt{c^2t^2-(\xi-x)^2-(\eta-y)^2}}$$

where $\mu=\sqrt{c^2t^2-(\xi-x)^2}$. Now carry out the integration with respect to η.

50 **(a)** In plane polar coordinates (r,θ) the homogeneous wave equation (4-44) takes the form

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right)$$

A solution independent of θ is called a *circularly symmetric wave function.* Consider a function

$$u(r,t)=f(r-ct)\qquad r=(x^2+y^2)^{1/2}$$

Such a function represents a circularly symmetric traveling wave which is propagated radially outward from the origin with speed c. Show that no solution of the wave equation of this form exists (apart from the trivial case $f=$ const).

(b) Show that there does not exist a nontrivial amplitude function $\eta(r,t)$ such that

$$u=\eta(r,t)f(r-ct)$$

is a solution of the homogeneous wave epuation for arbitrary choice of f.

(c) A solution of the homogeneous wave equation in polar coordinates of the form

$$u=R(r)e^{-i\omega t}\qquad \omega \text{ real and positive; } i=\sqrt{-1}$$

is called *harmonic time-dependent.* On each circle $r=$ const the variation of u is sinusoidal with time t. Substitute into the wave equation, and show the radial dependent factor R must satisfy

$$rR''+R'+\frac{\omega^2 rR}{c^2}=0$$

Bessel's equation of order zero. Thus derive the circularly symmetric wave functions

$$u=\left[AJ_0\left(\frac{\omega r}{c}\right)+BY_0\left(\frac{\omega r}{c}\right)\right]e^{-i\omega t}$$

where A, B are arbitrary real constants, and J_0, Y_0 denote the Bessel functions of the first and second kind, respectively, of order zero. Recall that Y_0 has a logarithmic singularity at $r=0$. Hence if the region in the xy plane in which solutions are desired includes the origin, the choice $B=0$ is necessary.

51 The homogeneous *two-dimensional damped-wave equation* is

$$u_{tt}+2\gamma u_t-c^2\,\Delta u=0$$

where γ is a real positive constant and Δu denotes the two-dimensional laplacian of u. Assume a solution of the form

$$u=e^{-\gamma t}v$$

and show that v must satisfy

$$v_{tt}-\gamma^2 v-c^2\,\Delta v=0$$

Assume a solution of this equation of the form

$$v = \psi(x,y)e^{-i\omega t} \qquad \omega \text{ real and positive; } i = \sqrt{-1}$$

Show that ψ must satisfy the scalar Helmholtz equation

$$\Delta\psi + \mu^2\psi = 0 \qquad \mu^2 = \frac{\omega^2 + \gamma^2}{c^2}$$

Let

$$\mathbf{k} = k_x\mathbf{i} + k_y\mathbf{j}$$

be a vector in the xy plane such that

$$|\mathbf{k}| = \mu = \frac{(\omega^2 + \gamma^2)^{1/2}}{c}$$

Show that

$$\psi = e^{i\mathbf{k}\cdot\mathbf{r}} \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

satisfies the scalar Helmholtz equation. Thus derive the solutions

$$u = e^{-\gamma t}e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

of the damped-wave equation. These represent two-dimensional damped traveling waves which move with speed $c' = \omega/\mu < c$ in the direction of the vector **k**.

Sec. 4-5

52 **(a)** Consider problem (4-79) for the freely vibrating membrane with fastened edges when the boundary C is a rectangle. Assume the membrane at rest occupies the domain defined by

$$0 \le x \le a \qquad 0 \le y \le b$$

The boundary conditions can be written as

$$u(x,0,t) = 0 \qquad u(x,b,t) = 0 \qquad 0 \le x \le a;\ t \ge 0$$
$$u(0,y,t) = 0 \qquad u(a,y,t) = 0 \qquad 0 \le y \le b;\ t \ge 0$$

If a separable solution of the form $u = \varphi(x,y)T(t)$ is assumed, the boundary conditions on u imply that the space-dependent factor φ must satisfy the boundary conditions

$$\varphi(x,0) = 0 \qquad \varphi(x,b) = 0 \qquad 0 \le x \le a$$
$$\varphi(0,y) = 0 \qquad \varphi(a,y) = 0 \qquad 0 \le y \le b$$

These conditions together with the Helmholtz equation (4-64), written in rectangular coordinates, constitute an eigenvalue problem for the operator Δ. This eigenvalue problem was solved in Example 3-5 in connection with finding an eigenfunction expansion for the solution of Poisson's equation on the rectangle with the same boundary conditions. Review Example 3-5, and verify that the eigenfunctions of the problem are

$$\varphi_{nm}(x,y) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \qquad n = 1, 2, \ldots;\ m = 1, 2, \ldots$$

corresponding to the eigenvalues

$$\lambda_{nm} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \qquad n = 1, 2, \ldots;\ m = 1, 2, \ldots$$

The eigenfunctions are orthogonal over the domain

$$\int_0^a \int_0^b \varphi_{nm}(x,y)\varphi_{pq}(x,y)\,dx\,dy = 0 \qquad (n,m) \neq (p,q)$$

These eigenfunctions are not normalized, since

$$\int_0^a \int_0^b \varphi_{nm}{}^2(x,y)\,dx\,dy = \frac{ab}{4} \qquad n = 1, 2, \ldots ; m = 1, 2, \ldots$$

The time-dependent factor corresponding to φ_{nm} is

$$T_{nm}(t) = A_{nm} \cos \omega_{nm} t + B_{nm} \sin \omega_{nm} t$$

where the ω_{nm} are the characteristic frequencies:

$$\omega_{nm} = c\sqrt{\lambda_{nm}} = c\pi\left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]^{1/2} \qquad n = 1, 2, \ldots ; m = 1, 2, \ldots$$

Observe that in contrast with the vibrating string the vibrating membrane possesses a doubly infinite sequence of characteristic frequencies. The normal modes of vibration are

$$u_{nm}(x,y,t) = (A_{nm} \cos \omega_{nm} t + B_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

(b) In order to satisfy the initial conditions consider a superposition of normal modes

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (A_{nm} \cos \omega_{nm} t + B_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

If the series converges suitably, it is clear that the function u so defined satisfies the boundary conditions. The initial conditions are satisfied if

$$u(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} = f(x,y)$$

$$u_t(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{nm} B_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} = g(x,y)$$

$0 \leq x \leq a$, $0 \leq y \leq b$. The series on the left must be the double sine series for f and g respectively. It follows that the coefficients A_{nm}, B_{nm} are given by

$$A_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}\,dx\,dy$$

$$B_{nm} = \frac{4}{ab\omega_{nm}} \int_0^a \int_0^b g(x,y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}\,dx\,dy$$

(c) Assume the membrane is released from rest with the initial displacement

$$u(x,y,0) = Axy(a - x)(b - y) \qquad A = \text{const}$$

Determine the formal series expression for the resulting motion.

(d) The membrane is released from rest with the initial displacement

$$u(x,y,0) = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \qquad A = \text{const}$$

Determine the subsequent motion. What is the speed of the midpoint at time t?

53 **(a)** The rectangular membrane with fastened edges executes forced vibrations under the driving force

$$F(x,y,t) = F_0(x,y) \sin \omega t$$

Assume that $\omega \neq \omega_{nm}$, all n, m, where the ω_{nm} are the characteristic frequencies of the rectangular membrane derived in Prob. 52. The motion starts from rest with zero displacement. Utilize the eigenfunctions obtained in Prob. 52 and Eq. (4-78) to derive the expression for the subsequent motion in the form

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \frac{\omega \sin \omega_{nm} t - \omega_{nm} \sin \omega t}{\omega_{nm}(\omega^2 - \omega_{nm}{}^2)} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

where

$$B_{nm} = \frac{4}{ab} \int_0^a \int_0^b F_0(x,y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \, dx \, dy$$

(b) In part **a** it was assumed that the frequency of the driving force does not coincide with a characteristic frequency. Derive the series solution if $\omega = \omega_{pq}$, p, q a given fixed pair of positive integers. This is the case of *resonance*.

(c) Derive a formal series expression for the motion of the rectangular membrane if the initial conditions are

$$u(x,y,0) = \left(1 - \cos \frac{2\pi x}{a}\right)\left(1 - \cos \frac{2\pi y}{b}\right) \qquad u_t(x,y,0) = 0$$

and the external force of gravity acts on the membrane.

54 **(a)** Recall the properties of the unit impulse function given in Sec. 3-5. A concentrated impulsive force per unit mass and of unit magnitude applied at time $t = \tau$ at the point (ξ,η) on the rectangular membrane can be represented

$$F(x,y,t) = \delta(x - \xi)\delta(y - \eta)\delta(t - \tau)$$

Thus

$$F(x,y,t) = 0 \qquad (x,y,t) \neq (\xi,\eta,\tau)$$

and

$$\int_0^t \int_0^a \int_0^b F(x,y,t) \, dx \, dy \, dt = 1 \qquad t > \tau$$

Let the edge be fastened, and assume zero initial conditions. Use Eq. (4-78), and derive the formal series

$$G(x,y,t;\xi,\eta,\tau) = \frac{4}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\omega_{nm}} \sin \frac{n\pi x}{a} \sin \frac{n\pi \xi}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi \eta}{b} \sin [\omega_{nm}(t - \tau)]$$

for the subsequent displacement. Here the ω_{nm} are the characteristic frequencies of the freely vibrating rectangular membrane. It can be shown that the series converges uniformly on the rectangle and defines a continuous function. The function G is called the *Green's function* of the problem. In order to simulate the complete motion the function G is defined to be zero for $t \leq \tau$. Note that G satisfies the boundary condition and has the symmetry

$$G(\xi,\eta,t;x,y,\tau) = G(x,y,t;\xi,\eta,\tau)$$

(b) For the rectangle $0 \le x \le a$, $0 \le y \le b$ the solution (4-78) of problem (4-80) is

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{1}{\omega_{nm}} \int_0^t F_{nm}(\tau) \sin [\omega_{nm}(t - \tau)]\, d\tau \right\} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

where

$$F_{nm}(\tau) = \frac{4}{ab} \int_0^a \int_0^b F(x,y,\tau) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}\, dx\, dy$$

Use formal interchange of the operations of summation and integration, and show the solution can be rewritten

$$u(x,y,t) = \int_0^t \int_0^a \int_0^b G(x,y,t;\xi,\eta,\tau) F(\xi,\eta,\tau)\, d\xi\, d\eta\, d\tau$$

(c) The rectangular membrane with fastened edges is at rest in its equilibrium configuration. At time $t = 0$ the membrane is struck in such a manner that the point (ξ,η) receives a velocity impulse of unit magnitude. Thus the initial conditions are

$$u(x,y,0) = 0 \qquad u_t(x,y,0) = \delta(x - \xi)\delta(y - \eta)$$

Apply the formulas for the coefficients A_{nm}, B_{nm} derived in Prob 52**b**, together with the properties of the δ function, and show that the subsequent motion is given by

$$u(x,y,t) = G(x,y,t;\xi,\eta,0)$$

(d) Verify that

$$u(x,y,t) = \int_0^a \int_0^b G_t(x,y,t;\xi,\eta,0) f(\xi,\eta)\, d\xi\, d\eta + \int_0^a \int_0^b G(x,y,t;\xi,\eta,0) g(\xi,\eta)\, d\xi\, d\eta$$

furnishes the solution of problem (4-79) for the case of the fastened rectangle by substitution of the series for G and G_t evaluated at $\tau = 0$, formal interchange of the operations of summation and integration, and comparison with the series solution derived in Prob. 52**b.**

55 If the edge of the membrane is elastically constrained, the boundary condition is

$$\frac{\partial u}{\partial n} + \sigma u = 0 \qquad \text{on } C$$

where $\partial/\partial n$ denotes the derivative in the direction of the exterior normal to C and σ is a positive constant. Let the initial conditions be $u(x,y,0) = f(x,y)$ and $u_t(x,y,0) = g(x,y)$. Assume that a known external force $F(x,y,t)$ per unit mass acts on the membrane. Derive the expression for the motion. Also derive the series expression for the Green's function of the problem.

56 **(a)** If the membrane vibrates in a medium, a frictional force occurs. In the case where the retarding force is proportional to the speed, the equation of motion is

$$u_{tt} + 2\gamma u_t - c^2 \Delta u = F$$

the *two-dimensional damped-wave equation*. Here γ is a positive constant, $c = \sqrt{T_0/\rho}$, and F is an external force per unit mass. It was observed in Prob. 51 that solutions of the corresponding homogeneous equation may represent two-dimensional traveling waves whose amplitudes decrease with increasing time t. Let the membrane be rectangular:

$0 \le x \le a$, $0 \le y \le b$, with fastened edges along the boundary C. Derive the normal modes

$$u_{nm} = e^{-\gamma t}(A'_{nm} \cos \nu_{nm} t + B'_{nm} \sin \nu_{nm} t) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

and the characteristic frequencies

$$\nu_{nm} = (\omega_{nm}{}^2 - \gamma^2)^{1/2} \qquad n = 1, 2, \ldots; m = 1, 2, \ldots$$

where the ω_{nm} are the characteristic frequencies of the undamped rectangular membrane with fastened edges (see Prob. 52**a**) and the A'_{nm}, B'_{nm} are arbitrary constants. Observe that the normal modes are time harmonic if the damping constant $\gamma < \omega_{11}$. Assume henceforth that this is the case.

(b) Let the initial conditions be $u(x,y,0) = f(x,y)$ and $u_t(x,y,0) = g(x,y)$. Derive the series solution

$$u = e^{-\gamma t} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (A'_{nm} \cos \nu_{nm} t + B'_{nm} \sin \nu_{nm} t) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

where $A'_{nm} = A_{nm}$ and $B'_{nm} = \gamma A_{nm}/\nu_{nm} + B_{nm}$, the coefficients A_{nm}, B_{nm} being those defined in Prob. 52**b**, except that ν_{nm} replaces ω_{nm} in the expression for B_{nm}.

(c) Consider the problem

$$u_{tt} + 2\gamma u_t - c^2 \Delta u = F(x,y,t)$$

$$u = 0 \qquad \text{on } C$$

$$u(x,y,0) = 0 \qquad u_t(x,y,0) = 0 \qquad (x,y) \text{ in } \bar{\mathscr{R}}$$

Assume a solution

$$u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \psi_{nm}(t) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

Follow the method of the text, and derive the solution

$$u(x,y,t) = e^{-\gamma t} v(x,y,t)$$

$$v(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{1}{\nu_{nm}} \int_0^t e^{\gamma t} F_{nm}(\tau) \sin [\nu_{nm}(t - \tau)\, d\tau \right\} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

where $F_{nm}(t)$ is defined in Prob. 54**b**.

(d) Derive the *Green's function* of the problem as the series

$$G(x,y,t;\xi,\eta,\tau) = \frac{4}{ab} e^{-\gamma(t-\tau)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\nu_{nm}} \sin \frac{n\pi x}{a} \sin \frac{n\pi \xi}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi \eta}{b} \sin [\nu_{nm}(t - \tau)]$$

57 Derive the series expression for the vibrations of a circular membrane of radius a with fastened edge if the external force per unit mass

$$F(r,t) = \varphi(r) \sin \omega t$$

acts on the membrane. Here $\varphi(r)$ is a given continuous function. The membrane is released from rest with zero displacement. Distinguish two cases: **(i)** $\omega \ne \omega_{nm}$, all n, m; **(ii)** $\omega = \omega_{pq}$ for a given pair of integers $p, q, p \ge 0, q \ge 1$ (case of resonance). Here the ω_{nm} are the characteristic frequencies derived in Example 4-2. Write the solution for the particular case $\varphi(r) = F_0 = \text{const}$.

58 The circular membrane described in Example 4-2 is released from rest with the initial displacement

$$u(r,\theta,0) = A(a^2 - r^2) \qquad A = \text{const}$$

Instead of being fastened, the edge is elastically constrained. Thus the boundary condition

$$\frac{\partial u}{\partial r} + \sigma u = 0 \qquad \text{on } r = a$$

replaces the fixed-edge condition. Here σ is a real positive constant. Assume the external force of gravity acts on the membrane. Derive the series expression for the vibrations of the membrane.

59 Assume no external force acts on the surface of the membrane described in Example 4-2. However, by external means the edge of the membrane is forced to vibrate at a fixed frequency ω, so that the boundary condition

$$u(a,\theta,t) = \varphi(\theta) \sin \omega t \qquad 0 \le \theta \le 2\pi;\ t \ge 0$$

replaces the fixed-edge condition. Here $\varphi(\theta)$ is a given piecewise smooth function which is periodic, of period 2π. The motion starts with zero displacement. Derive the expression for the subsequent vibrations.

60 A thin elastic circular membrane of radius a has its edge fastened. An external force of magnitude $F_0 \sin \omega t$ (F_0, ω positive constants), confined to a concentric circle of radius $b < a$, is applied to the membrane for $t \ge 0$. The membrane starts from rest and with zero displacement. Determine the series expression for the subsequent motion.

61 For the circular membrane described in Example 4-2 assume a frictional force proportional to the speed. Then the equation of motion is

$$u_{tt} + 2\gamma u_t - c^2\left(u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}\right) = F(r,\theta,t)$$

where γ is a given positive constant and F is a known external force per unit mass. Assume the edge of the membrane is fastened. Give a formal derivation of the solution of the boundary- and initial-value problem. Derive the series representation of the Green's function of the problem. Find the motion for the particular case where the membrane is released from rest with zero displacement and

$$F(r,\theta,t) = F_0 \sin \omega t \qquad F_0,\ \omega \text{ real positive constants}$$

62 **(a)** Let β be a fixed angle, $0 < \beta < 2\pi$, and let $\mathscr{R}$ be the sector whose boundary consists of the ray segments

$$\theta = 0 \quad 0 \le r \le a \qquad\qquad \theta = \beta \quad 0 \le r \le a$$

together with the circular arc

$$r = a \qquad 0 \le \theta \le \beta$$

Here r, θ are polar coordinates in the plane. Assume an elastic membrane occupies $\mathscr{R}$ in its equilibrium configuration and the edge of the membrane is fastened along the boundary C of $\mathscr{R}$. The membrane is given an initial displacement and speed

$$u(r,\theta,0) = f(r,\theta) \qquad u_t(r,\theta,0) = g(r,\theta)$$

A known exterior force $F(r,\theta,t)$ per unit mass acts on the membrane. Derive a formal series expression for the subsequent motion. Also derive the series representation of the Green's function of the problem.

(b) The circular membrane described in Example 4-2 is fastened along the ray segment

$$\theta = 0 \qquad 0 \le r \le a$$

as well as on the circle $r = a$. Use the results of **a** to obtain a series expression for the subsequent motion of the membrane.

63 Let $\mathcal{R}$ be the region bounded by the concentric circles $r = a$, $r = b$, where $0 < a < b$. An elastic membrane occupies $\mathcal{R}$ in its equilibrium configuration. The edge of the membrane is fastened along the circles $r = a$, $r = b$. Give a formal discussion of the problem. Derive the series representation of the Green's function of the problem. Find the expression for the subsequent motion if the membrane is released from rest with zero displacement and the external force per unit mass $F(r,\theta,t) = F_0 \sin \omega t$ acts on the membrane, where F_0, ω are given positive constants.

64 (a) The kinetic energy of an element dA of the vibrating membrane is $(\rho u_t^2\, dA)/2$, where ρ is the density (mass/unit area). Hence the total kinetic energy of motion is

$$K = \frac{\rho}{2} \iint_{\mathcal{R}} u_t^2\, dA$$

To obtain an expression for the potential energy consider an element dA when the membrane lies in its equilibrium configuration. Let dA' be the area of the corresponding element at a subsequent time $t > 0$. The change in area is

$$dA' - dA = (1 + |\nabla u|^2)^{1/2}\, dA - dA \approx |\nabla u|^2 \frac{dA}{2}$$

This deformation occurs in the presence of an elastic restoring force of magnitude T_0. Thus the potential energy due to tension is

$$T_0\, |\nabla u|^2 \frac{dA}{2}$$

The total potential energy is

$$V = \frac{T_0}{2} \iint_{\mathcal{R}} |\nabla u|^2\, dA$$

The total energy is

$$E = K + V = \frac{1}{2} \iint_{\mathcal{R}} (\rho u_t^2 + T_0\, |\nabla u|^2)\, dA$$

(b) Show that in a traveling wave

$$u = f(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

where

$$\mathbf{k} = k_1\mathbf{i} + k_2\mathbf{j} \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

and $\omega = |\mathbf{k}|\, c$, the kinetic energy equals the potential energy.

(c) The nth normal mode of vibration for the membrane with fastened edge is

$$u_n = \varphi_n(x,y)T_n(t) = \varphi_n(x,y)(A_n \cos \omega_n t + B_n \sin \omega_n t)$$
$$= C_n \cos (\omega_n t - \epsilon_n)$$

where

$$C_n = (A_n{}^2 + B_n{}^2)^{1/2} \qquad \tan \epsilon_n = B_n A_n$$

The space factor φ_n satisfies

$$\Delta\varphi_n + \lambda_n\varphi_n = 0 \quad \text{in } \bar{\mathscr{R}} \qquad \varphi_n = 0 \quad \text{on } C$$

The nth characteristic frequency is $\omega_n = c\sqrt{\lambda_n}$. Show that the energy in the nth normal mode is

$$E_n = \frac{\rho\omega_n{}^2 C_n{}^2 \, \|\varphi_n\|^2}{2}$$

a constant proportional to the square of the characteristic frequency and proportional to the square of the amplitude C_n.

(d) If the edge of membrane is fastened, the initial conditions are

$$u(x,y,0) = f(x,y) \qquad u_t(x,y,0) = g(x,y)$$

and no external forces act on the membrane, then the total energy of motion is the constant

$$E = \frac{\rho}{2}\iint_{\mathscr{R}} \{[g(x,y)]^2 + c^2\,|\nabla f|^2\}\, dA$$

Use this result and the result in **c** above to show that the total energy of motion is the sum of the energies of the individual modes

$$E = \sum_{n=1}^{\infty} E_n = \frac{\rho}{2}\sum_{n=1}^{\infty} \omega_n{}^2 C_n{}^2 \, \|\varphi_n\|^2$$

Hint: For **c**,

$$E_n = \frac{\rho}{2}\iint_{\mathscr{R}} (\dot{T}_n{}^2\varphi_n{}^2 + c^2 T_n{}^2\,|\nabla\varphi_n|^2)\, dA$$

Apply Green's formula derived in Prob. 12**a**, Chap. 3, and show

$$\iint_{\mathscr{R}} |\nabla\varphi_n|^2\, dA = \lambda_n \iint \varphi_n{}^2\, dA$$

so that $E_n = \rho(\dot{T}_n{}^2 + \omega_n{}^2 T_n{}^2)\, \|\varphi_n\|^2/2$.

65 Prove the following existence theorem for problem (4-79) when $\bar{\mathscr{R}}$ is the rectangle $a \leq x \leq b$, $c \leq y \leq d$. Let the given functions f and g be four times continuously differentiable and such that

$$f = \Delta f = g = \Delta g = 0 \qquad \text{on the boundary } C$$

Then the formal series solution derived in Prob. 52**b** converges uniformly on $\bar{\mathscr{R}}$ and defines a twice continuously differentiable function u which satisfies the homogeneous wave equation and the boundary and initial conditions. *Hint:* Recall

$$\iint_{\mathscr{R}} \psi \, \Delta\varphi \, dx \, dy = \iint_{\mathscr{R}} \varphi \, \Delta\psi \, dx \, dy$$

holds for every pair of twice continuously differentiable functions on $\bar{\mathscr{R}}$ which vanish on the boundary C. Let φ_{nm} be the eigenfunction derived in Prob. 52**a**. From Prob. 52**b**, the fact that $\Delta\varphi_{nm} = -\lambda_{nm}\varphi_{nm}$, and the above, it follows that

$$A_{nm} = \frac{4}{ab} \iint_{\mathscr{R}} f\varphi_{nm} \, dx \, dy = -\frac{4}{ab\lambda_{nm}} \iint_{\mathscr{R}} f \, \Delta\varphi_{nm} \, dx \, dy$$

$$= -\frac{4}{ab\lambda_{nm}} \iint_{\mathscr{R}} \varphi_{nm} \, \Delta f \, dx \, dy = \frac{4}{ab\lambda_{nm}{}^2} \iint_{\mathscr{R}} \Delta\varphi_{nm} \, \Delta f \, dx \, dy$$

$$= \frac{4}{ab\lambda_{nm}{}^2} \iint_{\mathscr{R}} \varphi_{nm} \, \Delta\Delta f \, dx \, dy$$

Hence

$$|A_{nm}| \le \frac{4}{ab\lambda_{nm}{}^2} \iint_{\mathscr{R}} |\varphi_{nm}| \, |\Delta\Delta f| \, dx \, dy \le \frac{M}{\lambda_{nm}{}^2}$$

where M is a positive constant independent of n and m. Similarly $|B_{nm}| \le M/\omega_{nm}\lambda_{nm}{}^2 \le M/\lambda_{nm}{}^2$. Thus

$$|(A_{nm} \cos \omega_{nm}t + B_{nm} \sin \omega_{nm}t)\varphi_{nm}(x,y)| \le |A_{nm}| + |B_{nm}| \le \frac{2M}{\lambda_{nm}{}^2}$$

for all x, y, t. A dominant convergent series is

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{2M}{\lambda_{nm}{}^2} \qquad \lambda_{nm} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

Accordingly the series of Prob. 52**b** converges uniformly on $\bar{\mathscr{R}}$. Now show that the series obtained by differentiating termwise twice with respect to t converges uniformly and the series obtained by differentiating termwise twice with respect to x, and also with respect to y, converges uniformly on $\bar{\mathscr{R}}$. Hence show u satisfies the homogeneous wave equation. Since the series for u converges on $\bar{\mathscr{R}}$ for $t \ge 0$, it is clear that u satisfies the boundary conditions. To show u satisfies the initial conditions one can apply theorems on double Fourier series analogous to those stated in Sec. 2 of Appendix 2. Alternatively the completeness of the set $\{\varphi_{nm}(x,y)\}$ in the space of all functions continuous on $\bar{\mathscr{R}}$ and vanishing on the boundary implies the satisfaction of the initial conditions. Since the series converges uniformly in x and y at $t = 0$,

$$u(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm}\varphi_{nm}(x,y)$$

Let p, q be a fixed but otherwise arbitrarily chosen pair of positive integers. Multiply both sides of the above equation by φ_{pq}, integrate over $\bar{\mathscr{R}}$, interchange the order of summation and integration, and apply the orthogonality properties of the sequence $\{\varphi_{nm}\}$. The result is

$$A_{pq} = \frac{4}{ab} \iint_{\mathscr{R}} u(x,y,0)\varphi_{pq}(x,y)\, dx\, dy$$

But the coefficients A_{nm} are the Fourier coefficients of f with respect to the φ_{nm}. Hence

$$\iint_{\mathscr{R}} [u(x,y,0) - f(x,y)]\varphi_{nm}(x,y)\, dx\, dy = 0$$

for all n, m. Since f vanishes on the boundary of $\mathscr{R}$, so does the difference $u(x,y,0) - f(x,y)$. Hence the completeness property implies

$$u(x,y,0) = f(x,y) \qquad a \le x \le b;\ c \le y \le d$$

The remaining initial condition is shown in the same way.

66 Prove the following existence theorem for problem (4-80) when $\overline{\mathscr{R}}$ is the rectangle $a \le x \le b$, $c \le y \le d$. Let the given function F be four times continuously differentiable with respect to the independent variables and such that $F = \Delta F = 0$ on the boundary for $t \ge 0$. Then the formal series solution written in Prob. 54**b** converges uniformly on $\overline{\mathscr{R}}$ and defines a twice continuously differentiable function u which satisfies the inhomogeneous wave equation and the homogeneous boundary and initial conditions.

Sec. 4-6

67 Let the source function in problem (4-104) be a *point source*, of *strength* $F_0(t)$, located at the fixed point (x_0, y_0, z_0):

$$F(x,y,z,t) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)F_0(t)$$

Then

$$F(x,y,z,t) = 0 \qquad (x,y,z) \ne (x_0,y_0,z_0)$$

and

$$\iiint_V F(x,y,z,t)\, dx\, dy\, dz = F_0(t)$$

whenever V includes the point (x_0,y_0,z_0). Use Eq. (4-107) to show that the resulting field at a point $P(x,y,z)$ distinct from (x_0,y_0,z_0) is

$$u(x,y,z,t) = \begin{cases} 0 & 0 \le ct < r \\ \dfrac{F_0(t - r/c)}{4\pi c^2 r} & ct \ge r \end{cases}$$

where $r^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$. Observe that u defines a spherically symmetric wave. Accordingly, the field due to a time-harmonic point source

$$F(x,y,z,t) = F_0\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)e^{-i\omega t}$$

which vanishes, together with its first derivative with respect to t, at $t = 0$ is given by

$$u(x,y,z,t) = \begin{cases} 0 & 0 \le ct < r \\ \dfrac{F_0 e^{-i\omega(t-r/c)}}{4\pi c^2 r} & ct \ge r \end{cases}$$

Show that the field due to an instantaneous point source located at (x_0,y_0,z_0) at time t_0, of strength F_0, is given by

$$u(x,y,z,t) = \begin{cases} 0 & 0 \leq ct < r \\ \dfrac{F_0\delta[(t-t_0) - r/c]}{4\pi c^2 r} & ct \geq r \end{cases}$$

68 Let the source function in the two-dimensional problem (4-58) be a point source of strength $F_0(t)$ located at (x_0,y_0):

$$F(x,y,t) = \delta(x - x_0)\delta(y - y_0)F_0(t)$$

Apply Eq. (4-59) and show that the resulting field at a point $P(x,y)$ distinct from (x_0,y_0) is given by

$$u(x,y,t) = \begin{cases} 0 & 0 \leq ct < r \\ \dfrac{1}{2\pi c}\displaystyle\int_0^{t-r/c} \dfrac{F_0(\tau)}{\sqrt{c^2(t-\tau)^2 - r^2}}\,d\tau & ct \geq r \end{cases}$$

where $r^2 = (x - x_0)^2 + (y - y_0)^2$. Show that the field due to an instantaneous point source located at (x_0,y_0) at time t_0, of strength F_0 = const, is given by

$$u(x,y,t) = \begin{cases} 0 & 0 \leq c(t-t_0) < r \\ \dfrac{F_0}{2\pi c\sqrt{c^2(t-t_0)^2 - r^2}} & c(t-t_0) \geq r \end{cases}$$

69 **(a)** Let $r^2 = x^2 + y^2 + z^2$. Find a solution of the homogeneous wave equation for $r > 0$ such that

$$u|_{t=0} = \frac{A}{r} \qquad u_t|_{t=0} = B \qquad A, B \text{ real positive constants}$$

(b) Find a solution of the homogeneous wave equation for $r > 0$ such that

$$u|_{t=0} = u_0 \qquad u_t|_{t=0} = \frac{u_1}{r} \qquad u_0, u_1 \text{ real positive constants}$$

70 In spherical coordinates the homogeneous wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}\right]$$

As discussed in the text, a function of the form

$$u = \psi(r - ct)$$

represents a spherical wave propagated radially outward from the origin with speed c. Show that no nontrivial function of this form can be a solution of the homogeneous wave equation. On the other hand, a function

$$u = \eta(r,t)f(r - ct)$$

represents a spherical wave traveling outward from the origin but with amplitude $\eta(r,t)$ a function of position and time. Assume a solution of this form with f arbitrary (twice differentiable). Show that necessarily

$$\eta(r,t) = \frac{\psi(r - ct)}{r}$$

for some function ψ. Thus for each choice of ψ the pair of functions

$$\eta = \frac{\psi(r - ct)}{r} \qquad \zeta = r - ct$$

constitutes a functionally invariant pair of the wave equation in spherical coordinates. *Hint:* Show that η must satisfy

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{c^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \eta}{\partial r}\right) \qquad c\frac{\partial}{\partial r}(r\eta) + \frac{\partial \eta}{\partial t} = 0$$

Let $v = r$.

71 Derive the solution (4-59) of the two-dimensional initial-value problem (4-58) from Eq. (4-107).

Sec. 4-7

72 **(a)** Consider problem (4-114) when $\mathscr{V}$ is the rectangular parallelepiped defined by the inequalities

$$0 \leq x \leq a \qquad 0 \leq y \leq b \qquad 0 \leq z \leq h$$

Let the boundary condition be the Dirichlet condition

$$u = 0 \qquad \text{on } S;\ t \geq 0$$

where S is the surface which bounds $\mathscr{V}$. Use separation of variables to derive the ∞^3 eigenfunctions

$$\psi_{nmq}(x,y,z) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{q\pi z}{h} \qquad n = 1, 2, \ldots;\ m = 1, 2, \ldots;\ q = 1, 2, \ldots$$

corresponding to the eigenvalues

$$\lambda_{nmq} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} + \frac{q^2\pi^2}{h^2}$$

Show directly that the eigenfunctions have the properties

$$\int_0^a\int_0^b\int_0^h \psi_{nmq}\psi_{n'm'q'}\,dx\,dy\,dz = 0 \qquad (n,m,q) \neq (n',m',q')$$

$$\|\psi_{nmq}\|^2 = \frac{abh}{8}$$

Outline the proof of the fact that (apart from constant factors) the set $\{\psi_{nmq}\}$ constitutes all the eigenfunctions of problem (4-117) in the present case.

(b) Obtain the normal modes of vibration

$$u_{nmq}(x,y,z,t) = \psi_{nmq}(x,y,z)(A_{nmq}\cos\omega_{nmq}t + B_{nmq}\sin\omega_{nmq}t)$$

where the characteristic frequencies

$$\omega_{nmq} = c\pi\left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2 + \left(\frac{q}{h}\right)^2\right]^{1/2}$$

(c) Derive the solution of problem (4-114) when F is identically zero in the form

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{q=1}^{\infty}(A_{nmq}\cos\omega_{nmq}t + B_{nmq}\sin\omega_{nmq}t)\psi_{nmq}(x,y,z)$$

$$A_{nmq} = \frac{8}{abh}\int_0^a\int_0^b\int_0^h f(x,y,z)\psi_{nmq}(x,y,z)\,dx\,dy\,dz$$

$$B_{nmq} = \frac{8}{abh\omega_{nmq}}\int_0^a\int_0^b\int_0^h g(x,y,z)\psi_{nmq}(x,y,z)\,dx\,dy\,dz$$

(d) Derive the solution of problem (4-114) when f and g are identically zero in the form

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{q=1}^{\infty}\left\{\frac{1}{\omega_{nmq}}\int_0^t F_{nmq}(\tau)\sin[\omega_{nmq}(t-\tau)]\,d\tau\right\}\psi_{nmq}(x,y,z)$$

where

$$F_{nmq}(t) = \frac{8}{abh}\int_0^a\int_0^b\int_0^h F(x,y,z,t)\psi_{nmq}(x,y,z)\,dx\,dy\,dz$$

(e) Derive the *Green's function* of the problem

$$G(x,y,z,t;\xi,\eta,\zeta,\tau) = \frac{8}{abh}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{q=1}^{\infty}\frac{\psi_{nmq}(x,y,z)\psi_{nmq}(\xi,\eta,\zeta)}{\omega_{nmq}}\sin[\omega_{nmq}(t-\tau)]$$

for $t > \tau$. Define G to be identically zero for $t \le \tau$.

(f) Derive the solution of the problem

$$u_{tt} - c^2\,\Delta u = A\sin\omega t \qquad \text{in } \bar{\mathscr{V}};\ t \ge 0$$

$$u = B\sin\nu t \qquad \text{on } S;\ t \ge 0$$

$$u(x,y,z,0) = 0 \qquad u_t(x,y,z,0) = B\nu \qquad \text{in } \bar{\mathscr{V}}$$

where A, B, ω, and ν are given real positive constants and $\omega \ne \omega_{nmq}$, $\nu \ne \omega_{nmq}$, all n, m, q.

73 Let $\bar{\mathscr{V}}$ be the rectangular parallepiped of Prob. 72. Derive the formal series solution of the problem

$$u_{tt} + 2\gamma u_t + \beta u - c^2\,\Delta u = F_0\sin\omega t$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } S;\ t \ge 0$$

$$u(x,y,z,0) = \cos\frac{\pi x}{a}\cos\frac{\pi y}{b}\cos\frac{\pi z}{h} \qquad u_t(x,y,z,0) = 0$$

where F_0, ω, and γ are given real positive constants and β is a real nonnegative constant. Assume also

$$\gamma^2 < \omega^2 + \pi^2 c^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{h^2}\right)$$

74 In the study of small amplitude vibrations of an ideal gas confined to the interior of a rigid spherical surface of radius a about the origin it is shown that the *velocity potential* u (a function such that the velocity vector $\mathbf{v} = -\nabla u$) satisfies

$$u_{tt} = c^2 \, \Delta u \qquad \text{in } \bar{\mathscr{V}};\ t \geq 0$$

$$\frac{\partial u}{\partial r} = 0 \qquad r = a \qquad t \geq 0$$

$$u(r,0) = f(r) \qquad u_t(r,0) = g(r) \qquad \text{in } \bar{\mathscr{V}}$$

where (r,θ,φ) are spherical coordinates and $\bar{\mathscr{V}}$ is the sphere of radius a about the origin. Derive the formal series solution

$$u(r,t) = A_0 + B_0 t + \frac{1}{r} \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{\mu_n r}{a}$$

where $\{\mu_n\}$ is the sequence of real positive roots of the transcendental equation

$$\tan \mu = \mu$$

and

$$\omega_n = \frac{c\mu_n}{a} \qquad n = 1, 2, \ldots$$

are the characteristic frequencies of vibration. The coefficients in the series are determined by

$$A_0 = \frac{3}{a^3} \int_0^a r^2 f(r) \, dr \qquad B_0 = \frac{3}{a^3} \int_0^a r^2 g(r) \, dr$$

$$A_n = \frac{2(1 + 1/\mu_n^2)}{a} \int_0^a r f(r) \sin \frac{\mu_n r}{a} \, dr \qquad n \geq 1$$

$$B_n = \frac{2(1 + 1/\mu_n^2)}{a\mu_n} \int_0^a r g(r) \sin \frac{\mu_n r}{a} \, dr \qquad n \geq 1$$

75 Let r, θ, φ be spherical coordinates and let $\mathscr{V}$ denote the hemisphere defined by the inequalities

$$0 \leq r \leq a \qquad 0 \leq \theta \leq \frac{\pi}{2} \qquad 0 \leq \varphi \leq 2\pi$$

Derive the *Green's function* of the problem

$$u_{tt} - c^2 \, \Delta u = F(r,\theta,t) \qquad \text{in } \bar{\mathscr{V}};\ t \geq 0$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } S;\ t \geq 0$$

$$u(r,\theta,0) = f(r,\theta) \qquad u_t(r,\theta,0) = g(r,\theta) \qquad \text{in } \mathscr{V}$$

where S denotes the surface which bounds $\bar{\mathscr{V}}$ and $\partial u/\partial n$ is the derivative of u in the direction of the exterior normal on S. Obtain the solution of the boundary- and initial-value problem in the particular case

$$F(r,\theta,t) = F_0 \sin \omega t \qquad f(r,\theta) = h(r) \cos \theta \qquad g = 0$$

F_0, ω positive constants.

76 With reference to Prob. 74, assume the gas is confined to the region bounded by concentric spheres of inner radius a and outer radius b. Then the boundary- and initial-value problem for the velocity potential u is

$$u_{tt} = c^2 \Delta u \qquad \text{in } \bar{\mathscr{V}};\ t \geq 0$$

$$\frac{\partial u}{\partial r} = 0 \qquad r = a;\ r = b;\ t \geq 0$$

$$u(r,0) = f(r) \qquad u_t(r,0) = g(r) \qquad \text{in } \mathscr{V}$$

where $\bar{\mathscr{V}}$ is the volume defined by the inequalities

$$0 < a \leq r \leq b \qquad 0 \leq \theta \leq \pi \qquad 0 \leq \varphi \leq 2\pi$$

Show that the eigenvalues for the present case are

$$\lambda_0 = 0 \qquad \lambda_n = \xi_n^2 \qquad n = 1, 2, \ldots$$

where $\{\xi_n\}$ is the sequence of positive roots of the transcendental equation

$$\tan\,[\xi(b - a)] = \frac{\xi(b - a)}{1 + ab\xi^2}$$

Obtain the corresponding eigenfunctions

$$\psi_0 = 1 \qquad \psi_n = \frac{\sin \xi_n r + \alpha_n \cos \xi_n r}{r} \qquad n = 1, 2, \ldots$$

where

$$\alpha_n = \frac{a\xi_n \cos \xi_n a - \sin \xi_n a}{a\xi_n \sin \xi_n a + \cos \xi_n a}$$

Verify that the eigenfunctions are orthogonal on $[a,b]$ with weight function r^2:

$$\int_0^b r^2 \psi_n(r)\psi_m(r)\, dr = 0 \qquad n \neq m$$

The characteristic frequencies of vibration are $\omega_n = c\xi_n$, $n = 1, 2, \ldots$. Derive the formal series solution of the boundary- and initial-value problem in the form

$$u = A_0 + B_0 t + \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t)\psi_n(r)$$

where the coefficients are defined by

$$A_0 = \frac{3}{b^3 - a^3}\int_a^b r^2 f(r)\, dr \qquad B_0 = \frac{3}{b^3 - a^3}\int_a^b r^2 g(r)\, dr$$

$$A_n = \frac{1}{\|\psi_n\|^2}\int_a^b r^2 f(r)\psi_n(r)\, dr \qquad B_n = \frac{1}{\omega_n\ \|\psi_n\|^2}\int_a^b r^2 g(r)\psi_n(r)\, dr \qquad n \geq 1$$

77 Let $\bar{\mathscr{V}}$ be the cone defined by the inequalities

$$0 \leq r \leq a \qquad 0 \leq \theta \leq \beta \qquad 0 \leq \varphi \leq 2\pi$$

where r, θ, φ are spherical coordinates and β is a fixed angle such that $0 < \beta < \pi/2$. Consider the boundary- and initial-value problem

$$u_{tt} - c^2 \Delta u = F(r,t) \qquad \text{in } \mathscr{V};\ t \geq 0$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } S;\ t \geq 0$$

$$u(r,\theta,\varphi,0) = f(r) \qquad u_t(r,\theta,\varphi,0) = g(r) \qquad \text{in } \mathscr{V}$$

Derive the form of the eigenfunctions and eigenvalues for this case. Derive the formal series solution of the boundary- and initial-value problem.

78 Let $\mathscr{V}$ be the cylinder of radius a and altitude h defined by the inequalities

$$0 \leq r \leq a \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq z \leq h$$

where r, θ, z are cylindrical coordinates. Consider the boundary- and initial-value problem

$$u_{tt} - c^2 \Delta u = F(r,\theta,z,t) \qquad \text{in } \mathscr{V};\ t \geq 0$$

$$u = 0 \qquad \text{on } S;\ t \geq 0$$

$$u(r,\theta,z,0) = f(r,\theta,z) \qquad u_t(r,\theta,z,0) = g(r,\theta,z) \qquad \text{in } \mathscr{V}.$$

(a) Derive the eigenfunctions

$$\psi_{nmq}^{(e)} = J_n\left(\frac{\xi_{nm}r}{a}\right) \cos n\theta \sin\left(\frac{q\pi z}{h}\right) \qquad \psi_{nmq}^{(0)} = J_n\left(\frac{\xi_{nm}r}{a}\right) \sin n\theta \sin\frac{q\pi z}{h}$$

corresponding to the ∞^3 eigenvalues

$$\lambda_{nmq} = \frac{\xi_{nm}{}^2}{a^2} + \frac{q^2\pi^2}{h^2} \qquad n = 0, 1, \ldots;\ m = 1, 2, \ldots;\ q = 1, 2, \ldots$$

where ξ_{nm} denotes the mth positive zero of the Bessel function $J_n(\xi)$. Outline the proof of the fact that these are all the eigenvalues of the problem. Show that the eigenfunctions have the orthogonality properties

$$\iiint_{\mathscr{V}} \psi_{nmq}^{(e)}\psi_{n'm'q'}^{(e)}\, dV = 0 \qquad \iiint_{\mathscr{V}} \psi_{nmq}^{(0)}\psi_{nmq}^{(0)}\, dV = 0 \qquad (n,m,q) \neq (n',m',q')$$

$$\iiint_{\mathscr{V}} \psi_{nmq}^{(e)}\psi_{n'm'q'}^{(0)}\, dV = 0 \qquad \text{all } (n,m,q),\ (n',m',q')$$

where $dV = r\, dr\, d\theta\, dz$. Show also that

$$\|\psi_{nmq}^{(e)}\|^2 = \|\psi_{nmq}^{(0)}\|^2 = \frac{a^2h[J_n'(\xi_{nm})]^2}{4} \qquad n \geq 1;\ m \geq 1;\ q \geq 1$$

$$\|\psi_{0mq}^{(e)}\|^2 = \frac{a^2h[J_0'(\xi_{0m})]^2}{2} \qquad m \geq 1;\ q \geq 1$$

(b) Obtain the ∞^3 normal modes of vibration

$$u_{nmq}^{(e)}(r,\theta,z,t) = \psi_{nmq}^{(e)}(r,\theta,z)(A_{nmq}^{(e)} \cos \omega_{nmq}t + B_{nmq}^{(e)} \sin \omega_{nmq}t)$$

$$u_{nmq}^{(0)}(r,\theta,z,t) = \psi_{nmq}^{(0)}(r,\theta,z)(A_{nmq}^{(0)} \cos \omega_{nmq}t + B_{nmq}^{(0)} \sin \omega_{nmq}t)$$

where the characteristic frequencies are

$$\omega_{nmq} = c\sqrt{\lambda_{nmq}} \qquad n = 0, 1, \ldots ; m = 1, 2, \ldots ; q = 1, 2, \ldots$$

(c) Derive the *Green's function*

$$G(r,\theta,z,t;r_0,\theta_0,z_0,\tau) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \frac{J_n(\xi_{nm}r/a)J_n(\xi_{nm}r_0/a)}{\omega_{nmq}\,\|\psi_{nmq}\|^2} \cos\,[n(\theta-\theta_0)] \sin\frac{q\pi z}{h} \sin\frac{q\pi z_0}{h} \sin\,[\omega_{nmq}(t-\tau)]$$

(d) Derive the series solution of the boundary- and initial-value problem if $g = 0$ and

$$F(r,\theta,z,t) = F_0 \sin \omega t \qquad f(r,\theta,z) = (a-r)\sin\frac{\pi z}{h}$$

where F_0, ω are given positive constants.

79 Let $\bar{\mathscr{V}}$ be the cylinder in Prob. 78. Solve the problem

$$u_{tt} = c^2\,\Delta u \qquad \text{in } \bar{\mathscr{V}};\ t \ge 0$$

$$\frac{\partial u}{\partial r} = 0 \qquad r = a \qquad\qquad \frac{\partial u}{\partial z} = 0 \qquad z = 0$$

$$\frac{\partial u}{\partial z} = \varphi(r)\cos p\theta \cos \omega t \qquad z = h$$

$$\rho u(r,\theta,z,0) = 0 \qquad u_t(r,\theta,z,0) = 0$$

where φ is a given function, p a given positive integer, and ω a given positive constant.

80 Let S be a simple closed piecewise smooth surface which bounds a region $\mathscr{V}$ of xyz space. Let $\bar{\mathscr{V}}$ denote $\mathscr{V}$ together with S. Assume the divergence theorem is applicable to $\bar{\mathscr{V}}$. Let p_1, p_2, p_3 be given functions, defined and positive-valued on $\bar{\mathscr{V}}$. Assume also these are continuously differentiable on $\bar{\mathscr{V}}$. Define the elliptic operator L by

$$L\varphi = (p_1\varphi_x)_x + (p_2\varphi_y)_y + (p_3\varphi_z)_z$$

for every twice continuously differentiable function φ on $\bar{\mathscr{V}}$. In the particular case where $p_i = 1$, $i = 1, 2, 3$, the operator reduces to the laplacian. Consider the boundary- and initial-value problem.

$$\rho u_{tt} + 2\gamma u_t + \beta u - Lu = F(x,y,z,t) \qquad \text{in } \bar{\mathscr{V}};\ t \ge 0$$

$$B(u) = 0 \qquad \text{on } S;\ t \ge 0$$

$$u(x,y,z,0) = f(x,y,z) \qquad u_t(x,y,z,0) = g(x,y,z) \qquad \text{in } \bar{\mathscr{V}}$$

where ρ is a given positive valued continuously differentiable function on $\bar{\mathscr{V}}$, γ and β are given real nonnegative constants, and F, f, g are given functions. Here $B(u) = 0$ symbolizes one of the three types of boundary conditions

$$u = 0 \qquad \frac{\partial u}{\partial n} = 0 \qquad \frac{\partial u}{\partial n} + \sigma u = 0 \qquad \sigma > 0$$

(a) If $v(x,y,z,t)$ is a twice continuously differentiable function on $\bar{\mathscr{V}}$ and $t \geq 0$, define the vector

$$\mathbf{P} = p_1 v_x \mathbf{i} + p_2 v_y \mathbf{j} + p_3 v_z \mathbf{k}$$

and the *energy integral*

$$E(t) = \frac{1}{2}\iiint_{\mathscr{V}} (\rho v_t^2 + \beta v^2 + |\mathbf{P}|^2)\, d\tau$$

where $d\tau$ is the volume element in $\mathscr{V}$. Show that

$$\dot{E}(t) = \iiint_{\mathscr{V}} (\rho v_{tt} + \beta v) v_t\, d\tau + \iiint_{\mathscr{V}} \mathbf{P} \cdot \nabla v_t\, d\tau$$

Now make use of the vector identity

$$\nabla \cdot (w\mathbf{A}) = \nabla w \cdot \mathbf{A} + w \nabla \cdot \mathbf{A}$$

and show that

$$\dot{E}(t) = \iiint_{\mathscr{V}} (\rho v_{tt} + \beta v - Lv) v_t\, d\tau + \iint_S v_t \mathbf{P} \cdot \mathbf{n}\, dS$$

In particular if u is a solution of the partial differential equation in the boundary- and initial-value problem, then the energy rate of change is

$$\dot{E}(t) = \iiint_{\mathscr{V}} u_t F(x,y,z,t)\, d\tau - 2\gamma \iiint_{\mathscr{V}} u_t^2\, d\tau + \iint_S u_t \mathbf{P} \cdot \mathbf{n}\, dS$$

(b) Prove the following uniqueness theorem for the boundary- and initial-value problem. If the boundary condition is the Dirichlet condition $u = 0$ on S, there can be at most one twice continuously differentiable solution. If $p_1 = p_2 = p_3$ on $\bar{\mathscr{V}}$ and $\beta > 0$, the same is true for the Neumann boundary condition $\partial u/\partial n = 0$. If $p_1 = p_2 = p_3$ on $\bar{\mathscr{V}}$ and $\beta = 0$, a solution is unique to within an additive constant. If $p_1 = p_2 = p_3$ on $\bar{\mathscr{V}}$ and the boundary condition is the mixed condition $\partial u/\partial n + \sigma u = 0$ on S, there is at most one twice continuously differentiable solution. *Hint:* Suppose u_1, u_2 are twice continuously differentiable solutions of the problem. Let $u = u_1 - u_2$. By the linearity of things u satisfies the boundary- and initial-value problem with $F, f,$ and g replaced by zero. Hence the energy rate is

$$\dot{E}(t) = -2\gamma \iiint_{\mathscr{V}} u_t^2\, d\tau + \iint_S u_t \mathbf{P} \cdot \mathbf{n}\, dS$$

If the boundary condition is $u = 0$ on S, $t \geq 0$, then $u_t = 0$ on S, $t \geq 0$. Hence

$$\dot{E}(t) \leq 0 \qquad t \geq 0$$

This implies $E(t) \leq E(0)$, $t \geq 0$. But $E(0) = 0$. Hence $E(t) \leq 0$, $t \geq 0$. But an examination of the energy integral shows $E(t) \geq 0$, $t \geq 0$. Hence $E(t) = 0$, $t \geq 0$. Show this implies $u(x,y,z,t) = 0$, (x,y,z) in $\bar{\mathscr{V}}$, $t \geq 0$. Suppose now $p_1 = p_2 = p_3$ in $\bar{\mathscr{V}}$. Then

$$u_t \mathbf{P} \cdot \mathbf{n} = u_t p \frac{\partial u}{\partial n} \qquad p = p_1 = p_2 = p_3$$

If the boundary condition is $\partial u/\partial n = 0$ on S, then again $E(t) = 0$, $t \geq 0$. Show that if $\beta > 0$, this implies $u = 0$ in $\bar{\mathscr{V}}$, $t \geq 0$. On the other hand, if $\beta = 0$, then $u =$ const is possible. If $p_1 = p_2 = p_3 = p$ and the boundary condition is the mixed condition, then

$$\dot{E}(t) = -2\gamma \iiint\limits_{\mathscr{V}} u_t^2 \, d\tau - \sigma \iint\limits_{S} u u_t \, dS$$

so that

$$E(t) - E(0) = -2\gamma \int_0^t dt \iiint u_t^2 \, d\tau - \frac{\sigma}{2} \iint\limits_{S} u^2 \, dS \leq 0$$

Again $E(t) = 0$, $t \geq 0$. Show this implies that $u = 0$ on for $t \geq 0$.

representation (5-47) involving the *Green's function*. These remarks about problem (5-52) hold as well for the general problem (5-27).

PROBLEMS

Sec. 5-2

1 Let ξ, τ, ω be fixed values, $\omega > 0$. Verify that the function of x and t defined by

$$u(x,t;\xi,\omega) = \cos\,[\omega(x - \xi)]e^{-\kappa\omega^2(t-\tau)}$$

satisfies the one-dimensional homogeneous heat equation. Interpret $t = \tau$ as an "initial time." Then u represents the subsequent temperature in the infinitely long bar due to the initial temperature

$$u(x,\tau;\xi,\tau,\omega) = \cos\,[\omega(x - \xi)]$$

This distribution is sinusoidal along the length of the bar with frequency ω. The summation over all frequencies is

$$v(x,t;\xi,\tau) = \frac{1}{\pi}\int_0^\infty \cos\,[\omega(x - \xi)]e^{-\kappa\omega^2(t-\tau)}\,d\omega$$

Show that the improper integral converges for $t > \tau$ in such a manner that in calculating v_t and v_{xx} differentiation within the integral with respect to x and t is permissible, and hence show that v is a solution of the heat equation on the open half plane. Now make use of the integration formula

$$\int_0^\infty e^{-x^2}\cos\,2bx\,dx = \tfrac{1}{2}\sqrt{\pi}\,e^{-b^2}$$

and obtain the expression (5-13) for the fundamental solution.

2 The initial temperature in the infinitely long bar is

$$f(x) = u_0 \qquad |x| \le \delta \qquad\qquad f(x) = 0 \qquad |x| > \delta$$

where u_0, δ are given positive constants. There are no heat sources within the bar. Show that the subsequent temperature distribution for $t > 0$ is

$$u(x,t) = \frac{u_0}{2}\left(\operatorname{erf}\frac{x+\delta}{\sqrt{4\pi\kappa t}} - \operatorname{erf}\frac{x-\delta}{\sqrt{4\pi\kappa t}}\right)$$

where the error function is defined by

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\eta^2}\,d\eta \qquad -\infty < x < \infty$$

3 Solve Prob. 2 if the initial temperature is given by

$$f(x) = \begin{cases} u_1 & -\delta_1 \le x \le \delta_1 \\ u_2 & -\delta_2 \le x < -\delta_1 \text{ or } \delta_1 < x \le \delta_2 \\ 0 & \text{elsewhere} \end{cases}$$

where the constants u_1, u_2, δ_1, δ_2 are such that

$$0 < u_2 < u_1 \qquad 0 < \delta_1 < \delta_2$$

4 A slender homogeneous conducting bar of uniform cross section extends along the x axis for $x \geq 0$. The lateral surface is insulated. The end $x = 0$ is held at temperature $h(t)$ for $t \geq 0$. The initial temperature is described by the given function $f(x)$, $x \geq 0$. The boundary- and initial-value problem is then

$$u_t - \kappa u_{xx} = F(x,t) \qquad x > 0;\ t > 0$$
$$u(0,t) = h(t) \qquad t \geq 0 \qquad\qquad u(x,0) = f(x) \qquad x \geq 0$$

To solve the problem let v be the function defined by Eq. (5-13) and define the function w by

$$w(x,t;\xi,\tau) = v(x,t;\xi,\tau) - v(x,t;-\xi,\tau) \qquad -\infty < x < \infty;\ t > \tau$$
$$w(x,\tau;\xi,\tau) = 0 \qquad -\infty < x < \infty$$

Then w is the *fundamental solution* of the problem of the heat flow in the semi-infinite bar. Observe w is a twice continuously differentiable solution of the homogeneous heat equation for $t > \tau$ and is continuous in x and t everywhere on the half plane $t \geq \tau$ except at the points (ξ,τ), $(-\xi,\tau)$, where it suffers infinite discontinuities. Show that

$$\lim_{\substack{(x,t)\to(x_0,\tau)\\ t>\tau}} w = 0 \qquad x_0 \neq \xi;\ x_0 \neq -\xi$$

$$\lim_{\substack{t\to\tau\\ t>\tau}} w(\xi,t;\xi,\tau) = +\infty \qquad \lim_{\substack{t\to\tau\\ t>\tau}} w(-\xi,t;\xi,\tau) = -\infty$$

Note that w is an odd function of x (also odd in ξ), and is symmetric in x and ξ. Interpret w as the temperature in the infinitely long bar $(-\infty < x < \infty)$ for $t > \tau$ due to an initial temperature

$$f(x) = \delta(x - \xi) - \delta(x + \xi)$$

This is a point source at $x = \xi$ and a point sink at $x = -\xi$. Observe that

$$w(0,t;\xi,\tau) = 0 \qquad t > \tau$$

Following the method used in the text prior to Eq. (5-20) give a heuristic derivation of the solution of the problem in the form $u = u_1 + u_2 + u_3$, where

$$u_1(x,t) = \int_0^\infty w(x,t;\xi,0)f(\xi)\,d\xi \qquad u_2(x,t) = \kappa\int_0^t h(\tau)\left.\frac{\partial w}{\partial \xi}\right|_{\xi=0} d\tau$$

$$u_3(x,t) = \int_0^t d\tau \int_0^\infty w(x,t;\xi,\tau)F(\xi,\tau)\,d\xi$$

The function u_1 is the solution of the problem

$$u_t = \kappa u_{xx} \qquad x > 0;\ t > 0$$
$$u(0,t) = 0 \qquad t > 0 \qquad\qquad u(x,0) = f(x) \qquad x > 0$$

The function u_2 is the solution of the problem

$$u_t = \kappa u_{xx} \qquad x > 0;\ t > 0$$
$$u(0,t) = h(t) \qquad t > 0 \qquad\qquad u(x,0) = 0 \qquad x > 0$$

Show that u_2 can be rewritten

$$u_2(x,t) = \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4\kappa t}}^\infty h\left(t - \frac{x^2}{4\kappa\eta^2}\right) e^{-\eta^2}\,d\eta$$

The function u_3 is the solution of the problem

$$u_t - \kappa u_{xx} = F(x,t) \qquad x > 0;\ t > 0$$

$$u(0,t) = 0 \qquad t > 0 \qquad\qquad u(x,0) = 0 \qquad x > 0$$

5 The problem of heat flow in a semi-infinite slab is mathematically identical to the problem of heat flow in the semi-infinite bar. Assume that a homogeneous conductor occupies the half space $x \geq 0$ in xyz space. The temperature on the face $x = 0$ is $h(t)$ at time $t > 0$. The initial temperature in the slab is $f(x)$, $x > 0$. Thus the subsequent heat flow is one-dimensional. Assume there are no heat sources within the slab. Determine the subsequent temperature for each of the following cases.

(a) $h(t) = 0; f(x) = u_0$ (u_0 a positive constant)

(b) $h(t) = u_0; f(x) = 0$

(c) $h(t) = u_1; f(x) = u_2$ (u_1, u_2 nonzero constants)

(d) $h(t) = u_1$, $0 < t < t_1$, $h(t) = u_2$, $t_1 < t < t_2$, $h(t) = 0$ $t > t_2; f(x) = 0$

6 The face $x = 0$ of the semi-infinite slab described in Prob. 5 radiates heat into the exterior region $x < 0$ in such a way that the flux across the face is a constant q_0. The initial temperature is u_0, a constant. There are no heat sources within the slab. Show that the subsequent temperature in the slab is

$$u(x,t) = u_0 + \frac{q_0}{K}\left[x\left(\operatorname{erf}\frac{x}{\sqrt{4\kappa t}} - 1\right) + \frac{\sqrt{4\kappa t}}{\pi}\,(e^{-x^2/4\kappa t} - 1)\right]$$

Hint: Recall the flux $q = -Ku_x$. Show that if u satisfies the homogeneous heat equation, then so does q. Use the result of Prob. 5**b** to obtain the appropriate flux q. Now solve the partial differential equation $u_x = -q/K$ and apply the initial condition.

7 The model which describes the heat flow in the infinitely long bar in which the insulated surface is replaced by a condition of radiation into the exterior region (held at temperature zero) is

$$u_t + bu - \kappa u_{xx} = F(x,t) \qquad -\infty < x < \infty;\ t > 0$$

$$u(x,0) = f(x) \qquad -\infty < x < \infty$$

where b is a positive constant. Show that if w is a solution of problem (5-11), where $F(x,t)$ is replaced by $e^{-bt}F(x,t)$, then

$$u = e^{-bt}w$$

is a solution of the original problem. Thus the *fundamental solution* of the original problem is

$$\psi(x,t;\xi,\tau) = v(x,t;\xi,\tau)e^{-b(t-\tau)}$$

where v is defined by Eq. (5-13). Find the temperature in the bar for $t > 0$ if there are no heat sources and the initial temperature is u_0, a positive constant.

8 A thin homogeneous conducting plate of uniform thinness lies in the xy plane. The plate is of sufficiently large extent so that edge effects can be neglected. The initial temperature in the plate is

$$u(x,y,0) = \begin{cases} u_0 & |x| < a \text{ and } |y| < a \\ 0 & \text{elsewhere} \end{cases}$$

The heat source density in the plate is

$$F(x,y,t) = F_0\delta(x - x_0)\delta(y - y_0)$$

where (x_0,y_0) is a given point and F_0 is a positive constant. Find the subsequent temperature in the plate.

9 Let (r,θ,φ) be spherical coordinates. A temperature distribution $u(r,t)$ which is purely radially dependent satisfies

$$u_t - \kappa\frac{(r^2u_r)_r}{r^2} = F(r,t) \qquad r > 0;\ t > 0$$

and the initial condition $u(r,0) = f(r)$ for $r \geq 0$. Here F and f are given functions. Let $w = ru$ and obtain

$$w_t - \kappa w_{rr} = rF(r,t) \qquad r > 0;\ t > 0$$

$$w(0,t) = 0 \qquad t \geq 0 \qquad\qquad w(r,0) = rf(r) \qquad r \geq 0$$

Use the results of Prob. 4 to derive the solution of the original problem in the form

$$u(r,t) = \frac{1}{r\sqrt{4\pi\kappa t}}\int_0^\infty (e^{-(r-r_0)^2/4\kappa(t-\tau)} - e^{-(r+r_0)^2/4\kappa(t-\tau)})r_0 f(r_0)\,dr_0$$
$$+ \frac{1}{r}\int_0^t \frac{d\tau}{4\pi\kappa(t-\tau)}\int_0^\infty (e^{-(r-r_0)^2/4\kappa(t-\tau)} - e^{-(r+r_0)^2/4\kappa(t-\tau)})r_0F(r_0,\tau)\,dr_0$$
$$= \int_0^\infty\int_0^{2\pi}\int_0^\pi \psi(r,t;r_0,0)f(r_0)\,dV_0 + \int_0^t d\tau\int_0^\infty\int_0^{2\pi}\int_0^\pi \psi(r,t;r_0,\tau)F(r_0,\tau)\,dV_0$$

where $dV_0 = r_0^2\sin\theta_0\,d\theta_0\,d\varphi_0\,dr_0$ is the volume element and ψ is the *fundamental solution*

$$\psi(r,t;r_0,\tau) = \frac{v(r,t;r_0,\tau) - v(r,t;-r_0,\tau)}{8\pi^{3/2}rr_0}$$

where v is defined in Eq. (5-13). Find the solution of the original problem if the source density is zero and the initial temperature in the space is u_0, a positive constant.

10 Show that the solution of the problem

$$u_t = \kappa\left(u_{rr} + \frac{2u_r}{r}\right) \qquad r > a > 0;\ a = \text{const}$$

$$u(r,0) = u_0 \qquad r \geq a \qquad\qquad u(a,t) = U_0$$

is

$$u(r,t) = \frac{a}{r}(U_0 - u_0)\left(1 - \operatorname{erf}\frac{r-a}{\sqrt{4\kappa t}}\right)$$

where erf x is the error function defined in Prob. 2.

Sec. 5-3

11 A slender homogeneous conducting bar of uniform cross section lies along the x axis with ends at $x = 0$, $x = L$. The lateral surface is insulated. There are no heat sources within the bar. The bar has the indicated end conditions and initial temperature. Derive the series solution for the temperature distribution in the bar for $t > 0$.

(a) Ends are at zero temperature; $f(x) = x$, $0 \le x \le L/2$, $f(x) = L - x$, $L/2 \le x \le L$

(b) Ends are at zero temperature; $f(x) = 3 \sin 2\pi x/L$, $0 \le x \le L$

(c) $u(0,t) = u_1$, $u(L,t) = u_2$, u_1, u_2 nonzero constants; $u(x,0) = f(x)$, $0 \le x \le L$

(d) Ends are insulated; $f(x) = x(L - x)$, $0 \le x \le L$

(e) End $x = 0$ held at temperature zero, at the end $x = L$ there is a constant flux q_0; $u(x,0) = f(x)$, $0 \le x \le L$

12 Instead of being insulated, the lateral surface of the bar described in Prob. 11 radiates heat into the surroundings at temperature zero. The ends $x = 0$, $x = L$ are held at temperatures u_1 and u_2, respectively. The initial temperature is $u(x,0) = f(x)$, $0 \le x \le L$. Show that the subsequent temperature in the bar is

$$u(x,t) = \varphi(x) + e^{-bt} \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} e^{-\kappa\lambda_n t}$$

where

$$\varphi(x) = \frac{u_1 \sinh [\sqrt{b}(L - x)/\sqrt{\kappa}] + u_2 \sinh (\sqrt{b}\, x/\sqrt{\kappa})}{\sinh (\sqrt{b}\, L/\sqrt{\kappa})}$$

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad A_n = \frac{2}{L} \int_0^L [f(x) - \varphi(x)] \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \ldots$$

Hint: Find a solution of $u_t + bu = \kappa u_{xx}$ which satisfies the boundary conditions. Then superimpose with a solution which satisfies homogeneous boundary conditions and an appropriate initial condition.

13 The lateral surface of a slender homogeneous conducting bar of length L is insulated. The temperature at the end $x = 0$ is maintained at $A \sin \omega_1 t$ for $t \ge 0$, where A, ω_1 are positive constants. The end $x = L$ radiates heat into the exterior region, which is at temperature zero. The initial temperature in the bar is u_0, a constant. The heat-source density within the bar is

$$F(x,t) = F_0 \delta(x - x_0) \cos \omega t$$

where x_0 is a given point and F_0, ω are positive constants. Determine the temperature in the bar for $t > 0$.

14 An infinite slab is bounded by the parallel planes $x = 0$, $x = L$ in xyz space. The face $x = 0$ is insulated. Across the face $x = L$ there is a constant inward flux of magnitude q_0. The initial temperature in the slab is u_0 (a constant). Show that the temperature in the slab for $t > 0$ is

$$u(x,t) = u_0 + q_0 \left[\frac{\kappa t}{L} + \frac{3x^2 - L^2}{6L} + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos \frac{n\pi x}{L} e^{-\kappa n^2 \pi^2 t/L^2} \right]$$

15 The initial temperature of a solid homogeneous conducting cylinder of radius a is

$$u_0\left(1-\frac{r^2}{a^2}\right) \qquad 0 \le r \le a$$

where r is the distance from the axis. The temperature on the lateral surface is zero. Assume the cylinder is infinitely long. There are no heat sources within the cylinder. Determine the temperature distribution in the cylinder for $t > 0$.

16 An infinitely long tube has inner radius a and outer radius b. The material of the tube is homogeneous and contains no heat sources. The inner surface is insulated, and the outer surface is maintained at temperature u_0. The initial temperature is zero. Show that the temperature distribution for $t > 0$ and $a < r < b$ is

$$u(r,t) = u_0 + \pi u_0 \sum_{n=1}^{\infty} J_1^2(\xi_n a)\left[\frac{Y_0(\xi_n b)J_0(\xi_n r) - J_0(\xi_n b)Y_0(\xi_n r)}{J_0^2(\xi_n b) - J_1^2(\xi_n a)}\right]e^{-\kappa\xi_n^2 t}$$

where r is the distance from the axis and $\{\xi_n\}$ is the sequence of positive zeros of the function

$$\varphi(\xi) = J_0(b\xi)Y_1(a\xi) - J_1(a\xi)Y_0(b\xi)$$

17 A solid homogeneous circular cylinder of radius a and altitude b has its axis coincident with the z axis. The initial temperature is

$$u(r,\theta,z,0) = f(r,\theta,z)$$

where r, θ, z are cylindrical coordinates. The temperature of the base is held constant, at temperature u_0. The top and lateral surfaces are insulated. Determine the temperature in the cylinder for $t > 0$.

18 With reference to the cylinder described in Prob. 17 the top, bottom, and lateral surfaces all radiate heat into the exterior region, which is at temperature zero. The initial temperature

$$u(r,\theta,z,0) = f(r,z)$$

Show that the temperature in the cylinder for $t > 0$ is

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}J_0\left(\frac{\xi_n r}{a}\right)[\mu_m \cos \mu_m z + h \sin \mu_m z]e^{-\kappa\lambda_{mn}t}$$

where $\{\xi_n\}$ is the sequence of positive zeros of the function

$$\xi J_0'(\xi) + haJ_0(\xi)$$

and $\{\mu_m\}$ is the sequence of positive roots of the equation

$$\tan b\mu = \frac{2h\mu}{\mu^2 - h^2}$$

and

$$A_{nm} = \frac{1}{\|\psi_{nm}\|^2}\int_0^b\int_0^a rf(r,z)J_0\left(\frac{\xi_n r}{a}\right)[\mu_m \cos \mu_m z + h \sin \mu_m z]\,dr\,dz$$

$$\|\psi_{nm}\|^2 = \frac{a^2}{2\xi_n^2}(a^2h^2 + \xi_n^2)J_0^2(\xi_n)\left[h + \frac{b}{2}(\mu_m^2 + h^2)\right]$$

$$\lambda_{nm} = \frac{\xi_n^2}{a^2} + \mu_m^2 \qquad n = 1, 2, \ldots; m = 1, 2, \ldots$$

19 Let $\bar{\mathscr{V}}$ denote the domain defined by the inequalities

$$0 \leq r \leq a \qquad 0 \leq \theta \leq \theta_0 \qquad 0 \leq z \leq b$$

where r, θ, z are cylindrical coordinates and θ_0 is a given fixed angle, $0 < \theta_0 < \pi$. Let $\mathscr{V}$ denote the interior of $\bar{\mathscr{V}}$. Consider the problem

$$u_t - \kappa\, \Delta u = F(r,\theta,z,t) \qquad \text{in } \bar{\mathscr{V}};\ t > 0$$

$$\left.\frac{\partial u}{\partial \theta}\right|_{\theta=0} = 0 \qquad \left.\frac{\partial u}{\partial \theta}\right|_{\theta=\theta_0} = 0 \qquad u|_{r=a} = H(\theta,z,t) \qquad t \geq 0$$

$$u(r,\theta,z,0) = f(r,\theta,z) \qquad \text{in } \bar{\mathscr{V}}$$

Given a physical interpretation of the problem, derive the eigenfunctions and then the series representation of the Green's function of the problem. Find the series solution of the problem in the particular case where

$$F = 0 \qquad f = 0 \qquad H(\theta,z) = H_0 = \text{const}$$

20 A homogeneous solid sphere of radius a has the initial temperature distribution

$$a^2 - r^2 \qquad 0 \leq r \leq a$$

where r denotes distance measured from the center. The surface temperature is maintained at zero. Show that the temperature in the sphere for $t > 0$ is given by

$$u(r,t) = \frac{12a^3}{\pi^3 r} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin(n\pi r/a)}{n^3} e^{-\kappa n^2 \pi^2 t/a^2}$$

21 Solve Prob. 20 if instead of the surface temperature's being zero the surface radiates heat into the exterior region. The temperature of the exterior region is zero.

22 Let r, θ, φ be the usual spherical coordinates (see Fig. 3-3). Let $\bar{\mathscr{V}}$ denote the sphere of radius a about the origin, and let $\mathscr{V}$ be its interior. Consider the problem

$$u_t - \kappa\, \Delta u = F(r,\theta,\varphi,t) \qquad \text{in } \mathscr{V};\ t > 0$$

$$u(a,\theta,\varphi,t) = u_0 \qquad t \geq 0$$

$$u(r,\theta,\varphi,0) = f(r,\theta,\varphi) \qquad \text{in } \bar{\mathscr{V}}$$

where F and f are given functions and u_0 is a nonzero constant. Derive the formal series solution of the problem. Obtain the solution in the particular case where

$$F(r,\theta,\varphi,t) = \frac{F_0 \delta(r - r_0)\delta(\theta - \theta_0)\delta(\varphi - \varphi_0) \sin \omega t}{r^2 \sin \theta}$$

where F_0, ω are given positive numbers, (r_0,θ_0,φ_0) is a given point, and

$$f(r,\theta,\varphi) = (a^2 - r^2) \sin \theta$$

23 A spherical conducting shell has inner radius a and outer radius b. Through the inner wall from the interior there is a constant flux of heat q_0. The outer wall radiates into the exterior region which is at temperature zero. The initial temperature of the shell is a constant u_0. There are no heat sources within the shell material. Derive the formal series solution of the problem.

Appendix 4 ANSWERS

CHAPTER 1

1 **(a)** Linear second order, two independent variables.
(c) Nonlinear, almost linear, second order, two independent variables.
(e) Nonlinear, quasilinear, third order, three independent variables.

3 **(a)** $xp - z = 0$ **(c)** $xq - yp = 0$
(e) $yp - xq = 2xyz$ **(g)** $u_{tt} - c^2 u_{xx} = 0$

6 **(a)** $xp + yq = z \log z$ **(c)** $xp + yq = z$
(e) $u_{xx} + u_{yy} = 0$ **(g)** $u_t = u_{xx}$
(i) $z = xp + yq + p^2 + q^2$

7 **(a)** $(x - 1)p + yq = z$ **(c)** $xp + yq - pq = z$

8 **(a)** $z = x^2 y + \dfrac{y^3}{3} + f(x)$ **(c)** $z = f(y)e^{-x^2/2} + x^2 - 2 + 3y$

10 **(a)** $z = f(4x + 3y) + \dfrac{x^3}{9}$

(c) $z = e^{-x/5} f(4x - 5y) + x^3 - 15x^2 + 150x - 149 + \dfrac{2e^{3y}}{13}$

(e) $z = f(ax + y) + e^{mx}\dfrac{m \cos by - ab \sin by}{m^2 + a^2 b^2}$

11 **(a)** $z = f(x^2 y^4)$ **(c)** $z = f(x^7 y) - \dfrac{x^2 y}{5}$

(e) $z = x^{-c/a} F(x^b y^{-a}) + \dfrac{x^2}{2a + c} + \dfrac{y^2}{2b + c}$

12 **(a)** $z = \dfrac{f(x^2 + y^2)}{x}$ **(c)** $z = (x + a)^{-c} f\left(\dfrac{x + a}{y + b}\right)$

13 **(a)** $u = e^{-x/2} f(x - 2y - z)$

14 **(a)** $F(2y - x^2, ze^{-x}) = 0$ **(c)** $F\left(z, \dfrac{x + z}{y + z}\right) = 0$

(e) $F\left(x^2 - z^2, \dfrac{x + z}{y}\right) = 0$ **(g)** $z = \dfrac{x^2(x^2 + y^2)}{2} + f(x^2 + y^2)$

(i) $z = \dfrac{axy \log (x/y)}{x - y} + f\left(\dfrac{x - y}{xy}\right)$ **(k)** $xyz = f(x + y + z)$

(m) $z = x^3 \sin(y + 2x) + f(y + 2x)$ **(o)** $z = \sin y + f(\sin x - \sin y)$

(q) $\sin^{-1}\dfrac{z}{a} = xy + f\left(\dfrac{y}{x}\right)$

15 **(a)** $z = yf\left(\dfrac{x^2 + y^2 + z^2}{y}\right)$

17 $z = a + xf\left(\dfrac{x}{y}\right)$

18 **(a)** $u = e^{-Rx/P_1}f(P_2x - P_1y, P_3x - P_1z) \quad P_1 \neq 0$

(c) $u = f(5x - 3y, 5z + y) + \frac{1}{5}\sin y - 2e^{-z}$

(e) $u = f[z + 2\log y, z + 2\log(x + z - 2)] - 2e^z$

(g) $u = f(yz, x + y + z) + xy$ **(i)** $u = x^n f\left(\dfrac{x}{y}, \dfrac{z}{x}\right)$

(k) $u = f(x_1^2 - 2x_2, x_2^2 - 2x_3, x_3^2 - 2x_4)$

19 **(a)** $f(x^2 - 2y, y^2 - 2z, ue^{-z^2/2}) = 0$ **(c)** $u = xf\left(\dfrac{y}{x}, \dfrac{z}{x}\right) + \dfrac{xy\log x}{z}$

20 **(a)** $z = e^y \cos(x - y)$ **(c)** $z^2 = xy$

(e) $z = \dfrac{xy}{2x - y + xy}$ **(g)** $(x^2 + y^2 + z^2)^2 = 2a^2(x^2 + y^2 + xy)$

(i) $(x^2 + y^2)(a^2z^2 - h^2y^2) = a^2hx^2z$ **(k)** $(x + y + z)^3 = 27xyz$

(m) $z = \left(\dfrac{x^2 + y^2}{2}\right)^{1/2} \exp\dfrac{x^2 - y^2}{2}$

(o) $a\sin(y - a\sin x) - \log[\sin(y - a\sin x)] = az - \log\sin y$

22 **(a)**
$$z = 1 - \frac{3(x-1)}{2} + \frac{3(y-1)}{2} + \frac{15(x-1)^2/4 - 9(x-1)(y-1)/2 + 3(y-1)^2/4}{2} + \cdots$$

(c) No solution exists

23 **(a)** $u = (x + y + z)(xy/z)^{n-1}$

CHAPTER 2

4 **(a)** $z = f(x + y) + g(9x + y)$ **(c)** $z = f(y) + e^{-x}g(y) - \dfrac{x^2}{2} - xy$

(e) $z = f(x)\cos y + g(x)\sin y + \dfrac{e^{x+y}}{2}$

(g) $z = f(x) + e^{x/2}g(3x - 2y) + 2\sin(2x - 3y)$

(i) $z = f(2x + y) + g(2x - y) + x^4 + 2x^2 + \dfrac{\cos y}{4}$

(k) $z = f(x + y) + xg(x + y) + \frac{4}{9}e^{3y} - \cos x$

5 (a) $z = f(y - 2x) + g(y - 3x) + 7x^2 - 6xy + \frac{5y^2}{4} - \left(4x^2 - 3xy + \frac{y^2}{2}\right)\log(y - 2x)$

(c) $z = f(y - 2x) + g(y + 2x) + x \log y + y \log x$

6 (a) $z = f(x) + yg(x) + \frac{x^3y^2}{2} + \frac{y^5}{20}$ (c) $z = e^{xy}g(x) - \frac{f(x)}{x} - xy$

(e) $z = f(y) + \frac{g(x)}{y} + \sin(x + y)$ (g) $z = \frac{f(x) + g(y)}{x - y}$

8 (a) $z = f(xy) + x^3g\frac{y}{x}$ (c) $z = y^2f(x) + \frac{g(xy)}{x}$

(e) $z = xy \log x + f(xy) + xg\left(\frac{x}{y}\right)$ (g) $z = xf(y) + yg(x)$

10 (a) $z = f(e^x - e^y) + g(e^x + e^y)$ (c) $z = f(x + y) + g\left(\frac{y}{x}\right)$

11 (a) $u = xf(x - y, x + z) + g(x - y, x + z) + 2\sin x - x\cos x$
$+ 2\sin y - y\cos y + 2\sin z - z\cos z$

(c) $u = e^xf(x + z, y + z) + e^{-x}g(x - z, y - z) + \frac{e^{2x}}{3} - \cos y - z$

(e) $u = f(x + \sqrt{A}\,t, y + \sqrt{C}\,t) + g(x - \sqrt{A}\,t, y - \sqrt{C}\,t)$

12 (a) $z = \exp\left[hx + \frac{(1 - h^2)y}{1 - 2h}\right]$

(c) $z = \exp[hx + (h^2 - 2h)y] - \frac{x^2 + x}{4} - \frac{y^2}{2}$

(e) $u = \exp[h(x \pm iy)]$ $i = \sqrt{-1}$ (g) $u = \exp[hx \pm i(h^2 + k^2)^{1/2}y]$

(i) $u = \exp[\alpha x + \beta y \pm c(\alpha^2 + \beta^2)^{1/2}t]$

20 (a) $z = e^{-y}f(x) + e^{-x}g(y)$ (c) $z = \frac{xy}{x - y} + \frac{f(x) + g(y)}{x - y}$

22 (a) Parabolic; $z = f(x + y) + xg(x + y) + \frac{x^2}{8}$

(c) Parabolic; $z = \exp[\lambda_1(x + 2y)]f(x + y)$

$$+ \exp[\lambda_2(x + 2y)]g(x + y) + \frac{(x + 2y + a/c)^2}{c} + \frac{a^2 - 2c}{c^3}$$

where λ_1, λ_2 are the roots of $\lambda^2 - a\lambda + c = 0$

(e) Parabolic; $z = f\left(\frac{y}{x}\right) + yg\left(\frac{y}{x}\right) + 2x^2$

(g) Hyperbolic wherever $xy \neq 0$; $z = f(xy) + xg\left(\frac{x}{y}\right) + xy \log x$

(i) Hyperbolic; $z = f(x + y - \cos x) + g(x - y + \cos x)$

(k) Hyperbolic wherever $x^2 + y^2 \neq 0$; $z = f(x^2 + y^2) + g\left(\frac{y}{x}\right) - xy$

(m) Hyperbolic wherever $x \neq h$; $z = \dfrac{f(x - at) + g(x + at)}{h - x}$

23 (a) Hyperbolic (c) Elliptic (e) Elliptic

24 (a) Unique solution is $u = b + \sin x \sin t$

(c) If $f(\xi)$ is an arbitrary twice differentiable function such that $f'(0) = 1$, then $u = f(x - t) - f(0)$ is a solution.

(e) Unique solution is $u = x^2 + t - \dfrac{t^2}{9}$ (g) No solution

(i) Unique solution is $z = \dfrac{x^2 + y^2}{2} + y - 1 + e^{x-y}(1 - x)$

(k) Unique solution is $z = 4xy + \dfrac{y^2}{2} - \dfrac{2y^3}{3} - e^{-y}$

(m) Unique solution is $z = \dfrac{x^4}{12} + \dfrac{x^2y^2}{2} + 1$

25 (a) $u = x + y - 1$ (c) $u = \dfrac{x^2}{2} + 2y - 2$

(e) $u = (\log x + y - x)^2 + 2(\log x + y) + x^2(e^y - 1) - xe^y$

CHAPTER 3

1 (c) $a_{11} + a_{22} + a_{33} = 0$ (e) $b = \pm a$

3 In three dimensions $f(r) = a + b/r$, a, b constants; two dimensions $f(r) = a + b \log r$.

8 (a) $\max u(x,y) = 1$, $\min u(x,y) = -1$; max occurs at the points $(1,0)$, $(-\frac{1}{2},\sqrt{3}/2)$, $(-\frac{1}{2},-\sqrt{3}/2)$; min occurs at the points $(\frac{1}{2},\sqrt{3}/2)$, $(-1,0)$, $(\frac{1}{2},-\sqrt{3}/2)$.

17 (a) $u = \dfrac{400}{\pi} \displaystyle\sum_{k=1}^{\infty} \frac{\sin [(2k - 1)\pi x/a] \sinh [(2k - 1)\pi(b - y)/a]}{(2k - 1) \sinh [(2k - 1)\pi b/a]}$

(c) $u = -\dfrac{8a^2T}{\pi^3} \displaystyle\sum_{k=1}^{\infty} \frac{\sin [(2k - 1)\pi x/a] \sinh [(2k - 1)\pi(b - y)/a]}{(2k - 1)^3 \sinh [(2k - 1)\pi b/a]}$

21 (a) $u = \dfrac{2T \tan^{-1} [(\sin \pi x/a)(\sinh \pi y/a)]}{\pi}$

(b) $u = 2Th \displaystyle\sum_{k=1}^{\infty} \frac{1 - \cos (a\xi_k)}{\xi_k(ah + \cos^2 a\xi_k)} \sin (\xi_k x)e^{-\xi_k y}$

where $\{\xi_k\}$ denotes the sequence of positive roots of the transcendental equation $\tan a\xi = -\xi/h$.

23 $u = v + w_1 + w_2$ where $v = \dfrac{cx(x - a)}{2} + \dfrac{dy(y - b)}{2}$

$$w_1 = \sum_{n=1}^{\infty} A_n \frac{\sinh [(2n - 1)\pi(a - x)/b] + \sinh [(2n - 1)\pi x/b]}{\sinh [(2n - 1)\pi a/b]} \sin \frac{(2n - 1)\pi y}{b}$$

$$w_2 = \sum_{n=1}^{\infty} B_n \frac{\sinh [(2n - 1)\pi(b - y)/a] + \sinh [(2n - 1)\pi y/b]}{\sinh [(2n - 1)\pi b/a]} \sin \frac{(2n - 1)\pi x}{b}$$

where

$$A_n = \frac{4db^2}{\pi^3(2n-1)^3} \qquad B_n = \frac{4ca^2}{\pi^3(2n-1)^3}$$

27 $$u = 64Ta^2b^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(\mu_n x/a)\sin(\mu_m x/b)e^{-\gamma_{nm}z}}{\mu_n^{\,3}\mu_m^{\,3}}$$

where

$$\mu_k = (2k-1)\pi \qquad k = 1, 2, \ldots \qquad \gamma_{nm} = \left[\left(\frac{\mu_n}{a}\right)^2 + \left(\frac{\mu_m}{b}\right)^2\right]^{1/2}$$

29 $$u = \frac{c}{\pi}\left[\frac{\alpha}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{\sin n(\alpha-\theta) + \sin n\theta}{n}\right]$$

31 $$u = \frac{\pi - 2\theta}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{2m}\frac{\sin 2n\theta}{n}$$

33 $$u = \frac{8T}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\,\frac{(r/a)^n - (a/r)^n}{(b/a)^n - (a/b)^n}\sin n\theta$$

35 $$u = 100\frac{\theta}{\alpha} + \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{a^{\mu_n}}{a^{\mu_n} + b^{\mu_n}}\right)\left[\left(\frac{r}{a}\right)^{\mu_n} + \left(\frac{b}{r}\right)^{\mu_n}\right]\sin \mu_n\theta$$

where

$$\mu_n = \frac{n\pi}{\alpha} \qquad n = 1, 2, \ldots$$

37 (a) $$u = 200\sum_{k=1}^{\infty}\frac{J_0(\xi_k/a)\sinh[\xi_k(h-z)/a]}{\xi_k \sinh(\xi_k h/a)J_1(\xi_k)}$$

where $\{\xi_k\}$ denotes the sequence of positive zeros of $J_0(\xi)$

39 $u = A_{00} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} A_{nk}\cosh(\xi_{nk}h/a)J_{\nu_n}(\xi_{nk}r/a)\cos\nu_n\theta$ where, for each fixed n, $\{\xi_{nk}\}$ is the sequence of positive zeros of $J'_n(\xi)$, $\nu_n = n\pi/\beta$, and

$$A_{00} = \frac{2}{a^2\beta}\int_0^a\int_0^\beta f(r,\theta)r\,dr\,d\theta$$

$$A_{nk} = \frac{2\epsilon_n\xi_{nk}^{\,2}}{a^2\beta(\xi_{nk}^{\,2} - \nu_n^{\,2})\cosh(\xi_{nk}h/a)J_{\nu_n}^{\,2}(\xi_{nk})}\int_0^a\int_0^\beta f(r,\theta)J_{\nu_n}(\xi_{nk}r/a)\cos\nu_n\theta r\,dr\,d\theta$$

$$n = 0, 1, \ldots \qquad k = 1, 2, \ldots \qquad \epsilon_0 = 2;\ \epsilon_n = 1 \text{ if } n \geq 1$$

41 $$\psi = 4\pi(b-a) \qquad 0 \leq r \leq a \qquad\qquad \psi = 4\pi b - \frac{2\pi(a^2+r^2)}{r} \qquad a \leq r \leq b$$

$$\psi = \frac{M}{r} \qquad r \geq b$$

where total mass $M = 2\pi(b^2 - a^2)$

43 $$u = u_1 + \frac{ab^2h(u_2-u_1)}{a + hb(b-a)}\left(\frac{1}{a} - \frac{1}{r}\right)$$

47 $u = \frac{T}{3}\left[7 - 4\left(\frac{r}{a}\right)^2 P_2(\cos\theta)\right]$

49 $u = \frac{T}{2} + T\sum_{n=0}^{\infty}\frac{(4n+1)P_{2n}(0)}{(2n-1)(2n+2)}\left(\frac{r}{a}\right)^{2n} P_{2n}(\cos\theta)$

51 $u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$

$$A_{nm} = -\frac{4}{nm\pi^2}\{ac(-1)^n[1-(-1)^m] + bd(-1)^m[1-(-1)^n]\}$$

55 $\Psi = 4a^2\sum_{k=1}^{\infty}\frac{J_0(\xi_k r(a))}{\xi_k{}^3 J_1(\xi_k)}$

where $\{\xi_k\}$ denotes the sequence of positive zeros of $J_0(\xi)$.

CHAPTER 4

1 **(d)** **(i)** $u = \sin kx\cos\omega t \qquad \omega = kc$

(iii) $u = \exp[-(kx-\omega t)^2] - \exp[-(kx+\omega t)^2]$

(v) $u = (k\cos kx - \alpha\sin kx)\cos\omega t \qquad \omega = kc$

3 **(a)** $u = e^{-x}\cosh ct$

(c) $u = A\sin\omega x\cos\omega ct + \frac{B}{c\mu}\cos\mu x\sin\mu ct$

(e) $u = \frac{U(1-x+ct) - U(-1-x+ct) + U(1-x-ct) - U(-1-x-ct)}{2}$

where $U(\xi)$ is the *unit step function* defined by

$U(\xi) = 1 \qquad \xi \geq 0 \qquad\qquad U(\xi) = 0 \qquad \xi < 0$

(g) $u = \frac{[U(\pi/2 - x + ct) - U(-\pi/2 - x + ct)]\cos(x-ct)}{2}$

$$+ \frac{[U(\pi/2 - x - ct) - U(-\pi/2 - x - ct)]\cos(x+ct)}{2}$$

5 **(a)** $u = \sin\omega x\cos\omega ct + \frac{t^2}{2}$

(c) $u = \frac{\cosh bx\sinh bct}{bc} + 2xt^2 + \frac{t^3}{6}$

(e) $u = A\left[\frac{t\cos(kx-\omega t)}{2\omega} - \frac{\cos kx\sin\omega t}{2\omega^2}\right] \qquad \omega = kc$

7 **(a)** $u = 1 - \frac{1}{c}\cos x\sin ct \qquad 0 \leq ct \leq x$

$$u = \left(\frac{-1}{c}\right)\sin x\cos ct \qquad 0 \leq x \leq ct$$

8 **(a)** $u = \begin{cases} e^{-x}\cosh ct + t & 0 \leq ct \leq x \\ -e^{-ct}\sinh x + \frac{x}{c} + 1 & 0 \leq x \leq ct \end{cases}$

19 (a) $u = \dfrac{32h}{\pi^3} \sum_{n=1}^{\infty} \dfrac{\sin[(2n-1)\pi x/b] \cos[(2n-1)\pi ct/b]}{(2n-1)^3}$

(c) $u = \dfrac{2A \sin \omega b}{b} \sum_{n=1}^{\infty} [(-1)^{n+1}\mu_n/(\mu_n{}^2 - \omega^2)] \sin \mu_n x \cos \mu_n ct$

$$+ \frac{4}{bc} \sum_{n=1}^{\infty} \frac{\sin \mu_{2n-1}x \cos \mu_{2n-1}ct}{\mu_{2u-1}^2} \qquad \mu_n = \frac{n\pi}{b}$$

21 (a) For the case of nonresonance

$$u = \frac{4cF_0}{b} \sum_{n=1}^{\infty} \frac{\omega \sin \omega_{2n-1}t - \omega_{2n-1} \sin \omega t}{\omega_{2n-1}^2(\omega^2 - \omega_{2n-1}^2)} \sin \frac{(2n-1)\pi x}{b}$$

If $\omega = \omega_m$ (resonance case), m even, solution is given by the above series. If $\omega = \omega_{2q-1}$, then in the series the term involving ω_{2q-1} is replaced by the term

$$\frac{\sin \omega_{2q-1}t - \omega_{2q-1}t \cos \omega_{2q-1}t}{\omega_{2q-1}^3} \sin \frac{(2n-1)\pi x}{b}$$

23 $u = -\dfrac{4F_0}{\pi c} \sum_{n=1}^{\infty} \dfrac{\cos(n\pi/2) \sin(n\pi/6) \sin(n\pi x/b) \sin(n\pi ct/t)}{n}$

25 (d) If $\omega \neq n\pi c/b$, $n = 1, 2, \ldots,$ then $u = v + w$, where

$$v = X(x) \cos \omega t$$

$$X = \frac{h_0 \sin[\omega(b-x)/c] + h_1 \sin(\omega x/c) + X_p(x) \sin(\omega b/c) - \sin(\omega x/c)X_p(b)}{\sin(\omega b/c)}$$

$$X_p(x) = -F_0 \frac{[c\nu \sin(\omega x/c) - \omega \sin \nu x]/(c^2\nu^2 - \omega^2)}{\omega}$$

$$w = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{b} \cos \frac{n\pi ct}{b}$$

$$A_n = \frac{2}{b} \int_0^b [x(b-x) - X(x)] \sin \frac{n\pi x}{b}\, dx$$

27 Let $\{\mu_n\}$ be the sequence of positive roots (in increasing order of magnitude) of the transcendental equation

$$\tan b\mu = \frac{\mu(h_1 + h_2)}{\mu^2 - h_1h_2}$$

Let $\omega_n = c\mu_n$, $n = 1, 2, \ldots.$ Then

$$u = \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin(\mu_n x + \theta_n)$$

$$A_n = \frac{1}{\alpha_n} \int_0^b f(x) \sin(\mu x_n + \theta_n)\, dx \qquad B_n = \frac{1}{\alpha_n \omega_n} \int_0^b g(x) \sin(\mu x_n + \theta_n)\, dx$$

$$\theta_n = \tan^{-1} \frac{\mu_n}{h_1} \qquad \alpha_n = \tfrac{1}{2}\left[b + \frac{(\mu_n{}^2 + h_1h_2)(h_1 + h_2)}{(\mu_n{}^2 + h_1{}^2)(\mu_n{}^2 + h_2{}^2)}\right]$$

29 (a) $u = \left(1 - \frac{x}{b}\right) h_0 \sin \omega t + w_1 + w_2$

$$w_1 = \frac{2\omega h_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin (n\pi x/b) \sin (n\pi ct/b)}{n\omega_n}$$

$$w_2 = -\frac{2}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{g[1 - (-1)^n](1 - \cos \omega_n t)}{n\omega_n^2} + h_0\omega^2 \frac{\omega_n \sin \omega t - \omega \sin \omega_n t}{n(\omega^2 - \omega_n^2)\omega_n} \right\} \sin \frac{n\pi x}{b} \qquad \omega_n = \frac{n\pi c}{b}$$

31 $u = v + w$, where

$$v = \frac{8b^2 e^{-\gamma t}}{\pi^3} \sum_{n=1}^{\infty} \sin \frac{(2n-1)\pi x}{b} \left[\frac{\cos \omega_{2n-1} t + (\gamma/\omega_{2n-1}) \sin \omega_{2n-1} t}{(2n-1)^3} \right]$$

$$w = \frac{2g}{\pi} \sum_{n=1}^{\infty} C_n(t) \sin \frac{n\pi x}{b}$$

$$C_n(t) = [1 - (-1)^n] \frac{e^{-\gamma t}(\gamma \sin \omega_n t + \omega_n \cos \omega_n t) - \omega_n}{n\omega_n(\gamma^2 + \omega_n^2)}$$

$$\omega_n = \left(\frac{n^2\pi^2c^2}{b^2} - \gamma^2 \right)^{1/2} \qquad n = 1, 2, \ldots$$

35 Let $\mu_n = (2n - 1)\pi/2b$, $\omega_n = (c^2\mu_n^2 - \gamma^2)^{1/2}$, $n = 1, 2, \ldots.$ Then

$$u = h_1 x \sin \omega t + v$$

$$v(x,t) = e^{-\gamma t} \left[\frac{2h_1\omega}{b} \sum_{n=1}^{\infty} \frac{(-1)^n}{\omega_n \mu_n^2} (\sin \omega_n t \sin \mu_n x) + w \right]$$

$$w(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{1}{\omega_n} \int_0^t e^{\gamma\xi} \sin [\omega_n(t - \xi)] F_n(\xi)\, d\xi \right\} \sin \mu_n x$$

$$F_n(t) = \frac{2}{b\mu_n^2} [\mu_n F_0 \sin \nu t + \omega h_1(\omega \sin \omega t - 2\gamma \cos \omega t)(-1)^{n+1}]$$

37 $\xi = \sum_{n=1}^{\infty} A_n X_n(x) \cos \omega_n t$

where $X_n(x) = \sin \mu_n x$ and $\{\mu_n\}$ denotes the sequence of positive roots of

$$\tan b\mu = -\frac{EA}{Mc^2\mu}$$

$$A_n = \frac{b_1 - b}{b\alpha_n\mu_n^2} \sin \mu_n b \qquad \alpha_n = \frac{1}{2}\left(b - \frac{Mc^2 \sin^2 \mu_n b}{EA} \right) \qquad \omega_n = c\mu_n;\ n = 1, 2, \ldots$$

39 (a) Let $\omega_n = (n^2\pi^2c^2/b^2 - \omega^2)^{1/2}$, $n = 1, 2, \ldots.$ Then $u = v + w$ where

$$v(x,t) = \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{n\pi x}{b}$$

$$w(x,t) = \frac{4F_0}{\pi} \sum_{n=1}^{\infty} \left[\frac{\nu \sin \omega_{2n-1} t - \omega_{2n-1} \sin \nu t}{(2n-1)\omega_{2n-1}(\nu^2 - \omega_{2n-1}^2)} \right] \sin \left[(2n-1) \frac{\pi x}{b} \right]$$

$$A_n = \frac{2}{b} \int_0^b f(x) \sin \frac{n\pi x}{b}\, dx \qquad B_n = \frac{2}{b\omega_n} \int_0^b g(x) \sin \frac{n\pi x}{b}\, dx$$

41 $u = \sum_{n=1}^{\infty} J_0\left(\xi_n \sqrt{\frac{x}{b}}\right)(A_n \cos \omega_n t + B_n \sin \omega_n t)$

$$A_n = \frac{1}{bJ_1^2(\xi_n)} \int_0^b J_0\left(\xi_n \sqrt{\frac{x}{b}}\right) f(x)\, dx$$

$$B_n = \frac{1}{b\omega_n J_1^2(\xi_n)} \int_0^b J_0\left(\xi_n \sqrt{\frac{x}{b}}\right) g(x)\, dx$$

where $\{\xi_n\}$ denotes the positive zeros of $J_0(\xi)$ arranged in increasing order, and $\omega_n = c\xi_n/2\sqrt{b}$, $n = 1, 2, \ldots$.

53 (c) $u = v + w$, where

$$\varphi_{nm} = \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

$$v = -\frac{16g}{\pi^2} \sum_{n \text{ odd}} \sum_{m \text{ odd}} \left(\frac{1 - \cos \omega_{nm} t}{nm\omega_{nm}^{\ 2}}\right) \varphi_{nm}(x,y)$$

$$w = \frac{256}{\pi^2} \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{\varphi_{nm}(x,y) \cos \omega_{nm} t}{nm(n^2 - 4)(m^2 - 4)}$$

$$\omega_{nm} = c\pi \left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]^{1/2}$$

57 If $F(r,t) = F_0 \sin \omega t$, F_0 constant, and $\omega \neq \omega_n$, all n, then

$$u = 2F_0 \sum_{n=1}^{\infty} \left[(\omega \sin \omega_n t - \omega_n \sin \omega t) \frac{J_0(\xi_n r/a)}{\xi_n \omega_n (\omega^2 - \omega_n^2) J_1(\xi_n)}\right]$$

where $\{\xi_n\}$ denotes the positive zeros of $J_0(\xi)$ and $\omega_n = c\xi_n/a$, $n = 1, 2, \ldots$.

59 If $\omega \neq \omega_{nm} = \xi_{nm}/a$, where $\{\xi_{nm}\}$ is the sequence of positive zeros of $J_n(\xi)$, $n = 0, 1, \ldots$, then $u = \varphi(\theta) \sin \omega t + w(r,\theta,t)$, where

$$w = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\omega \sin \omega_{nm} t - \omega_{nm} \sin \omega t}{\omega_{nm}(\omega^2 - \omega_{nm}^2)} [\alpha_{nm}^{(e)} \varphi_{mn}^{(e)}(r,\theta) + \alpha_{nm}^{(0)} \varphi_{nm}^{(0)}(r,\theta)]$$

$$\alpha_{nm}^{(a)} = \frac{1}{\|\varphi_{nm}^{(e)}\|^2} \int_0^a \int_0^{2\pi} \left[-\omega^2 \varphi(\theta) + \frac{c^2 \varphi''(\theta)}{r^2}\right] \varphi_{nm}^{(e)}(r,\theta) r\, dr\, d\theta$$

with $\alpha_{nm}^{(0)}$ defined similarly by replacing (e) with (0) throughout,

$$\varphi_{nm}^{(e)} = J_n\left(\frac{\xi_{nm} r}{a}\right) \cos n\theta \qquad \varphi_{nm}^{(0)} = J_n\left(\frac{\xi_{nm} r}{a}\right) \sin n\theta$$

$$\|\varphi_{nm}^{(e)}\|^2 = \frac{\pi a^2 J_{n+1}^2(\xi_{nm})}{2} = \|\varphi_{nm}^{(0)}\|^2 \qquad n = 1, 2, \ldots;\ m = 1, 2, \ldots$$

$$\|\varphi_{om}^{(e)}\|^2 = \pi a^2 J_1^2(\xi_{0m}) \qquad m = 1, 2, \ldots$$

62 Eigenvalues $\lambda_{nm} = \xi_{nm}^2/a^2$, where $\{\xi_{nm}\}$ denotes the positive zeros of $J_{n\pi/\beta}(\xi)$; corresponding real-valued eigenfunctions are

$$\varphi_{nm} = J_{n\pi/\beta}\left(\frac{\xi_{nm}r}{a}\right) \sin\frac{n\pi\theta}{\beta} \qquad n = 0, 1, \ldots; m = 1, 2, \ldots$$

$$\|\varphi_{nm}\|^2 = a^2\beta \frac{[J_{n\pi/\beta+1}(\xi_{nm})]^2}{4}$$

Solution is $u = v + w$, where

$$\omega_{nm} = \frac{c\xi_{nm}}{a}$$

$$v = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \varphi_{nm}(r,\theta)(A_{nm} \cos \omega_{nm}t + B_{nm} \sin \omega_{nm}t)$$

$$A_{nm} = \frac{1}{\|\varphi_{nm}\|^2} \int_0^a \int_0^\beta f(r,\theta)\varphi_{nm}(r,\theta)r\, dr\, d\theta$$

$$B_{nm} = \frac{1}{\omega_{nm}\ \|\varphi_{nm}\|^2} \int_0^a \int_0^\beta g(r,\theta)\varphi_{nm}(r,\theta)r\, dr\, d\theta$$

$$w = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{\frac{1}{\omega_{nm}} \int_0^t F_{nm}(\xi) \sin [\omega_{nm}(t - \xi)]\, d\xi\right\} \varphi_{nm}(r,\theta)$$

$$F_{nm}(t) = \frac{1}{\|\varphi_{nm}\|^2} \int_0^a \int_0^\beta F(r,\theta,t)\varphi_{nm}(r,\theta)r\, dr\, d\theta$$

$$G(r,\theta,t;r_0,\theta_0,\tau) = \frac{4}{\beta a^2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\varphi_{nm}(r,\theta)\varphi_{nm}(r_0,\theta_0) \sin [\omega_{nm}(t - \tau)]}{\omega_{nm}\ \|\varphi_{nm}\|^2}$$

63 Eigenvalues $\lambda_{nm} = \xi_{nm}^2$, where, for each $n = 0, 1, \ldots$, $\{\xi_{nm}\}$ denotes the sequence of positive roots of the equation

$$J_n(a\xi)Y_n(b\xi) - J_n(b\xi)Y_n(a\xi) = 0$$

Corresponding real-valued eigenfunctions are

$$\varphi_{nm}^{(e)} = [J_n(\xi_{nm}r)Y_n(\xi_{nm}a) - J_n(\xi_{nm}a)Y_n(\xi_{nm}r)] \cos n\theta \qquad n = 0, 1, \ldots$$

$$\varphi_{nm}^{(0)} = [J_n(\xi_{nm}r)Y_n(\xi_{nm}a) - J_n(\xi_{nm}a)Y_n(\xi_{nm}r)] \sin n\theta \qquad n = 1, 2, \ldots$$

$$\|\varphi_{nm}\|^2 = 2\epsilon_n \frac{J_n^2(\xi_{nm}a) - J_n^2(\xi_{nm}b)}{\pi\xi_{nm}^2 J_n^2(\xi_{nm}b)}$$

where $\epsilon_0 = 2$, $\epsilon_n = 1$, $n \geq 1$. Characteristic frequencies $\omega_{nm} = c\xi_{nm}$, normal modes

$$u_{nm} = [A_{nm}\varphi_{nm}^{(e)}(r,\theta) + B_{nm}\varphi_{nm}^{(0)}(r,\theta)] \cos \omega_{nm}t + [C_{nm}\varphi_{nm}^{(e)}(r,\theta) + D_{nm}\varphi_{nm}^{(0)}(r,\theta)] \sin \omega_{nm}t$$

where it is understood that $\varphi_{0m}^{(e)} = 0$, $m = 1, 2, \ldots$. The formal series solution is identical in form to the solution obtained in Example 4-2 with the φ_{nm} defined above and the region of integration the annulus $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$.

69 (a) $u = \dfrac{A}{r} + Bt$

75 $G = 0$ if $t \le \tau$, and if $t > \tau$, then

$$G(r,\theta,t;r_0,\theta_0,\tau) = \frac{3(t-\tau)}{a^3} + \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} \frac{\psi_{2k,m}(r,\theta)\psi_{2k,m}(r_0,\theta_0)\sin[\omega_{2k,m}(t-\tau)]}{\omega_{2k,m}\,\|\psi_{2k,m}\|^2}$$

where

$$\psi_{2k,m} = j_{2k,m}\left(\frac{\xi_{2k,m}r}{a}\right)P_{2k}(\cos\theta) \qquad k = 0, 1, \ldots; m = 1, 2, \ldots$$

$$\|\psi_{2k,m}\|^2 = \frac{a^3[1 - 2k(2k+1)/\xi^2_{2k,m}]\,j_{2k}{}^2(\xi_{2k,m})}{2(4k+1)}$$

and, for each $k = 0, 1, \ldots$, $\{\xi_{2k,m}\}$ denotes the positive zeros of $j'_{2k}(\xi)$, $\omega_{2k,m} = c\xi_{2k,m}/a$ are the characteristic frequencies. If $F(r,\theta,t) = F_0 \sin \omega t$, $f(r,\theta) = h(r)\cos \omega t$, $g = 0$, then the solution is

$$u = \frac{F_0}{\omega^2}(\omega t - \sin \omega t) + v$$

$$v = A_0 + \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} A_{2k,m}\psi_{2k,m}(r,\theta)\cos\omega_{2k,m}t$$

$$A_0 = \frac{3}{2a^3}\int_0^a r^2 h(r)\,dr$$

$$A_{2k,m} = -\frac{P_{2k}(0)}{(2k-1)(2k+2)\,\|\psi_{2k,m}\|^2}\int_0^a r^2 h(r) j_{2k}\left(\frac{\xi_{2k,m}r}{a}\right)dr$$

$$P_0(0) = 1 \qquad P_{2k}(0) = (-1)^k\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdots(2k)} \qquad k \ge 1$$

79 $u = V(r,z)\cos p\theta \cos \omega t + W(r,z,t)\cos p\theta$

$$V(r,z) = \sum_{m=1}^{\infty} A_m J_p\left(\frac{\xi_m r}{a}\right)\cosh \mu_m z$$

$$W(r,z,t) = \sum_{m=1}^{\infty}\sum_{q=0}^{\infty} B_{mq} J_p\left(\frac{\xi_m r}{a}\right)\cos\frac{q\pi z}{h}\cos\omega_{mq}t$$

where $\{\xi_m\}$ denotes the positive zeros of $J'_p(\xi)$ arranged in order of increasing magnitude, $\mu_m = (\xi_m{}^2/a^2 - \omega^2/c^2)^{1/2}$, $\omega_{mq} = c(\xi_m{}^2/a^2 + q^2\pi^2/h^2)^{1/2}$,

$$A_m = \frac{2\xi_m{}^2}{a^2(\xi_m{}^2 - p^2)\mu_m \sinh(\mu_m h) J_p{}^2(\xi_m)}\int_0^a J_p\left(\frac{\xi_m r}{a}\right)\varphi(r) r\,dr$$

$$B_{mq} = \frac{2A_m(-1)^{q+1}\mu_m \sinh(\mu_m h)}{h(\mu_m{}^2 + q^2\pi^2/h^2)}$$

$$B_{m0} = -\frac{A_m \sinh \mu_m h}{h\mu_m} \qquad m = 1, 2, \ldots; q = 1, 2, \ldots$$

CHAPTER 5

3 $u = (u_1 - u_2)\operatorname{erf}\delta_1 + u_2 \operatorname{erf}\delta_2$

5 (a) $u = u_0 \operatorname{erf}\dfrac{x}{\sqrt{4\kappa t}}$ (c) $u = u_1 + (u_2 - u_1)\operatorname{erf}\dfrac{x}{\sqrt{4\kappa t}}$

7 $u = u_0 e^{-bt}\operatorname{erf}\dfrac{x}{\sqrt{4\kappa t}}$

11 (a) $u = \dfrac{4L}{\pi^2}\sum_{k=1}^{\infty}\dfrac{(-1)^{k+1}}{(2k-1)^2}\sin\dfrac{(2k-1)\pi x}{L}\, e^{-(2k-1)^2\pi^2\kappa t/L^2}$

(c) $u = u_1 + (u_2 - u_1)\dfrac{x}{L} + v(x)$

$$v(x) = \sum_{n=1}^{\infty}\left[B_n + 2\frac{(-1)^n u_2 - u_1}{n\pi}\right]\sin\frac{n\pi x}{L}\, e^{-n^2\pi^2\kappa t/L^2}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

(e) $u = \dfrac{q_0 x}{K} + \sum_{n=1}^{\infty} A_n \sin\dfrac{(2n-1)\pi x}{2L}\, e^{-(2n-1)^2\pi^2\kappa t/4L^2}$

$$A_n = \frac{-4Lq_0}{\pi^2 K(2n-1)^2} + \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx$$

13 $u = A[1 + h(L - x)]\dfrac{\sin\omega_1 t}{1 + hL} + v(x,t)$

$$v(x,t) = \sum_{n=1}^{\infty}[C_n(t) + B_n]\sin\mu_n x$$

where $\{\mu_n\}$ denotes the positive roots of $\tan\mu L = -\mu/h$, and

$$C_n(t) = \left[\frac{F_0\sin\mu_n x_0}{\kappa^2\mu_n^4 + \omega^2}(\kappa\mu_n^2\cos\omega t + \omega\sin\omega t - \kappa\mu_n^2 e^{-\kappa\mu_n^2 t}) - \frac{A\omega_1}{\mu_n(\kappa^2\mu_n^4 + \omega_1^2)}(\kappa\mu_n^2\cos\omega_1 t + \omega_1\sin\omega_1 t - \kappa\mu_n^2 e^{-\kappa\mu_n^2 t})\right]\Big/\|X_n\|^2$$

$$\|X_n\|^2 = \frac{L}{2} + \frac{\cos^2\mu_n L}{2h} \qquad B_n = \frac{1}{\|X_n\|^2}\int_0^L f(x)\sin\mu_n x\,dx$$

15 $u = 8u_0\sum_{n=1}^{\infty}\dfrac{J_0(\xi_n r/a)}{\xi_n^3 J_1(\xi_n)}\, e^{-\kappa\xi_n^2 t/a}$

where $\{\xi_n\}$ denotes the positive zeros of $J_0(\xi)$.

17 $u = u_0 + \sum_{q=1}^{\infty} A_q \sin\dfrac{(2q-1)\pi z}{2b}\, e^{\kappa\lambda_q t}$

$$+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{q=1}^{\infty}[A_{nmq}\psi_{nmq}^{(e)}(r,\theta,z) + B_{nmq}\psi_{nmq}^{(0)}(r,\theta,z)]e^{-\kappa\lambda_{nmq}t}$$

where

$$\lambda_q = \frac{(2q-1)^2\pi^2}{4b^2} \qquad \lambda_{nmq} = \frac{\xi_{nm}{}^2}{a^2} + \frac{(2q-1)^2\pi^2}{4b^2} \qquad n = 0, 1, \ldots; m = 1, 2, \ldots; \; q = 1, 2, \ldots$$

and, for each n, $\{\xi_{nm}\}$ denotes the sequence of positive zeros of $J_n'(\xi)$,

$$A_q = \frac{-4u_0}{(2q-1)\pi^2a^2} + \frac{2}{\pi a^2 b}\int_0^a\int_0^{2\pi}\int_0^b f(r,\theta,z)\sin\frac{(2q-1)\pi z}{2b}\, r\,dr\,d\theta\,dz$$

$$A_{nmq} = \frac{1}{\|\psi_{nmq}\|^2}\int_0^a\int_0^{2\pi}\int_0^b f(r,\theta,z)\psi_{nmq}^{(e)}(r,\theta,z)r\,dr\,d\theta\,dz \qquad n = 0, 1, \ldots; m = 1, 2, \ldots; \; q = 1, 2, \ldots$$

$$B_{nmq} = \frac{1}{\|\psi_{nmq}\|^2}\int_0^a\int_0^{2\pi}\int_0^b f(r,\theta,z)\psi_{nmq}^{(0)}(r,\theta,z)r\,dr\,d\theta\,dz \qquad n = 1, 2, \ldots; m = 1, 2, \ldots; \; q = 1, 2, \ldots$$

$$\psi_{nmq}^{(e)} = J_n\left(\frac{\xi_{nm}r}{a}\right)\cos n\theta \sin\frac{(2q-1)\pi z}{2b}$$

$$\psi_{nmq}^{(0)} = J_n\left(\frac{\xi_{nm}r}{a}\right)\sin n\theta \sin\frac{(2q-1)\pi z}{2b}$$

$$\|\psi_{nmq}^{(e)}\|^2 = \|\psi_{nmq}^{(0)}\|^2 = \frac{\pi a^2 b}{4\xi_{nm}{}^2}(\xi_{nm}{}^2 - n^2)J_n{}^2(\xi_{nm}) \qquad n = 1, 2, \ldots$$

$$\|\psi_{0mq}\|^2 = \frac{\pi a^2 b}{2}J_0{}^2(\xi_{0m}) \qquad m = 1, 2, \ldots; q = 1, 2, \ldots$$

21 $u = \dfrac{1}{r}\sum_{n=1}^{\infty} A_n \sin(\mu_n r)e^{-\kappa\mu_n{}^2 t}$ where $\{\mu_n\}$ denotes the sequence of positive roots of $\tan a\mu = a\mu/(1-ah)$ and

$$A_n = 4\frac{a^2\mu_n + (ah-1)^2}{a^2\mu_n{}^2 + ah(ah-1)}\;\frac{3h - a\mu_n{}^2}{\mu_n{}^4}\sin a\mu_n$$

23 $u = \dfrac{q_0a^2}{K}\left(\dfrac{bh-1}{b^2h} - \dfrac{1}{r}\right) + w(r,t)$

$$w(r,t) = \sum_{m=1}^{\infty} A_m\varphi_m(r)e^{-\kappa\xi_m{}^2 t}$$

where $\{\xi_m\}$ denotes the sequence of positive roots of the equation

$$\xi[j_0'(a\xi)y_0'(b\xi) - y_0'(a\xi)j_0'(b\xi)] + h[j_0'(a\xi)y_0(b\xi) - y_0'(a\xi)j_0(b\xi)] = 0$$

$$\varphi_m(r) = f_0(\xi_m r)y_0'(\xi_m a) - j_0'(\xi_m a)y_0(\xi_m r)$$

$$A_m = \frac{1}{\alpha_m}\left[u_0\int_a^b \varphi_m(r)r^2\,dr - \frac{q_0a^2}{K}\int_a^b\left(\frac{bh-1}{b^2h} - \frac{1}{r}\right)\varphi_m(r)r^2\,dr\right]$$

$$\alpha_m = \int_a^b \varphi_m{}^2(r)r^2\,dr \qquad j_0(x) = \sqrt{\frac{\pi}{2x}}J_{1/2}(x) \qquad y_0(x) = \sqrt{\frac{\pi}{2x}}Y_{1/2}(x)$$