# Computer Lab 1: Heat Diffusion in Rods using *Maple*

The diffusion of heat in a one-dimensional rod of length $L$ (with insulated surface, uniform heat conductivity, and no heat sources) satisfies the one-dimensional *diffusion equation*

$$u_t = k u_{xx}, \qquad \text{for } 0 < x < L \text{ and } t > 0. \tag{1}$$

Of course, we shall need to know the *initial temperature:*

$$u(x,0) = f(x), \qquad \text{for } 0 < x < L, \tag{2}$$

as well as the *boundary conditions (B.C.'s)* at $x = 0$ and $x = L$. When the ends of the rod are maintained at temperature zero, we have the boundary conditions

$$u(0,t) = 0 = u(L,t), \qquad \text{for } t > 0; \tag{3a}$$

when the ends of the rod are insulated, we have the boundary conditions

$$u_x(0,t) = 0 = u_x(L,t), \qquad \text{for } t > 0; \tag{3b}$$

The method of *separation of variables,* $u(x,t) = X(x)T(t)$, leads us to consider the eigenvalue problem

$$\begin{aligned} X'' + \lambda X = 0 \quad & \text{for } 0 < x < L \\ B.C.'s \quad & \text{at } x = 0 \text{ and } x = L. \end{aligned} \tag{4}$$

Finding the eigenvalues $\lambda_n$ and eigenfunctions $X_n$ for (4) will enable us to write our solution as an infinite series involving unknown coefficients; for (3a) the series involves sine functions, and for (3b) it involves cosine functions. The initial conditions are then used to evaluate the coefficients. By taking a *partial sum* of the series, we obtain a good approximation of the solution which may then be plotted using a graphics program such as *Maple.*

**I. Comments on Maple.** *Maple V* is available on NUNet, under Statistical and Computational Packages. The most recent version is R4=Release 4, but R3 is also available. You may use either version (or even R2 if you find it somewhere), although there are some significant differences. For one thing, in R4 plots are displayed on the worksheet (you'll know what I'm talking about when you try it) whereas in R3 they appear in a separate window. However, R4 takes a lot of RAM: plots can take a while to print, and the system can crash, so save your work often; or use R3. In fact, if you want to save time, you may use R3 and not even print the plots; just print the command window that shows you got the right plot (I've seen them before, after all).

Although *Maple V* is a very powerful computational program, it can be very tempermental about your input. If you do not give it commands in precisely the required form, it will not accept them, and probably will not tell you what is wrong; so be very careful.

If you open *Maple V,* you will find yourself in the command window. Enter

$$> \texttt{with(plots)};$$

This loads the main plotting package that we shall be using; its various plotting capabilities have been listed on the screen. (If you had typed `with(plots)` : instead, i.e. replace the semicolon with a colon, the plotting package would still be loaded, but the output is suppressed, i.e. the plotting capabilities are not displayed.)

**II. Heat Diffusion in a Rod with Ends Maintained at Zero Temperature.** In the case (3a), the eigenvalue problem (4) takes the form

$$X''+\lambda X = 0 \quad \text{for } 0 < x < L,$$
$$X(0) = 0 = X(L),$$

which has eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin\frac{n\pi x}{L}, \quad \text{for } n = 1, 2, \ldots$$

Having found the values of $\lambda$, we may solve the time equation $T' + k\lambda T = 0$ to obtain

$$T_n(t) = A_n e^{-k\lambda_n t} = A_n e^{-kn^2\pi^2 t/L^2}.$$

Finally, superposition yields the series solution

(5a) $$u(x,t) = \sum_{n=1}^{\infty} A_n\, e^{-kn^2\pi^2 t/L^2}\, \sin\frac{n\pi x}{L}.$$

We evaluate the $A_n$'s using the initial condition $u(x,0) = \phi(x)$ and the orthogonality of the sine functions:

(5b) $$A_n = \frac{2}{L}\int_0^L f(x)\, \sin\frac{n\pi x}{L}\, dx.$$

To be specific, let us take

(6) $$L = 1, \quad k = 2, \quad \text{and} \quad f(x) = x(1-x).$$

We cannot ask *Maple* to plot the graph of the infinite series (5ab), but if we consider the partial sum

(7) $$S_N(x,t) = \sum_{n=1}^{N} A_n\, e^{-2n^2\pi^2 t}\, \sin n\pi x, \quad A_n = 2\int_0^1 x(1-x)\, \sin n\pi x\, dx,$$

then we can ask *Maple* to graph $S_N(x,t)$. *Maple* uses `int(f(x),x=a..b)` for $\int_a^b f(x)dx$ and `sum(a(n),n=1..N)` for $\sum_{n=1}^N a(n)$. We therefore calculate (7) by entering

```
>A := n -> 2 * int(x * (1 - x) * sin(n * Pi * x), x = 0..1);
>S := (N, x, t) -> sum(A(n) * exp(-2 * n^2 * Pi^2 * t) * sin(n * Pi * x), n = 1..N);
```

**Warning:** You must use "Pi" and not "pi"; otherwise *Maple* treats it symbolically, not numerically.

In these commands, the `->` represents an arrow. In other words, $A_n$ is regarded as a map or function taking $n$ to the number $A_n$. Similarly, $S_N(x,t)$ is regarded as a mapping or function taking the triplet $(N,x,t)$ to the number $S_N(x,t)$. We specify $N$, say $N=3$, by entering

```
> S(3, x, t);
```

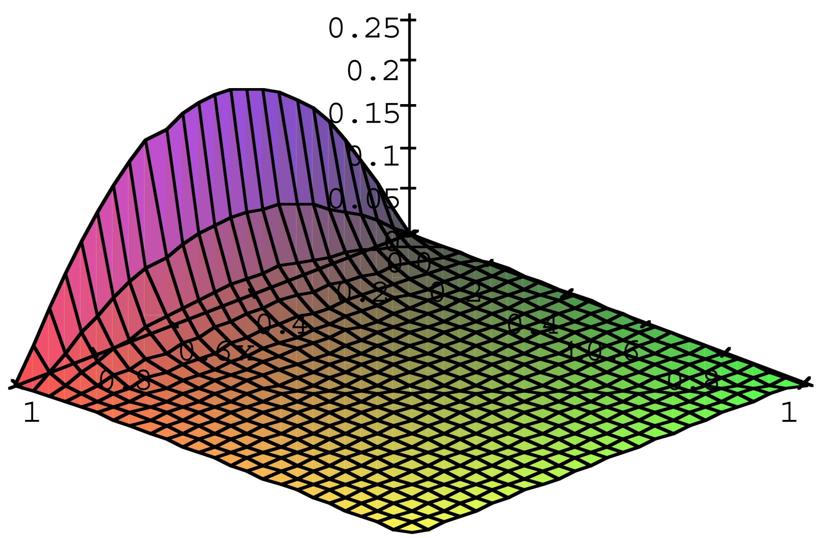
The result, $S_3(x,t) =$ `S(3, x, t)`, is a function of $x$ and $t$.

In order to plot $S_3(x,t)$ for $0 < x < 1$ and $0 < t < 1$, load the plotting package (if you have not already done so) and then enter

```
> plot3d(S(3, x, t), x = 0..1, t = 0..1, axes = normal);
```

The result appears as Figure 1a; as expected, the heat rapidly diffuses to zero.

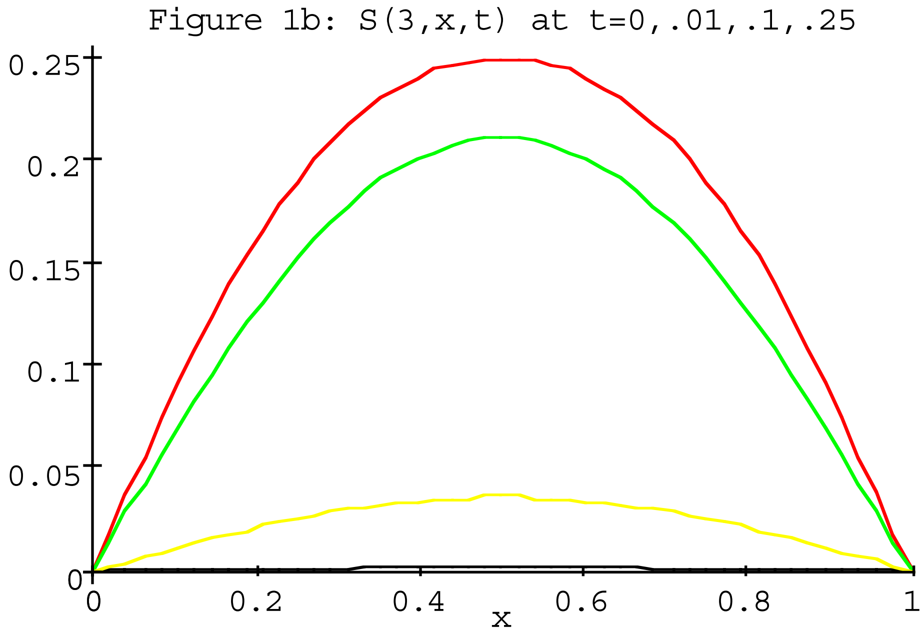
Figure 1: Solution to (4) with u(x,0)=x(1-x)




Another way to view the diffusion is by graphing $S_3(x,t)$ as a function of $x$ when $t$ is fixed; this amounts to taking a "snapshot" of the temperature at that time. For example, we may take $t = 0, .01, .1, .25$ and graph the snapshots $S_3(x,0)$, $S_3(x,.01)$, $S_3(x,.1)$, and $S_3(x,.25)$ on the same axes by

```
> plot({S(3, x, 0), S(3, x, .01), S(3, x, .1), S(3, x, .25)}, x = 0..1, axes = normal);
```

The result appears as Figure 1b.




Again we can see that the heat diffuses rapidly to zero. In fact, after only .01 seconds, the temperature at the middle of the bar ($x = .5$) has decreased from .25 to approximately .2, i.e., a loss of almost 20% of its temperature.

**Exercise 1.** Let $L = 2$, $k = 1$, and $f(x) = x^2(2 - x)$.

(a) Use *Maple* to obtain a 3d-plot of $S_3(x,t)$ for $0 < x < 2$ and $0 < t < 1$.

(b) Use *Maple* to plot snapshots of the diffusion at $t = 0$, $t = .01$, $t = .1$, $t = .25$, and $t = 1$.

(c) Describe what is happening. In particular, is the temperature always decreasing at every point $x$?

**HAND IN:** Printouts of your plots in (a) and (b), and a written answer (using complete sentences) to (c).

**III. Heat Diffusion in a Rod with Insulated Ends.** In the case (3b), the corresponding eigenvalue problem (4) is

$$X'' + \lambda X = 0 \quad \text{for } 0 < x < L,$$
$$X'(0) = 0 = X'(L),$$

which has eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \cos\frac{n\pi x}{L}, \quad \text{for } n = 0, 1, \ldots$$

Notice that $\lambda_0 = 0$ and $X_0(x) = 1$. The corresponding series solution is now

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\, e^{-kn^2\pi^2 t/L^2} \cos\frac{n\pi x}{L}, \tag{8a}$$

where

(8b) $$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}.$$

**Exercise 2.** Let $L = 2$, $k = 1$, and $f(x) = x^2(2 - x)$ for a rod with insulated ends.

(a) Use *Maple* to obtain a 3d-plot of $S_3(x, t)$ for $0 < x < 2$ and $0 < t < 1$.

(b) Use *Maple* to plot snapshots of the diffusion at $t = 0$, $t = .01$, $t = .1$, $t = .25$, and $t = 1$.

(c) Interpret your results physically, comparing and contrasting them with those of Exercise 1. In particular, what happens as $t \to \infty$?

**HAND IN:** Printouts of your plots in (a) and (b), and a written answer (using complete sentences) to (c).

# Computer Lab 2: Convergence of Fourier Series

The Fourier series for a function $f(x)$ defined on $(-L, L)$ is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos(\frac{n\pi x}{L}) + B_n \sin(\frac{n\pi x}{L})\right), \tag{1}$$

where

$$A_n = (1/L)\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx \quad (n = 0, 1, \ldots)$$
$$B_n = (1/L)\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx \quad (n = 1, 2, \ldots).$$

But (1) is an infinite series, so it is natural to wonder whether, and in what sense, the series actually converges to $f(x)$. To be more precise, let us define the partial sum

$$S_N(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N}\left(A_n \cos(\frac{n\pi x}{L}) + B_n \sin(\frac{n\pi x}{L})\right), \tag{2}$$

and then ask whether, and in what sense,

$$S_N(x) \to f(x) \qquad \text{as } N \to \infty. \tag{3}$$

A computational/graphics software package such as *Maple* is useful in graphing $S_N(x)$ and $f(x)$, which sheds some light on this issue.

**Example 1.** Let $f(x) = x$ for $-\pi < x < \pi$. Since $f$ is odd, we get a Fourier sine series:

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin(nx) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx), \tag{4}$$

where we have used $\sim$ to indicate that we are not yet certain whether the series actually converges to $f(x)$. Notice that the Fourier coefficients

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = 2\frac{(-1)^{n+1}}{n} \tag{5}$$

are computed by integrating by parts; but *Maple* can also compute them for you. Enter

```
>B := n -> (1/Pi) * int(x * sin(n * x), x = -Pi..Pi);
>B(n);
```

and *Maple* produces the output

$$-2\frac{-\sin(n\pi) + n\cos(n\pi)\pi}{\pi n^2}. \tag{6}$$

Of course, *Maple* doesn't know that $n$ is an integer, but using $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, we easily simplify (6) to obtain (5).

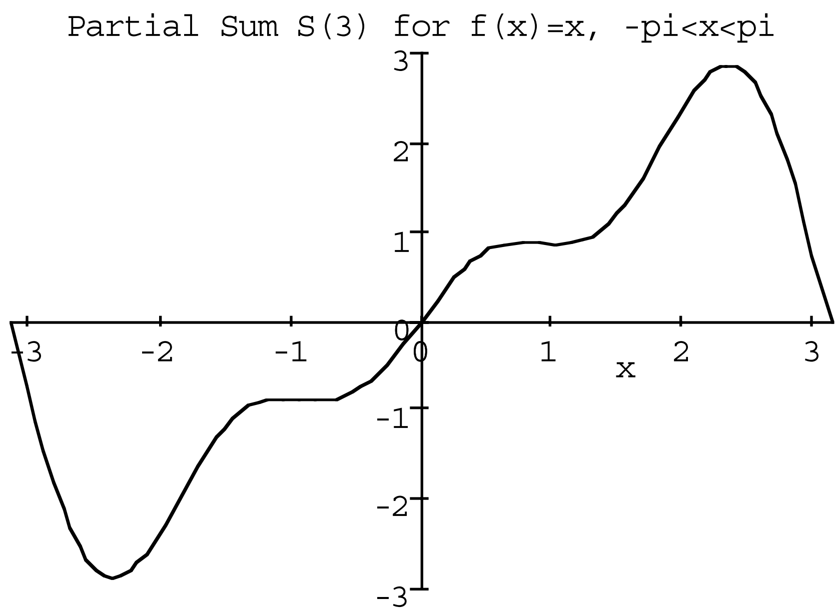
Now let us define the partial sums in *Maple:*

```
> S := N -> sum(B(n) * sin(n * x), n = 1..N);
```

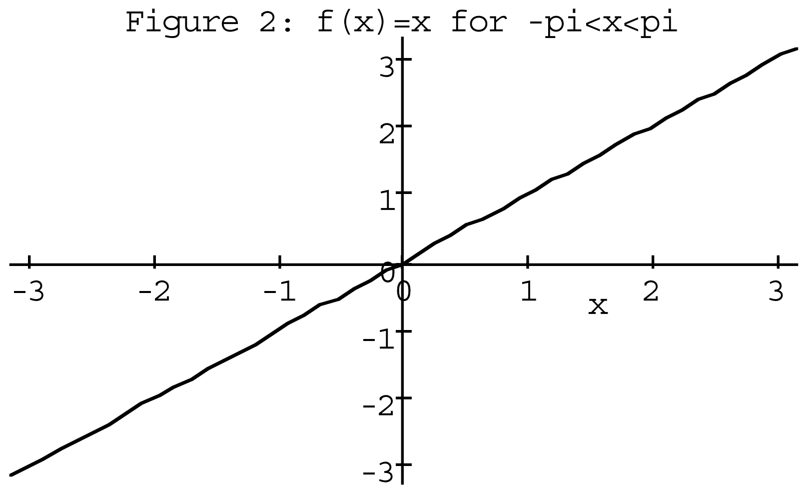
To plot the partial sums with $N = 3$, for example, load the plotting package and enter

```
>S(3);
>plot(", x = -Pi..Pi);
```

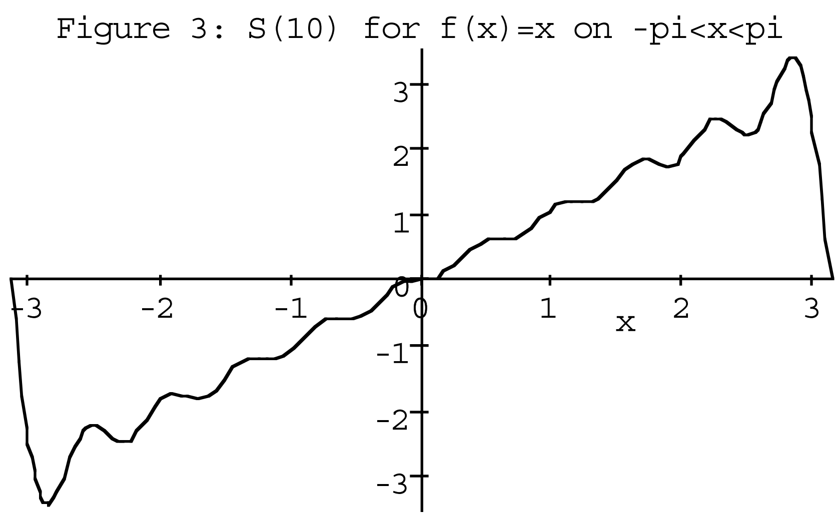
to obtain the 2-D plot in Figure 1. (Notice that " tells *Maple* to plot the previous line.)




Of course, we want this partial sum to be a rough approximation of $f(x) = x$, whose graph appears in Figure 2.




Comparing Figures 1 and 2, we see that the approximation is not too bad in the middle, but is extremely bad at the ends $x = \pm\pi$; this is because all the $\sin(nx)$ terms must vanish there even though $f(x) = x$ does not. If we take larger $N$ in our partial sums, we might expect the approximation to become better and better in the middle, but remain fairly poor at the ends $x = \pm\pi$. Indeed, we can use *Maple* to plot $S_{10}(x)$ (i.e., $S(10)$ as we have called it in *Maple*) to obtain Figure 3.



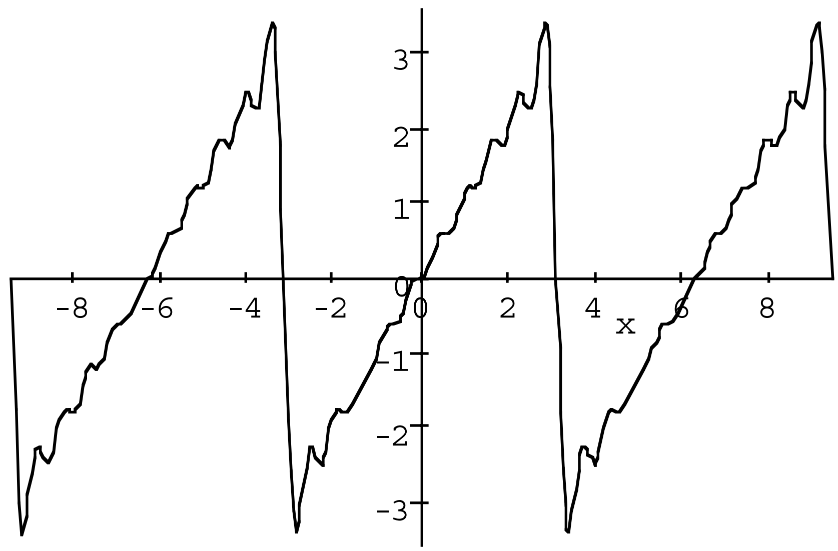

Notice that, as expected, this approximation to Figure 2 is pretty good in the middle, but lousy at the ends. In fact, no matter how large we take $N$, there will be points near $x = \pm\pi$ for which the approximation of $S_N(x)$ to $f(x)$ will be off by more than 3 (since the jump down to the $x$-axis is approximately $\pi$). For this reason, we say that $S_N(x)$ is *not uniformly close* to $f(x)$ on $(-\pi, \pi)$.

**Exercise 1.** Plot the partial sum $S_{20}(x)$ and $S_{30}(x)$ for $f(x) = x$ on $-\pi < x < \pi$. Would you say $S_N(x) \to x$ for every $x$ in $(-\pi, \pi)$? What happens to $S_N(\pm\pi)$ as $N \to \infty$? Is $S_N(x)$ uniformly close to $f(x)$ on $(-\pi, \pi)$?

**HAND IN:** Printouts of the plots for $S_{20}(x)$ and $S_{30}(x)$, and answers to the questions.

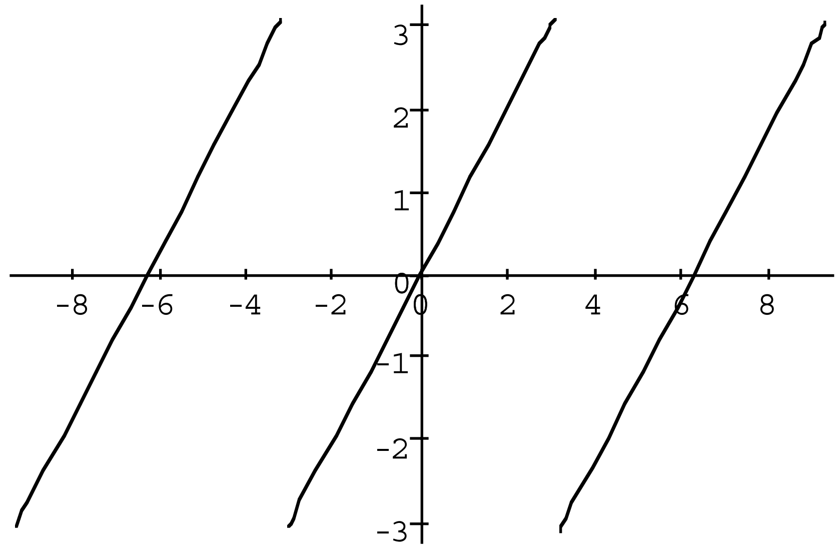
The convergence of $S_N(x)$ as $N \to \infty$ illustrates the convergence theorem ("Fourier's theorem") in Section 3.2 of the text by Haberman. To see this, let us recall that the Fourier series (1) can be viewed as a *periodic function* (with period $2L$) for $-\infty < x < \infty$. We have plotted the periodic function $S_{10}(x)$ for Example 1 in Figure 4.

Figure 4: S(10) for f(x)=x on -3pi<x<3pi




On the other hand, the function $f(x) = x$ on $(-L, L)$ also admits a periodic extension to $-\infty < x < \infty$. We have plotted this periodic extension of $f(x) = x$ in Figure 5.

Figure 5: Periodic Extension of Example 1



Notice that, where the periodic extension of $f(x) = x$ is continuous (namely for all points *except* $x = \pm\pi, \pm 3\pi, \ldots$), we see that $S_N(x) \to f(x)$. *But* for $x = \pm\pi, \pm 3\pi, \ldots$ we have

$$S_N(x) = 0 = \frac{f(x^+) + f(x^-)}{2}.$$

On the other hand, when the periodic extension of $f(x)$ is itself continuous, then the approximation of $S_N(x)$ to $f(x)$ may be uniformly good. We say that $S_N(x) \to f(x)$ *uniformly on* $(-L, L)$ if $|S_N(x) - f(x)| < \epsilon$ for all $-L < x < L$ provided $N$ is sufficiently large.

**Exercise 2.** Consider $f(x) = 1 - x^2$ on $-1 < x < 1$.

(a) Plot $f(x)$, $S_2(x)$, and $S_4(x)$ on $-1 < x < 1$.

(b) Pick a large $N$ and plot the graph of $S_N(x)$ on $(-1, 1)$. Would you say that $S_N(x) \to f(x)$ *uniformly on* $(-1, 1)$ *as* $N \to \infty$*?*

**HAND IN:** (a) A printout of your plots for $f(x)$, $S_2(x)$ and $S_4(x)$. (b) Your value of $N$, the plot of $S_N(x)$, and your answer to the question.

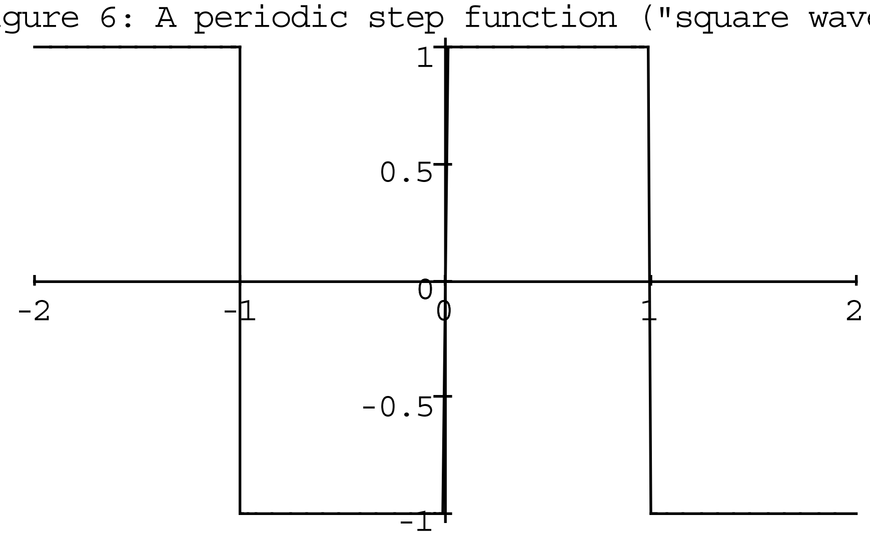
The jump phenomenon in Figures 4 and 5 is rather interesting, so let us focus on it in the next example.

**Example 2.** Let $f(x)$ be the "step function" (or "square wave") defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ -1 & \text{if } -1 < x < 0, \end{cases} \tag{7}$$

and then extended periodically (with period 2) to all values of $x$; see Figure 6.

Figure 6: A periodic step function ("square wave")




Notice that $L = 1$. Since $f(x)$ is an odd function, the Fourier series (1) is a sine series

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin(n\pi x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(1 - (-1)^n)}{n} \sin(n\pi x) \tag{8}$$

where the coefficients $B_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_0^1 \sin(n\pi x)\,dx - \int_{-1}^{0} \sin(n\pi x)\,dx$ are easily computed by hand or using *Maple;* notice that $B_2 = B_4 = \cdots = 0$. Now let us use *Maple* to plot some of the partial sums $S_N(x)$ to see how closely they approximate (7):

```
>S := N -> sum(2*(1-(-1)^n)*sin(n*Pi*x)/(Pi*n), n = 1..N);
>S(3);
>plot(", x = -2..2);
```

produces the plot of $S_3(x)$ in Figure 7a, and similarly we obtain the plots for $S_5(x)$ and $S_7(x)$ in Figures 7b and 7c respectively. Notice that $S_3(x)$ approximates $f(x) = 1$ for $0 < x < 1$ by an oscillation with 2 peaks; $S_5(x)$ has 3 peaks and $S_7(x)$ has 4 peaks in their oscillatory approximations of $f(x) = 1$.

**Exercise 3.** How many peaks does $S_4(x)$ have in its oscillatory approximation of $f(x) = 1$ on $0 < x < 1$? Compare $S_4(x)$ with $S_3(x)$ and $S_5(x)$; explain why.

**HAND IN:** No printout, just the number of peaks and your comparison/explanation.

Figure 7a: S(3) for Example 2

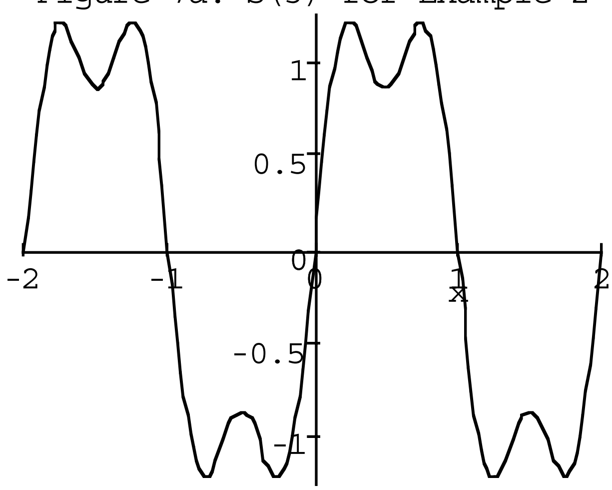



Figure 7b: S(5) for Example 2

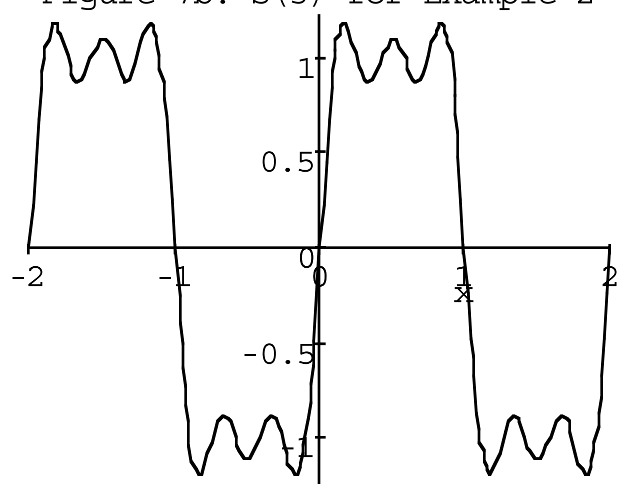
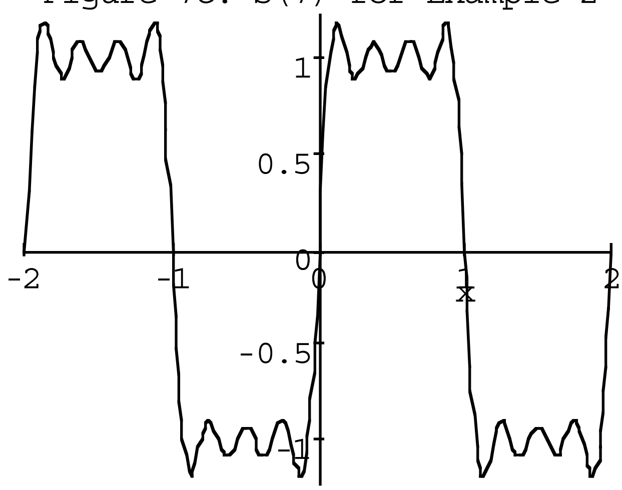



Figure 7c: S(7) for Example 2



There are some additional curious features about the graphs of $S_3$, $S_5$, and $S_7$:

(i) Notice that in each of them, $S_N(0) = 0$. This is, of course, because where $f(x)$ jumps from $-1$ to 1 (or vice-versa) the partial sum "averages" these two values to get 0.

(ii) The amplitude of the oscillations of the partial sums $S_N(x)$ on $0 < x < 1$ become smaller as $N$ increases, so they become better approximations for $f(x) = 1$. However, the first peak after the jump at $x = 0$ and the last peak before the jump at $x = 1$ are both higher than the peaks in between. It seems like jumps cause the approximation to "overshoot" the desired amplitude. This is called the *Gibbs phenomenon.*

**Exercise 4.** Plot $S_{15}(x)$ and $S_{25}(x)$ for $f(x)$ as in Example 2. Circle where the Gibbs phenomenon occurs. For what values of $x$ in $(-1, 1)$ do we have $S_N(x) \to f(x)$?

**HAND IN:** Your plots with Gibbs locations circled, and your answer to the question.

# Computer Lab 3: Vibrating Strings

A string which undergoes small transverse vibrations satisfies the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \qquad \text{for } 0 < x < L \text{ and } t > 0, \tag{1}$$

where $L$ is the length of the string and $c = \sqrt{T/\rho}$ is the propagation speed; $T$ is the tension and $\rho$ is the density of the string. We shall assume that the string is fixed at the ends, which leads to the boundary conditions

$$u(0,t) = 0 = u(L,t), \qquad \text{for } t > 0. \tag{2}$$

We also need initial conditions; because (1) is second-order in $t$, we need to know both the initial position and the initial velocity of the string:

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \quad \text{for } 0 < x < L. \tag{3}$$

The method of *separation of variables,* $u(x,t) = X(x)T(t)$, leads to the familiar eigenvalue problem $X'' + \lambda X = 0$ on $0 < x < L$ with $X(0) = 0 = X(L)$, so we obtain eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots \tag{4}$$

The corresponding time equation is $T'' + c^2\lambda T = 0$ which we solve for each $\lambda_n$ to find

$$T_n(t) = A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}. \tag{5}$$

Using superposition, we obtain the infinite series solution of (1)-(2):

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{cn\pi t}{L}\sin\frac{n\pi x}{L} + B_n \sin\frac{cn\pi t}{L}\sin\frac{n\pi x}{L} \tag{6}$$

The coefficients $A_n$ and $B_n$ are chosen in order to satisfy (3), i.e.

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \tag{7a}$$

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{L}\sin\frac{n\pi x}{L}. \tag{7b}$$

**I. Vibrations of a Plucked String.** Suppose a string has length $L$, uniform density $\rho$, and is fixed at its ends under a tension $T$; let $c^2 = T/\rho$. If the middle point is displaced $a$ units from its equilibrium position and then released, the displacement $u(x,t)$ satisfies (1)-(3), where

$$g(x) = 0 \quad \text{and} \quad f(x) = \begin{cases} 2ax/L & \text{for } 0 < x \leq L/2 \\ 2a(L-x)/L & \text{for } L/2 \leq x < L. \end{cases}$$

We conclude from (7b) that $B_n = 0$ for all $n$ in (6). The $A_n$, on the other hand, are the Fourier sine coefficients of $f(x)$; in order to compute them, let us take the specific values

$$L = 1 = c^2 = a.$$

Then

$$A_n = 2\left(\int_0^{1/2} 2x\sin(n\pi x)\,dx + \int_{1/2}^{1} 2(1-x)\sin(n\pi x)\,dx\right),$$

and we can take a partial sum in (6) to get an approximation for $u(x,t)$:

$$S_N(x,t) = \sum_{n=1}^{N} A_n \cos\frac{cn\pi t}{L}\sin\frac{n\pi x}{L}.$$

It is not difficult to compute $A_n$ (using integration by parts), but why not ask *Maple* to do it? Let us enter

```
>A := n -> 2 * (int(2 * x * sin(n * Pi * x), x = 0..0.5)
   +int(2 * (1 - x) * sin(n * Pi * x), x = 0.5..1));
>S := (N, x, t) -> sum(A(n) * cos(n * Pi * t) * sin(n * Pi * x), n = 1..N);
```

In order to plot $S_5(x,t)$ for $0 < x < 1$ and $0 < t < 4$, load the plotting package (if you have not already done so) and then enter

```
>S(5, x, t);
>plot3d(", x = 0..1, t = 0..4, axes = normal);
```

You should be able to see periodic wave motion. Do you know what the period is?

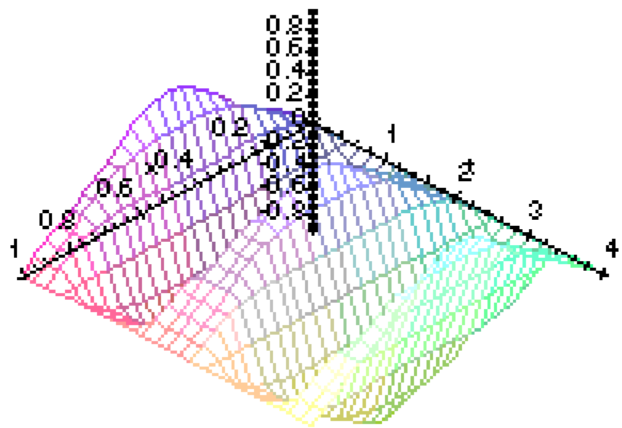
Alternatively, you can plot various "snapshots" of $u(x,t)$ by fixing $t$ at various values, say at $t = 0$, $t = .2$, $t = .4$, and $t = .6$:

```
> plot({S(5, x, 0), S(5, x, .2), S(5, x, .4), S(5, x, .6)}, x = 0..1, axes = normal);
```

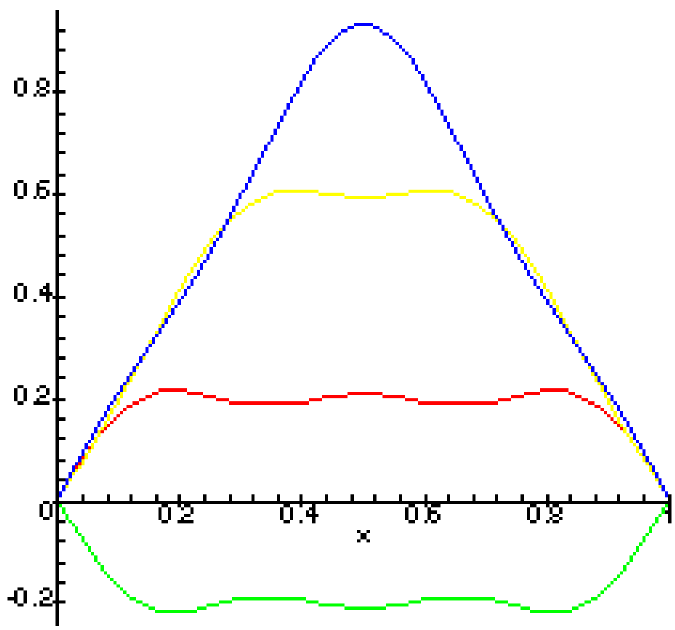
The period $t^*$ is the smallest positive value for which $u(x,t) = u(x,t+t^*)$ for all $t > 0$. When do you think $S(5,x,t^*) = S(5,x,0)$?

The plots appear below.

S(5,x,t) for Plucked String




Snapshots of Plucked String at t=0,.2,.4,.6




**Exercise 1.** Let us change the values above to $L = 2$ and $c^2 = 1 = a$.

(a) Obtain a 3d plot of the partial sum $S_5(x, t)$ for $0 < x < 2$ and $0 < t < 4$.

(b) Graph the snapshots at several values of $t$ for $S_5(x, t)$ on $0 < x < 2$.

(c) Is this motion periodic? In particular, is there a time $t^* > 0$ at which the displacement $u(x, t^*)$ looks *exactly* like $u(x, 0)$? If so, find the smallest such $t^* > 0$.

**HAND IN:** Your printouts for (a) and (b), and your written answer to (c).

**II. Vibrations of a Struck String.** Instead of plucking the string, let us now leave the string in its equilibrium position, but strike the string with a hammer. Let us suppose that the hammer has a circular head of radius $r$ and is moving with velocity $v$ when it strikes the string precisely at its midpoint $x = L/2$. Taking $t = 0$ to be the time of impact, we obtain the initial conditions (3) with

$$f(x) = 0 \quad \text{and} \quad g(x) = \begin{cases} v & \text{for } (L/2) - r \le x \le (L/2) + r \\ 0 & \text{otherwise.} \end{cases}$$

We conclude from (7a) that $A_n = 0$ for all $n$ in (6). The $B_n$, on the other hand, are found from the Fourier cosine coefficients of $g(x)$; in order to compute them, let us take the specific values $L = 4$ and $c^2 = 1 = v = r$, so

$$g(x) = \begin{cases} 1 & \text{for } 1 \le x \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

This means that

$$B_n \frac{n\pi}{4} = B_n \frac{cn\pi}{L} = \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx = \int_1^3 \sin \frac{n\pi x}{4}\, dx,$$

and we can take a partial sum in (6) to get an approximation for $u(x,t)$:

$$S_N(x,t) = \sum_{n=1}^{N} B_n \sin \frac{n\pi t}{4} \sin \frac{n\pi x}{4}$$

**Exercise 2.** (a) Obtain a 3d plot of the partial sum $S_5(x,t)$ for $0 < x < 4$ and $0 < t < 4$.

(b) Graph the time slices of $S_5(x,t)$ on $0 < x < 4$ for several fixed values of $t$.

(c) Is this motion periodic? In particular, is there a time $t^* > 0$ at which the displacement $u(x,t^*)$ looks *exactly* like $u(x,0)$? If so, find the smallest such $t^* > 0$.

(d) Suppose you used a larger hammer (larger $r$) or hit the string with more velocity (larger $v$). Would the value $t^*$ change? Why or why not?

**HAND IN:** Your printouts for (a) and (b), and your written answer to (c) and (d).

# Computer Lab 4: Boundary Conditions and Eigenvalues

Let us consider a metal rod of length $L$ which is insulated except at the ends, where heat is allowed to escape according to Newton's law of cooling; as we have seen, this means that a Robin boundary condition applies at each end. Let us suppose that the two ends have different Robin conditions. By separation of variables $u(x,t) = \phi(x)T(t)$, this leads us to study eigenvalue problems of the form

$$\begin{gathered} \phi'' + \lambda\phi = 0 \qquad \text{for} \quad 0 < x < L, \\ \phi'(0) - a_0\phi(0) = 0 = \phi'(L) + a_L\phi(L), \end{gathered} \tag{1}$$

where $a_0$ and $a_L$ are constants (which are both *positive* if heat is *radiating* out of each end).

## I. Positive Eigenvalues

Let us look for positive eigenvalues $\lambda$ for (1); when $a_0$ and $a_L$ are both positive, the text verifies that (1) only admits nontrivial solutions when $\lambda > 0$. The general solution

$$\phi(x) = C\cos\beta x + D\sin\beta x \qquad (\lambda = \beta^2,\ \beta > 0) \tag{2}$$

is differentiated ($\phi'(x) = -C\beta\sin\beta x + D\beta\cos\beta x$) and substituted into the boundary conditions of (1) to obtain

$$D\beta - a_0C = 0 = -C\beta\sin\beta L + D\beta\cos\beta L + a_L(C\cos\beta L + D\sin\beta L).$$

Rewriting this in matrix notation, we obtain

$$\begin{pmatrix} -a_0 & \beta \\ a_L\cos\beta L - \beta\sin\beta L & \beta\cos\beta L + a_L\sin\beta L \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3}$$

Since we want a nontrivial solution $C, D$ of (3), the $2\times 2$ matrix in (3) must be singular, i.e. have zero determinant:

$$-a_0(\beta\cos\beta L + a_L\sin\beta L) - \beta(a_L\cos\beta L - \beta\sin\beta L) = 0.$$

This last equation may be simplified to

$$(\beta^2 - a_0a_L)\tan\beta L = (a_0 + a_L)\beta. \tag{4}$$

We are interested in finding the solutions $\beta$ of (4). This is easily done using a computational software package such as *Maple.*

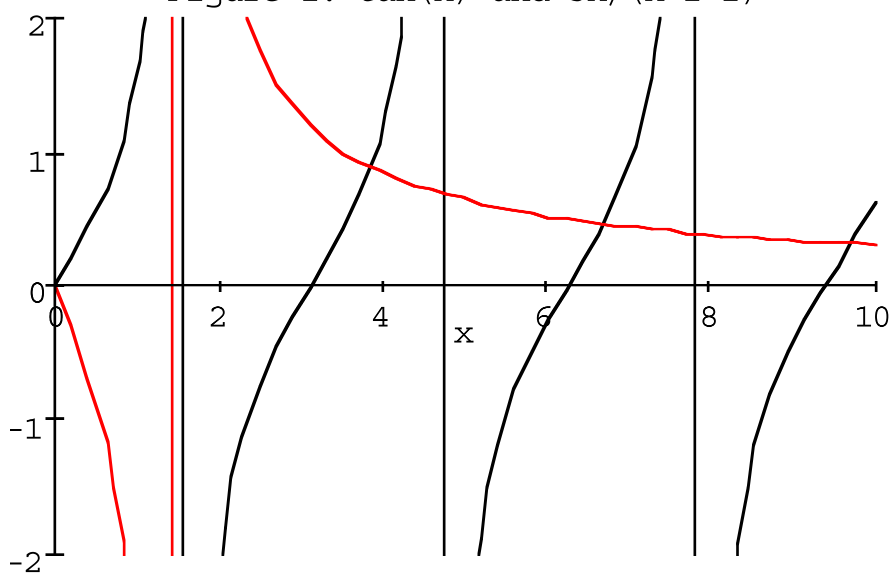
**Example 1.** Let us take $a_0 = 1$, $a_L = 2$, and $L = 1$. Then (4) becomes

$$\tan\beta = \frac{3\beta}{(\beta^2 - 2)} \tag{5}$$

Let us use *Maple* to view the solutions of (5). Enter

(6)
```
>with(plots);
>plot({tan(x), 3*x/(x^2 - 2)}, x = 0..10, -2..2);
```

in order to plot both $y = \tan x$ and $y = 3x/(x^2 - 2)$ on the same axes for the range of values $0 < x < 10$ and $-2 < y < 2$. The result appears in Figure 1.




Notice that the intersections of the curves represent solutions of (4). (Notice that we must ignore intersections with vertical lines which *Maple* has plotted to represent vertical asymptotes.) Since we are assuming $\beta > 0$, the first of these intersections seems to occur just before $x = 4$. To find a more accurate value, we can ask *Maple* to solve the equation $\tan x = 3x/(x^2 - 2)$ by entering

```
> fsolve(tan(x) = 3*x/(x^2 - 2));
```

(As you might expect from our experience in Lab 2, `f` in `fsolve` stands for "floating point arithmetic.") The 10-digit answer we obtain is

3.871244368,

and if we enter the command

```
> evalf((")^2);
```

we obtain

`14.98653296.`

This means that we have found values for $\beta_1$ and $\lambda_1 = \beta_1^2$:

$$\beta_1 \approx 3.871244368 \quad \text{and} \quad \lambda_1 \approx 14.98653296.$$

The next point of intersection of the curves in Figure 1 seems to occur between $x = 6$ and $x = 7$. In order to use *Maple* to find it, we must include the range $6 \le x \le 7$ to tell *Maple* to zero in on that particular solution:

```
> fsolve(tan(x) = 3*x/(x^2 - 2), x = 6..7);
```

The 10-digit answer we obtain is

`6.720171109,`

and if we square this as before we get

`45.16069973.`

This means that we have found values for $\beta_2$ and $\lambda_2 = \beta_2^2$:

$$\beta_2 \approx 6.720171109 \quad \text{and} \quad \lambda_2 \approx 45.16069973.$$

**Exercise 1.** Find $\beta_3$ and $\lambda_3$ to 10 decimal places (You need not print-out anything.)

We now see how to use (4) to generate the infinite sequence of eigenvalues; how do we find the associated eigenfunctions? We know they are of the form (2), but what are $C$ and $D$? Recall from (3) that $a_0 C = \beta D$, i.e., $D = a_0 C/\beta$. Substituting into (2), we find

$$\phi(x) = C\left(\cos\beta x + \frac{a_0}{\beta}\sin\beta x\right), \qquad \text{where } C \text{ is arbitrary.} \tag{7a}$$

Thus, for a particular eigenvalue $\lambda_n = \beta_n^2$, we find that

$$\phi_n(x) = \cos\beta_n x + \frac{a_0}{\beta_n}\sin\beta_n x \tag{7b}$$

is the associated eigenfunction.

**Example 1 (Revisited).** With $a_0 = 1$, we find (7b) becomes

$$\phi_n(x) = \cos\beta_n x + \frac{1}{\beta_n}\sin\beta_n x,$$

giving us the eigenfunctions for the specific values of $\beta_n$ that we have generated.

**Exercise 2.** For the values $a_0 = 1/2$, $a_L = 1$, and $L = 3$, generate a plot similar to Figure 1. Then find decimal approximations for the first three eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$, and find the associated eigenfunctions $\phi_1$, $\phi_2$, and $\phi_3$. (*Hint:* look for $0 < \beta_1 < 1$.)
**HAND IN.** A print-out of your plot; also the values for $\lambda_n$ and formulas for $\phi_n(x)$ (not printed-out).

## II. Negative Eigenvalues

It is possible, when at least one of $a_0$ or $a_L$ is negative, that there are negative eigenvalues $\lambda = -\beta^2 < 0$ for (1). The general solution is

$$\phi(x) = C \cosh \beta x + D \sinh \beta x \qquad (\beta > 0) \tag{8}$$

where $\cosh z = (e^z + e^{-z})/2$ and $\sinh z = (e^z - e^{-z})/2$. Differentiation yields $\phi'(x) = C\beta \sinh \beta x + D\beta \cosh \beta x$, and substitution into the boundary conditions of (1) yields

$$D\beta - a_0 C = 0 = C\beta \sinh \beta L + D\beta \cosh \beta L + a_L(C \cosh \beta L + D \sinh \beta L).$$

Writing this as a system, and taking the determinant as before yields

$$-a_0(\beta \cosh \beta L + a_L \sinh \beta L) - \beta(a_L \cosh \beta L + \beta \sinh \beta L) = 0.$$

Simplification yields

$$\tanh \beta L = -\frac{(a_0 + a_L)\beta}{\beta^2 + a_0 a_L}. \tag{9}$$

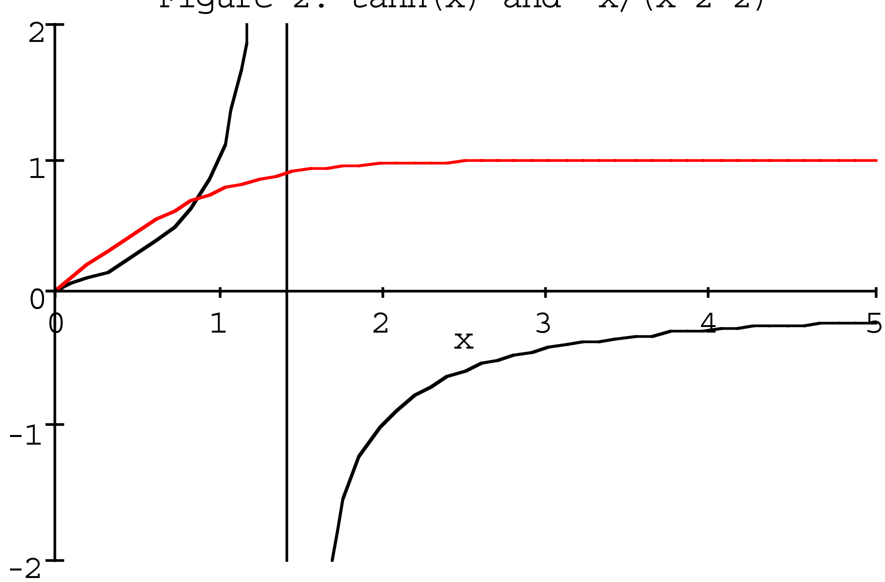
Depending on the values of $a_0$, $a_L$, and $L$, this may or may not have a solution $\beta$; if a solution $\beta$ *does* exist, then $\lambda = -\beta^2$ is a negative eigenvalue for (1) with eigenfunction given by (8) with $D = a_0 C/\beta$ as before. We shall denote the negative eigenvalue by "$\lambda_0$."

**Example 2.** $a_0 = -1$, $a_L = 2$, $L = 1$. Then (9) becomes

$$\tanh \beta = -\frac{\beta}{\beta^2 - 2}. \tag{10}$$

Let us use *Maple* to plot $y = \tanh x$ and $y = -x/(x^2 - 2)$; see Figure 2.

Figure 2: tanh(x) and -x/(x^2-2)



Moreover, we can use *Maple* to approximate the intersection point in Figure 2 which occurs between $x = .5$ and $x = 1$. If we enter

$$> \texttt{fsolve}(\texttt{tanh}(\texttt{x}) = -\texttt{x}/(\texttt{x\^{}2} - 2), \texttt{x} = .5..1);$$

we obtain for $\beta$ the value

```
.8711803575.
```

This means that $\lambda = -\beta^2$ has the value

$$\lambda_0 = -.789552153.$$

**Exercise 3.** For the values $a_0 = -1/2$, $a_L = 1$, and $L = 3$, generate a plot similar to Figure 2, and determine the negative eigenvalue $\lambda_0$ (if it exists).
**HAND IN.** A print-out of your plot; also the value for $\lambda_0$ (if it exists).

## III. Inside Maple's "Black Box"

**Exercise 4.** Be prepared to venture a verbal opinion in class as to how *Maple* is able to solve equations like (5) or (10). **(Not to hand in.)**

# Computer Lab 5: Diffusion and Vibrations in Rectangular Domains

We want to be able to solve the diffusion equation

$$u_t = k\Delta u, \tag{1}$$

and the wave equation

$$u_{tt} = c^2\Delta u, \tag{2}$$

on two-dimensional domains $D$. Of course, we shall need to impose *boundary conditions* (which for now we shall abbreviate as *B.C.'s on* $\partial D$*),* and *initial conditions* at $t = 0$. The method of *separation of variables,* $u(x, y, t) = v(x, y)T(t)$, leads us to consider the eigenvalue problem

$$\begin{aligned} \Delta v + \lambda v = 0 \quad &\text{in } D \\ B.C.'s \quad &\text{on } \partial D. \end{aligned} \tag{3}$$

Finding the eigenvalues $\lambda_n$ and eigenfunctions $v_n$ for (3) will enable us to write our solution as an infinite series involving unknown constants; the initial conditions are then used to evaluate the constants. By taking a *partial sum* of the series, we obtain a good approximation of the solution which may then be plotted using a graphics program such as *Maple.*

In this lab, we shall assume that the domain $D$ is a rectangle:

$$D = \{(x, y) : 0 < x < L,\ 0 < y < H\}. \tag{4}$$

Having introduced $u(x, y, t) = v(x, y)\,T(t)$, it is natural to use the separation of variables $v(x, y) = X(x)\,Y(y)$ in order to solve (3).

**I. Diffusion of Heat in a Rectangular Plate.**

The eigenvalue problem (3) will yield eigenvalues $\lambda_{mn}$ and eigenfunctions $v_{mn}(x, y)$. For (1), the associated time equation $T' + k\lambda T = 0$ may then be solved as usual to obtain $T_{mn}(t) = A_{mn}e^{-k\lambda_{mn}t}$, and hence series solutions of the form

$$u(x, y, t) = \sum_{m,n} A_{mn}e^{-k\lambda_{mn}t}v_{mn}(x, y). \tag{5}$$

As usual, the coefficients $A_{mn}$ are determined by the initial condition

$$u(x, y, 0) = f(x, y); \tag{6}$$

specifically, we find the eigenfunction expansion

$$f(x,y) = \sum_{mn} A_{mn} v_{mn}(x,y), \tag{7}$$

and use the coefficients $A_{mn}$ in (5).

Of course, the exact values $\lambda_{mn}$ and functions $v_{mn}$ depend on the $B.C.'s$ imposed. Let us consider an example.

**Example 1 (Plate Edges Maintained at Zero Temperature).** If we maintain the edges of the plate (4) at zero temperature, then the $B.C.'s$ on $v$ (and hence on $u$) are

$$v(x,y) = 0 \quad \text{for } (x,y) \text{ on } \partial D = \text{edges of } D. \tag{8}$$

The $\lambda_{mn}$ and $v_{mn}(x,y)$ are found by first solving

$$\begin{aligned} X'' + \mu X = 0, &\quad X(0) = 0 = X(L), \\ Y'' + \nu Y = 0, &\quad Y(0) = 0 = Y(H), \end{aligned} \tag{9}$$

to find

$$\begin{aligned} \mu_m = m^2\pi^2/L^2, &\quad X_m(x) = \sin(m\pi x/L), \quad m = 1, 2, \ldots \\ \nu_n = n^2\pi^2/H^2, &\quad Y_n(y) = \sin(n\pi y/H), \quad n = 1, 2, \ldots, \end{aligned} \tag{10}$$

and then letting $\lambda = \mu + \nu$ and $v(x,y) = X(x)\,Y(y)$ to find

$$\lambda_{mn} = \pi^2 \left( \frac{m^2}{L^2} + \frac{n^2}{H^2} \right), \quad v_{mn}(x,y) = \sin(\frac{m\pi x}{L})\,\sin(\frac{n\pi y}{H}). \tag{11}$$

Plugging into (5), we obtain

$$u(x,y,t) = \sum_{m,n=1}^{\infty} A_{mn} e^{-k\pi^2(\frac{m^2}{L^2}+\frac{n^2}{H^2})t} \sin(\frac{m\pi x}{L})\,\sin(\frac{n\pi y}{H}). \tag{12}$$

To be specific, let us take

$$k = 1, \quad L = H = 1, \quad f(x,y) = xy. \tag{13}$$

This means that the $A_{mn}$ are found by

$$A_{mn} = 4 \int_0^1 \int_0^1 xy \sin(m\pi x) \sin(n\pi y)\,dxdy$$

which can be easily computed by integration by parts (in each integral $\int dx$ and $\int dy$). But let us ask *Maple* to compute the $A_{mn}$ and then plot the partial sum

$$S_N(x,y,t) = \sum_{m,n=1}^{N} A_{mn} e^{-\pi^2(m^2+n^2)t} \sin(m\pi x) \sin(n\pi y).$$

For example, with $N = 3$ we have

```
>A := (m,n)-> 4*int(x*sin(m*Pi*x),x=0..1)
    *int(y*sin(n*Pi*y),y=0..1);
>S := (N,t)-> sum(sum(A(m,n)*exp(-Pi^2*(m^2+n^2)*t)
    *sin(m*Pi*x)*sin(n*Pi*y),m=1..N),n=1..N);
>with(plots):
>plot3d(S(3,.01),x=0..1,y=0..1,axes=normal,scaling=constrained);
>plot3d(S(3,.02),x=0..1,y=0..1,axes=normal,scaling=constrained);
>plot3d(S(3,.03),x=0..1,y=0..1,axes=normal,scaling=constrained);
```

which produces the series of plots on the next page (we have adjusted the viewing angle to $\varphi = 60°$).

As we might have expected, the heat is diffusing out of the plate, and the temperature is rapidly approaching zero as $t$ increases.

**Exercise 1 (Plate with Insulated Edges).** Let $k = 1$, and consider (1) in the plate (4) with $L = H = 1$ and boundary conditions:

$$\frac{\partial u}{\partial n} = 0 \quad \text{for } (x, y) \text{ on } \partial D, \tag{14}$$

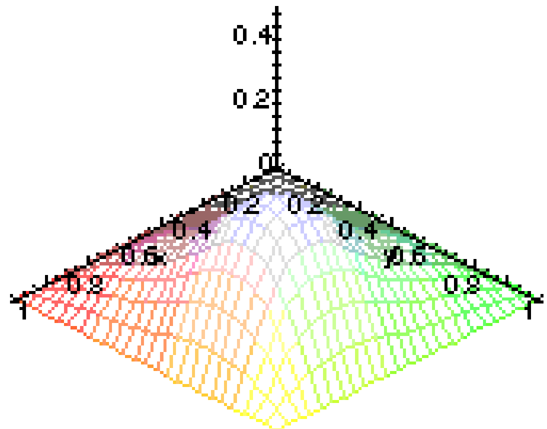
where $n$ denotes the normal direction along $\partial D$. Suppose the initial temperature is

$$u(x, y, 0) = f(x, y) = xy. \tag{15}$$

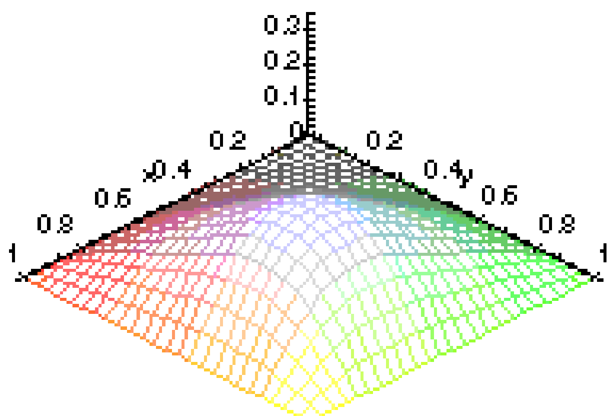
(a) Modify the above analysis for (8) to apply to the B.C. (14).
(b) Use *Maple* to obtain 3d snapshots of the partial sum $S_3(x, y, t)$ for several small values of $t$.
(c) Describe in words what is happening to the temperature. In particular, what happens as $t \to \infty$?

**HAND IN:** Your printouts for (a) and (b), and your written answer to (c).

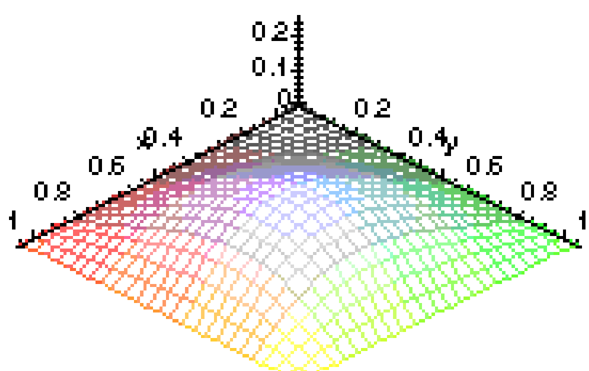
Example 1: Temperature at t=.01




Example 1: Temperature at t=.02




Example 1: Temperature at t=.03



**II. Vibrations of a Rectangular Drum.** If the domain $D$ in (4) represents a rectangular drumhead which is fixed along its boundary, then vibrations of the drumhead are found by solving (2) with the boundary conditions (8). This means that the $\lambda$ and $v(x,y)$ are again given by (11), and it only remains to solve the time equation $T'' + c^2\lambda T = 0$ for each $\lambda_{mn}$. We obtain

$$T_{mn}(t) = A_{mn}\cos(c\sqrt{\lambda_{mn}}\,t) + B_{mn}\sin(c\sqrt{\lambda_{mn}}\,t).$$

Using superposition $u(x,y,t) = \sum_{m,n} v_{mn}(x,y)\,T_{mn}(t)$, we obtain our series solution

$$(16)\quad u(x,y,t) = \sum_{m,n=1}^{\infty}\Big(A_{mn}\cos(c\sqrt{\lambda_{mn}}\,t) + B_{mn}\sin(c\sqrt{\lambda_{mn}}\,t)\Big)\sin(\frac{m\pi x}{L})\,\sin(\frac{n\pi y}{H}).$$

As usual, the coefficients $A_{mn}, B_{mn}$ are determined from the initial conditions.

**Example 2 (Drum is Struck from Below).** Suppose the drum is initially at equilibrium ($u = 0$), but is *struck from below* by an impulse creating the initial conditions

$$(17)\qquad u(x,y,0) = 0 \quad \text{and} \quad u_t(x,y,0) = 1.$$

From the first initial condition in (17), we obtain $A_{mn} = 0$ in (16); the second initial condition in (17) will enable us to compute the $B_{mn}$ in (16).

To be specific, let us take

$$(18)\qquad c = 1, \quad L = 1 = H.$$

The second initial condition then yields

$$u_t(x,y,0) = \sum_{m,n}^{\infty} B_{mn}\sqrt{\lambda_{mn}}\sin(m\pi x)\,\sin(n\pi y) = 1,$$

so that $B_{mn}\sqrt{\lambda_{mn}}$ are the Fourier coefficients for the double sine series for 1:

$$B_{mn}\sqrt{\lambda_{mn}} = 4\int_0^1\int_0^1 \sin(m\pi x)\,\sin(n\pi y)\,dxdy = 4\,\frac{(1-(-1)^m)(1-(-1)^n)}{mn\pi^2}.$$

(Notice that $B_{mn} = 0$ unless $m$ and $n$ are *both odd.*) Solving for $B_{mn}$, using $\sqrt{\lambda_{mn}} = \pi\sqrt{m^2+n^2}$, and plugging into (16), we obtain

$$u(x,y,t) = \sum_{m,n=1}^{\infty} B_{mn}\sin(\pi\sqrt{m^2+n^2}t)\sin(m\pi x)\,\sin(n\pi y)$$

$$\text{where } B_{mn} = 4\,\frac{(1-(-1)^m)(1-(-1)^n)}{mn\pi^3\sqrt{m^2+n^2}}.$$

To get a feeling for the behavior of $u(x, y, t)$, let us plot the partial sum

$$S_N(x, y, t) = \sum_{m,n=1}^{N} B_{mn} \sin(\pi\sqrt{m^2 + n^2}t) \sin(m\pi x) \sin(n\pi y), \tag{19}$$

since, for large $N$, we expect $S_N(x, y, t)$ to be a good approximation to $u(x, y, t)$. Let us use "snapshots" of the vibration at different times, in order to get a feeling for the evolution of the vibration (like watching a movie frame by frame). To achieve this using *Maple,* let us use the commands

```
>B := (m,n) -> 4*(1-(-1)^n)*(1-(-1)^m)/(m*n*Pi^3*sqrt(m^2+n^2))
>S := (N,t) -> sum(sum(B(m,n)*sin(Pi*sqrt(m^2+n^2)*t)
    *sin(m*Pi*x)*sin(n*Pi*y),m=1..N),n=1..N);
```

As far as *Maple* is concerned, `S(N,t)` is a function of $x$ and $y$ which can then be plotted using `plot3d`.

For example, let us take $N = 5$. (For the diffusions above, we generally used $N = 3$, but for vibrations one generally should use larger $N$; can you guess why?) Let us also take $t = 1/4 = .25$:

```
>plot3d(S(5,.25),x=0..1,y=0..1,axes=normal);
```

The result appears in Figure 2a; notice that, in response to the initial impulse, the center of the drumhead has raised upward; of course the edges remain clamped in place.

**Exercise 2.** In Figure 2a, notice that the raised portion of the drumhead looks rather "bumpy." Is this a feature of the vibration, or caused by our truncation $N = 5$? To investigate this, replace $N = 5$ by a larger number; e.g. with $N = 15$, try plotting $S(15, .25)$. Have the "bumps" disappeared? What is your conclusion about the cause of the bumps for $N = 5$?

**HAND IN:** A copy of your plot and a written conclusion about the cause of the "bumps" in Figure 2a.

Next let us consider $t = .5$:

```
>plot3d(S(5,.5),x=0..1,y=0..1,axes=normal);
```

The result is Figure 2b; the change of scale on the vertical axis shows that the middle of the plate has risen higher than in Figure 2a. In Figure 2c we see $t = .75$; by now the vibration forces the drumhead to stretch downward (the edges still clamped in place).

Figure 2a: S(5,.25)

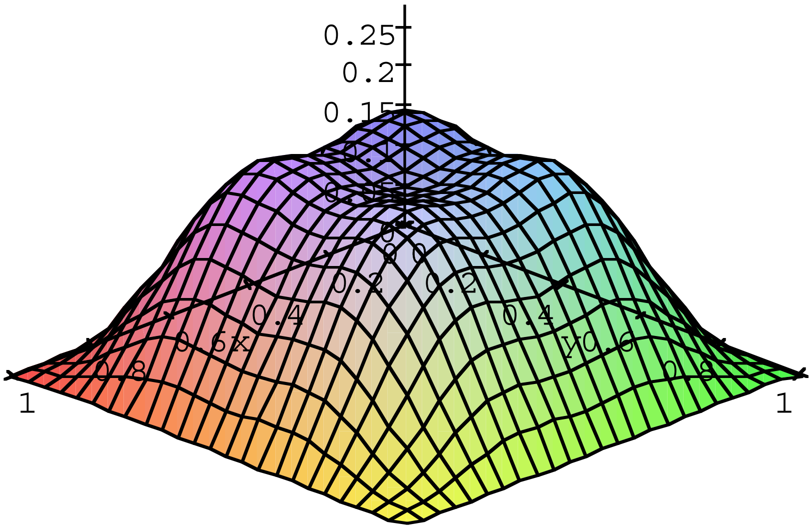



Figure 2b: S(5,.5)

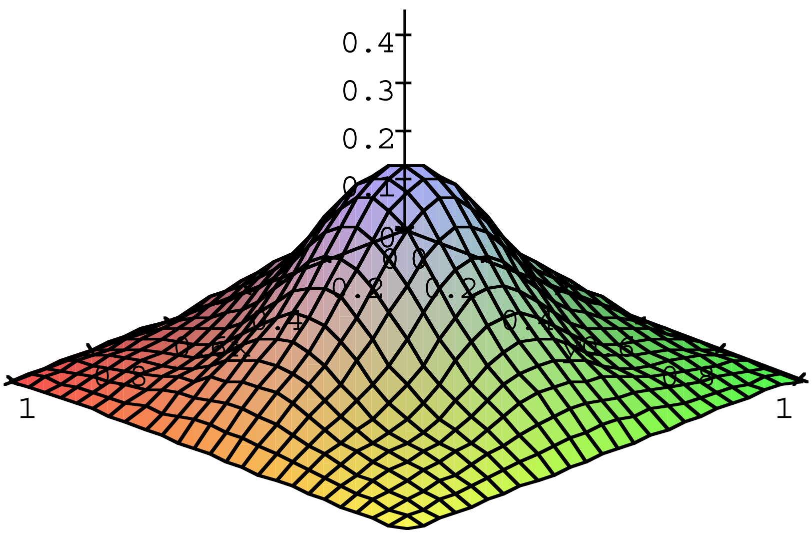
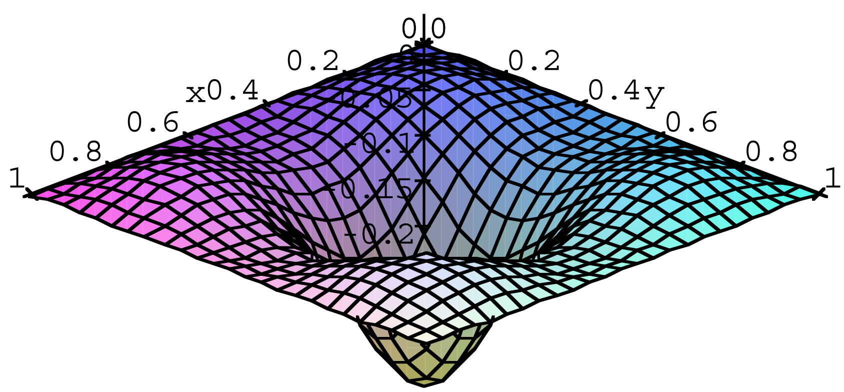



Figure 2c: S(5,.75)



**Exercise 3.** (a) Plot `S(5,t)` for the values $t = 1$, $t = 1.5$, and $t = 2$.

(b) Describe the vibration in words, referring to the snapshots.

(c) Do you think that the "peaks" in $S(5, 2)$ are effects of the approximation with $N$ rather small? Try taking $N = 15$. What do you observe? What do you conclude?

(d) Is this motion periodic? In particular, is there a time $t^* > 0$ at which the plate is *exactly flat* (as it was when $t = 0$)? If you think the motion is periodic, try using *Maple* to find $t^*$; if you think it is not periodic, refer to (18) to explain why not.

**HAND IN:** Printouts of your snapshots in (a), clearly labeled with the value of $t$. Also, your written answers to (b), (c), and (d).

# Computer Lab 6: Applications to Circular Domains

If we consider the diffusion equation

$$u_t = k\Delta u, \tag{1}$$

or the wave equation

$$u_{tt} = c^2\Delta u, \tag{2}$$

on a circular domain (a disk) $D = \{(x,y) : x^2 + y^2 < a^2\}$, it is natural to use polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ so that $\Delta u = u_{xx} + u_{yy} = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$. The method of *separation of variables,* $u(r,\theta,t) = v(r,\theta)T(t)$, leads us to consider the eigenvalue problem

$$\begin{aligned} \Delta v + \lambda v = 0 \quad \text{in } D \\ B.C.'s \quad \text{on } \partial D. \end{aligned} \tag{3}$$

Finding the eigenvalues $\lambda_n$ and eigenfunctions $v_n$ for (3) will enable us to write our solution as an infinite series involving unknown constants; the initial conditions are then used to evaluate the constants. In particular, to solve (3) we shall assume $v(r,\theta) = R(r)g(\theta)$.

## I. Vibrations of a Circular Drumhead

The vibrations of a circular drumhead (fixed on its perimeter) are given by

$$\begin{cases} u_{tt} = c^2\Delta u \quad \text{in } D, \\ u = 0 \quad \text{on } \partial D, \\ u = \phi \text{ and } u_t = \psi \quad \text{in } D \text{ at } t = 0. \end{cases} \tag{4}$$

The assumption $u(r,\theta,t) = v(r,\theta)\,T(t) = R(r)\,g(\theta)\,T(t)$ leads to the equations

$$T'' + c^2\lambda T = 0 \tag{5a}$$

$$g'' + \mu g = 0, \quad g \text{ has period } 2\pi, \tag{5b}$$

$$r^2R'' + rR' + (\lambda r^2 - \mu)R = 0, \quad R(0) \text{ finite}, \ R(a) = 0. \tag{5c}$$

As usual, the equation (5b) for $g$ produces eigenvalues $\mu_m = m^2$ for $m = 0, 1, \ldots$ with eigenfunctions

$$g_m(\theta) = A_m\cos m\theta + B_m\sin m\theta \quad \text{for } m = 0, 1, 2, \ldots. \tag{6}$$

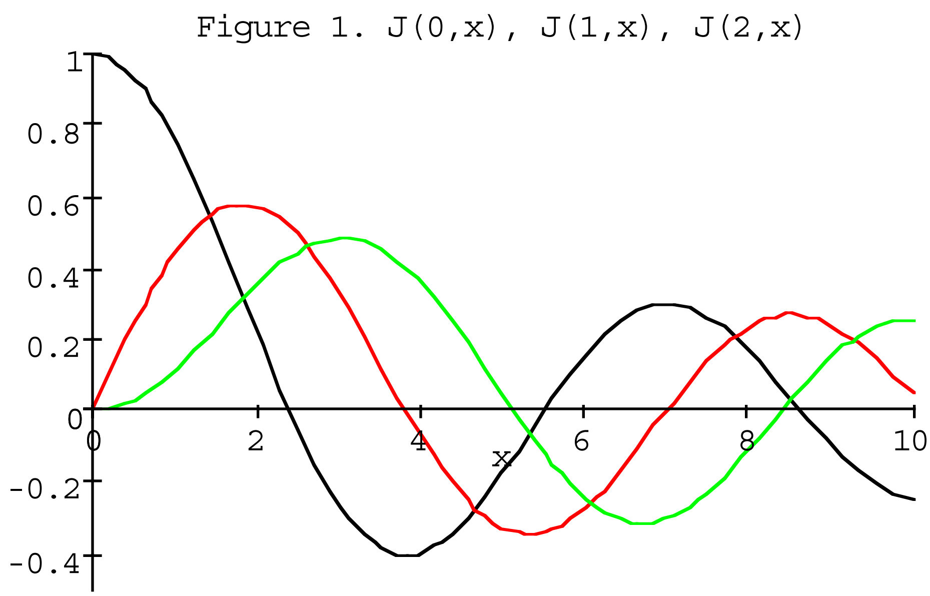
Now we must solve the equation (5c) for each $\mu_m = m^2$. The first step is to make the substitution $\rho = \sqrt{\lambda}r$, so that the chain rule implies $R_r = \sqrt{\lambda}R_\rho$, $R_{rr} = \lambda R_{\rho\rho}$, and (5c) becomes

$$\rho^2R_{\rho\rho} + \rho R_\rho + (\rho^2 - m^2)R = 0, \tag{7}$$

which is *Bessel's differential equation of order m.* This equation cannot be explicitly solved, but its solutions have naturally received alot of study. In particular, for each integer $m$, there is (up to scalar multiple) exactly one solution which is finite at $\rho = 0$; it is called the *Bessel function of order m,* and is denoted by $J_m(\rho)$. In fact, *Maple* is familiar with these functions; it uses the name `BesselJ(m,x)` for $J_m(x)$. Let us ask *Maple* to graph $J_0(x)$, $J_1(x)$, and $J_2(x)$, simultaneously on $0 \le x \le 10$ and $-.5 \le y \le 1$ by entering:

$$> \mathtt{plot(\{BesselJ(0,x), BesselJ(1,x), BesselJ(2,x)\}, x = 0..10, -.5..1);}$$

The result is the following:




Notice that $J_0(0) = 1$ and $J_1(0) = 0 = J_2(0)$. Moreover, each $J_m(x)$ "oscillates" by crossing the $x$-axis an infinite number of times at certain points $x_{m1} < x_{m2} < x_{m3}, \ldots \to \infty$; for example, with $m = 0$, $x_{01}$ lies between $x = 2$ and $x = 3$, $x_{02}$ lies between $x = 5$ and $x = 6$, etc. These values are important for satisfying the condition $R(a) = 0$ of (5c); namely we now want to let $\lambda_{mn} = x_{mn}^2/a^2$ so that

$$J_m(\sqrt{\lambda_{mn}}\, a) = 0. \tag{8}$$

Now we let $R_{mn}(r) = J_m(\sqrt{\lambda_{mn}}\, r)$, so that for $m = 0, 1, \ldots$ and $n = 1, 2, \ldots,$

$$v_{mn}(r, \theta) = R_{mn}(r) g_m(\theta) = J_m(\sqrt{\lambda_{mn}}\, r)(A_{mn} \cos m\theta + B_{mn} \sin m\theta) \tag{9}$$

satisfies (3) with $\lambda = \lambda_{mn}$ and the Dirichlet condition $v(a, \theta) = 0$.

Finally, we solve (5a) for these $\lambda_{mn}$ to find

$$T_{mn}(t) = C_{mn} \cos(\sqrt{\lambda_{mn}}\, ct) + D_{mn} \sin(\sqrt{\lambda_{mn}}\, ct). \tag{10}$$

Combining (9) and (10) by superposition, we find that $u = \sum v_{mn}T_{mn}$ may be written as

$$u(r,\theta,t) = \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_{0n}}r)(C_{0n}\cos(\sqrt{\lambda_{0n}}ct) + D_{0n}\sin(\sqrt{\lambda_{0n}}ct)) + \tag{11}$$
$$\sum_{m,n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)(C_{mn}\cos(\sqrt{\lambda_{mn}}ct) + D_{mn}\sin(\sqrt{\lambda_{mn}}ct)).$$

The constants $A_{mn}$, $B_{mn}$, $C_{mn}$, and $D_{mn}$ are of course determined by the initial conditions $\phi$ and $\psi$, but before we get into that, let us investigate some special cases of (11).

For concreteness, let us henceforth take

$$a = 1 = c. \tag{12}$$

In particular, this means that $\sqrt{\lambda_{mn}} = x_{mn}$. The numbers $x_{mn}$ are easily found using *Maple.* For example, to find $x_{01}$, we look for the solution of $J_0(x) = 0$ between $x = 2$ and $x = 3$:

```
> fsolve(BesselJ(0, x) = 0, x = 2..3);
```

produces the value to 10-digits as

```
2.404825558
```

We shall round this off to three decimal places: $x_{01} \approx 2.405$

**Exercise 1.** (a) Show that $x_{02} \approx 5.520$, $x_{03} \approx 8.654$, and $x_{04} \approx 11.7915$.

(b) Find three decimal approximations for $x_{11}$, $x_{12}$, and $x_{13}$.

**HAND IN:** Your three decimal place approximations for $x_{11}$, $x_{12}$, and $x_{13}$.

Now let us consider (11) with

$$C_{01} = 1 \quad \text{and all other coefficients } = 0, \tag{13a}$$

so that

$$u(r,\theta,t) = J_0(x_{01}r)\cos(x_{01}t). \tag{13b}$$

Suppose we want to graph some "snapshots" of this solution at various values of $t$. The problem that the function is defined in terms of polar coordinates may be overcome using the "cylinderplot" option in *Maple:*

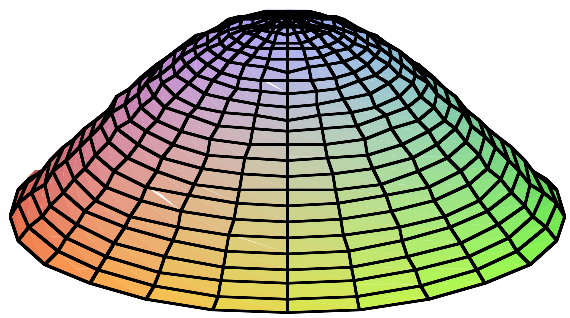
```
> cylinderplot([r, theta, f(r, theta)], r = a..b, theta = c..d);
```

will plot $z = f(r,\theta)$ for $a \le r \le b$ and $c \le \theta \le d$. Thus, to graph $u(r,\theta,.25)$, we use the commands:

```
>with(plots);
>u := t-> BesselJ(0, 2.405 * r) * cos(2.405 * t);
>cylinderplot([r, theta, u(.25)], r = 0..1, theta = 0..2 * Pi);
```

The result appears in Figure 2.

Figure 2. Graph of (13b) at t=.25



For viewing and comparison purposes, it may be better to use some options for the graph. If we replace the third command by

$$\texttt{>cylinderplot([r, theta, u(.25)], r = 0..1, theta = 0..2 * Pi,}$$
$$\texttt{scaling = constrained, orientation = [40, 70], axes = boxed);}$$

then the vertical axis has the same scaling as the horizontal axes, the viewing orientation is convenient, and the axes "box" the figure for clarity.

**Exercise 2.** Plot the snapshots of (13b) at the values $t = .25$, $t = .5$, $t = .75$, and $t = 1$ with this scaling, orientation, and boxed axes. Can you "see" the vibration?
**HAND IN:** Printouts of the snapshots at $t = .25$, $t = .5$, $t = .75$, and $t = 1$.

For another special case of (11), let us take

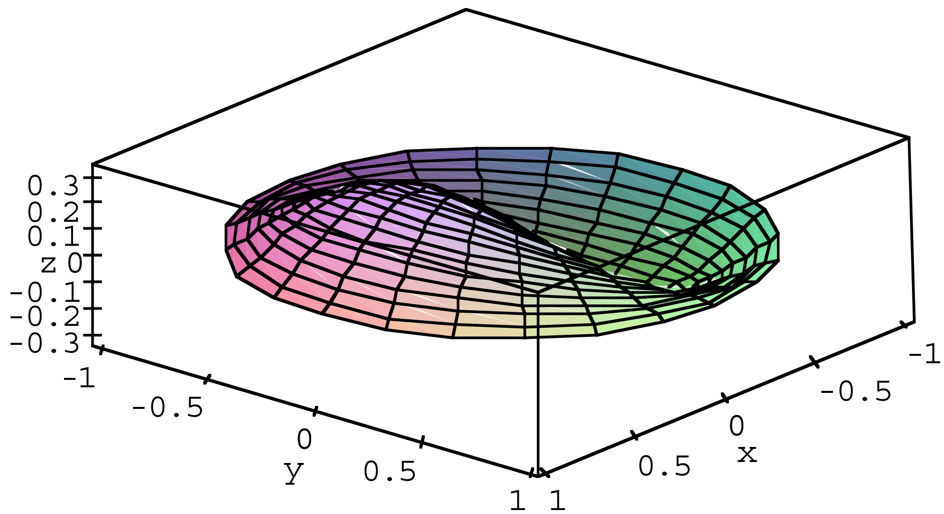
(14a) $$A_{11} = 1 = C_{11} \quad \text{and all other coefficients } = 0,$$

so that

(14b) $$u(r, \theta, t) = J_1(x_{11} r) \cos(\theta) \cos(x_{11} t).$$

Using the value of $x_{11}$ found in Exercise 1, *Maple* may again be used to plot (14b) at various values of $t$. At $t = .25$ we get the snapshot in Figure 3.

Figure 3. Graph of (14b) at t=.25




This is quite interesting; one part of the drum rises as the other falls. What do you think happens as $t$ increases?

**Exercise 3.** Plot the snapshots of (14b) at the values $t = .25$, $t = .5$, $t = .75$, and $t = 1$ with the scaling, orientation, and boxed axes. Can you "see" the vibration?
**HAND IN:** Printouts of the snapshots at $t = .25$, $t = .5$, $t = .75$, and $t = 1$.

As a final example of a special case of (11), consider

$$u(r, \theta, t) = J_2(x_{21} r) \cos(2\theta) \cos(x_{21} t). \tag{15}$$

**Exercise 4.**
(a) What are the values of the $A$'s, $B$'s, $C$'s, and $D$'s in (11) that produce (15)?
(b) Based on your experience of (13b) and (14b), what would you *expect* the snapshots of (15) to look like?
(c) Use *Maple* to approximate the value of $x_{21}$, and to plot (15) at the times $t = .25$, $t = .5$, $t = .75$, and $t = 1$. Was your expectation in (b) confirmed?

**HAND IN:** Your approximate value of $x_{21}$, printouts of the snapshots of (15) at $t = .25$, $t = .5$, $t = .75$, and $t = 1$, and your answers to the questions in (a), (b), and (c).

## II. Initial-Value Problems

Now let us try to solve (4) with specific initial conditions $\phi$ and $\psi$; in other words, we now want to use $\phi$ and $\psi$ to evaluate the constants in (11). An important fact that we need in this quest is the orthogonality of the eigenfunctions $v_{mn}(r,\theta)$ of (9): we claim that

$$\int\int_D v_{mn}(r,\theta)\, v_{m'n'}(r,\theta)\, r\, dr\, d\theta = 0 \quad \text{if } (m,n) \neq (m',n'). \tag{16}$$

If $m \neq m'$, then the orthogonality of both $\cos m\theta$ and $\sin m\theta$ to both $\cos m'\theta$ and $\sin m'\theta$ means that we need only consider $m = m'$ and $n \neq n'$, i.e. we must verify that

$$\int_0^a J_m(\sqrt{\lambda_{mn}}r) J_m(\sqrt{\lambda_{mn'}}r)\, r\, dr = 0 \quad \text{whenever } n \neq n'. \tag{17}$$

This is a simple consequence of general Sturm-Liouville theory; however, we can use *Maple* to see it in action. For example, let us take $a = 1$ and compute $\int_0^1 J_0(x_{01}r)J_0(x_{02}r)rdr$ numerically. For this purpose, let us use the approximations $x_{01} \approx 2.405$ and $x_{02} \approx 5.520$, and enter the commands

```
>int(BesselJ(0,2.405*r)*BesselJ(0,5.520*r)*r,r=0..1);
>evalf(");
```

*Maple* produces the value $.82\times 10^{-5}$, which is pretty small. However, if we perform the same calculation with the improved approximations $x_{01} \approx 2.404825558$ and $x_{02} \approx 5.520078110$, we obtain the value $.18 \times 10^{-10}$ which is indeed very close to zero.

**Exercise 5.** Compute $\int_0^1 J_1(x_{11}r)\, J_1(x_{12}r)\, r\, dr$ using both (i) a 3 decimal place and (iv) a 9 decimal place approximations for $x_{11}$ and $x_{12}$.
**HAND IN:** Your evaluation of the integral for both approximations (i) and (ii).

When we calculate the integral in (17) with $n = n'$, we shall get a nonzero number which we denote $j_{mn}$, i.e.

$$j_{mn} := \int_0^a \left[J_m(\sqrt{\lambda_{mn}}r)\right]^2 r\, dr. \tag{18}$$

Now suppose we want to write a given function $f(r,\theta)$ as

$$f(r,\theta) = \frac{1}{2}\sum_{n=1}^{\infty} a_{0n} J_0(\sqrt{\lambda_{0n}}r) + \sum_{m,n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\, r)(a_{mn}\cos m\theta + b_{mn}\sin m\theta). \tag{19}$$

Multiply both sides of (19) by $J_{m'}(\sqrt{\lambda_{m'n'}})\cos(m\theta)\, r\, dr$ or $J_{m'}(\sqrt{\lambda_{m'n'}})\sin(m\theta)\, r\, dr$ and integrate $r$ from 0 to $a$; passing the integrals under the summation, using (18), and the orthogonality properites discussed, we find

$$\begin{aligned} a_{mn} &= \frac{1}{\pi j_{mn}} \int_0^{2\pi}\int_0^a J_m(\sqrt{\lambda_{mn}}r)\, \cos(m\theta)\, f(r,\theta)\, r\, dr d\theta, \\ b_{mn} &= \frac{1}{\pi j_{mn}} \int_0^{2\pi}\int_0^a J_m(\sqrt{\lambda_{mn}}r)\, \sin(m\theta)\, f(r,\theta)\, r\, dr d\theta \end{aligned} \tag{20}$$

where $m = 0, \ldots$ and $n = 1, \ldots$ (although $b_{0n} = 0$).

At last we are in a position to use $\phi$ and $\psi$ to evaluate the constants in (11). First of all,

$$\begin{aligned} &\phi(r,\theta) = u(r,\theta,0) = \\ &\sum_{n=1}^{\infty} C_{0n} J_0(\sqrt{\lambda_{0n}}r) + \sum_{m,n=1}^{\infty} C_{mn} J_m(\sqrt{\lambda_{mn}}r)(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)), \end{aligned} \tag{21}$$

so from the expansion (19),(20) for $\phi(r,\theta)$, we find

$$\begin{aligned} C_{0n} = \frac{a_{0n}}{2} &= \frac{1}{2\pi j_{0n}} \int_0^{2\pi} \int_0^a J_0(\sqrt{\lambda_{0n}}r)\, \phi(r,\theta)\, r\, dr d\theta \\ C_{mn}A_{mn} = a_{mn} &= \frac{1}{\pi j_{mn}} \int_0^{2\pi} \int_0^a J_m(\sqrt{\lambda_{mn}}r)\, \cos(m\theta)\, \phi(r,\theta)\, r\, dr d\theta \\ C_{mn}B_{mn} = a_{mn} &= \frac{1}{\pi j_{mn}} \int_0^{2\pi} \int_0^a J_m(\sqrt{\lambda_{mn}}r)\, \sin(m\theta)\, \phi(r,\theta)\, r\, dr d\theta. \end{aligned} \tag{22}$$

Similarly,

$$\begin{aligned} \psi(r,\theta) = u_t(r,\theta,0) = \sum_{m=1}^{\infty} D_{0n}\sqrt{\lambda_{0n}} J_0(\sqrt{\lambda_{0n}}r) + \\ \sum_{m,n=1}^{\infty} D_{mn}\sqrt{\lambda_{mn}} J_m(\sqrt{\lambda_{mn}}r)(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)), \end{aligned} \tag{23}$$

so we obtain

$$\begin{aligned} D_{0n}\sqrt{\lambda_{0n}} &= \frac{1}{2\pi j_{0n}} \int_0^{2\pi} \int_0^a J_0(\sqrt{\lambda_{0n}}r)\, \psi(r,\theta)\, r\, dr d\theta \\ D_{mn}A_{mn}\sqrt{\lambda_{mn}} &= \frac{1}{\pi j_{mn}} \int_0^{2\pi} \int_0^a J_m(\sqrt{\lambda_{mn}}r)\, \cos(m\theta)\, \psi(r,\theta)\, r\, dr d\theta \\ D_{mn}B_{mn}\sqrt{\lambda_{mn}} &= \frac{1}{\pi j_{mn}} \int_0^{2\pi} \int_0^a J_m(\sqrt{\lambda_{mn}}r)\, \sin(m\theta)\, \psi(r,\theta)\, r\, dr d\theta. \end{aligned} \tag{24}$$

**Example. Vibrations of a Circular Drumhead, Struck from Below.** Let us assume $a = 1 = c$ as in (12), and the initial conditions

$$\phi(r,\theta) = 0 \quad \text{and} \quad \psi(r,\theta) = 1, \tag{25}$$

which correspond to striking a stationary drumhead with a uniform impulse from below. In particular, we might ask the following.

*Question: Is the motion periodic? In particular, will the drumhead pass through its original flat state at some time $t^* > 0$?*

We considered this question for a one-dimensional vibration (a string) in Lab 3 and found the answer was "yes"; we also considered the question for a two-dimensional vibration (a rectangular drumhead) in Lab 5 and found the answer was "no"; what happens for circular drumheads? Maybe *Maple* can help us discover the answer.

From $\phi(r, \theta) = 0$ and (22) we find

(26a) $$C_{0n} = 0 = C_{mn}A_{mn} = C_{mn}B_{mn} \quad \text{for all } m, n = 1, \dots.$$

Moreover, since $\psi(r, \theta) = 1$ is independent of $\theta$, the integrals in (24) involving $\cos(m\theta)$ and $\sin(m\theta)$ all vanish, so

(26b) $$D_{mn}A_{mn} = D_{mn}B_{mn} = 0 \quad \text{for all } m, n = 1, \dots,$$

and we are left with

(26c) $$D_{0n} = \frac{1}{2\pi j_{0n}\sqrt{\lambda_{0n}}} \int_0^1 J_0(\sqrt{\lambda_{0n}}r)\, r\, dr$$

Our solution is therefore independent of $\theta$ and may be written as

(27) $$u(r, t) = \sum_{n=1}^{\infty} D_{0n} J_0(\sqrt{\lambda_{0n}}r) \sin(\sqrt{\lambda_{0n}}t).$$

**Exercise 6.** (a) Use *Maple* to calculate $D_{01}$, $D_{02}$, and $D_{03}$.

(b) What is your answer to the *Question* above? Why?

**HAND IN:** Your numerical values for $D_{01}$, $D_{02}$, and $D_{03}$, and your answers to the questions in (b).