Partial fraction

Let $h(t) \xleftrightarrow{\mathcal{L}} H(s)$

$$H(s) = \frac{N(s)}{D(s)}$$ order of $N(s)$ is Less than the order of $D(N)$

Polynomials

$$H(s) = \frac{N(s)}{(s-P_1)(s-P_2)\cdots(s-P_n)} = \frac{A_1}{s-P_1} + \frac{A_2}{s-P_2} + \frac{A_3}{s-P_3} + \cdots + \frac{A_n}{s-P_n}$$

to find A_i

$$A_i = \left[(s-P_i)\,H(s)\right]\Big|_{s=P_i}$$

A_i is called "residue"

Example 1:

$$H(s) = \frac{s+2}{s^3+4s^2+3s} = \frac{s+2}{s(s+1)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+3}$$

Now: $$A_1 = \left[s\,H(s)\right]\Big|_{s=0} = \frac{s+2}{(s+1)(s+3)}\Big|_{s=0} = \frac{2}{3}$$

$$A_2 = \left[(s+1)\,H(s)\right]\Big|_{s=-1} = \frac{s+2}{s(s+3)}\Big|_{s=-1} = -1/2$$

$$A_3 = \left[(s+3)\,H(s)\right]\Big|_{s=-3} = \frac{s+2}{s(s+1)} = -1/6$$

finally, rewrite $H(s)$ in terms of A_1, A_2, A_3

$$H(s) = \frac{2/3}{s} + \frac{-1/2}{s+1} + \frac{-1/6}{s+3}$$

. Complex Poles

Example 2

$$H(s) = \frac{s^2 - 2s + 1}{s^3 + 3s^2 + 4s + 2} = \frac{s^2 - 2s + 1}{(s+1-j)(s+1+j)(s+1)}$$

$$= \frac{A_1}{(s+1-j)} + \frac{A_2}{(s+1+j)} + \frac{A_3}{s+1}$$

Now: $A_1 = \left[(s+1-j)\, H(s) \right] \Big|_{s=-1+j} = \frac{s^2 - 2s + 1}{(s+1+j)(s+1)} \Big|_{s=-1+j} = -3/2 + j2$

$A_2 = \left[(s+1+j)\, H(s) \right] \Big|_{s=-1-j} = \frac{s^2 - 2s + 1}{(s+1-j)(s+1)} \Big|_{s=-1-j} = -3/2 - j2$

$A_3 = \left[(s+1)\, H(s) \right] \Big|_{s=-1} = \frac{s^2 - 2s + 1}{(s+1-j)(s+1+j)} \Big|_{s=-1} = 4$

$$\Longrightarrow H(s) = \frac{-3/2 + j2}{s+1-j} + \frac{-3/2 - j2}{s+1+j} + \frac{4}{s+1}$$

Repeated Poles

$$H(s) = \frac{N(s)}{(s-P_1)^r \cdots (s-P_n)}$$

~~example~~ $$H(s) = \frac{N(s)}{(s-P_1)^r \cdots (s-P_n)} = \frac{A_1}{(s-P_1)} + \frac{A_2}{(s-P_1)^2} + \cdots + \frac{A_i}{(s-P_1)^r} + \cdots + \frac{A_n}{(s-P_n)}$$

Example:

$$H(s) = \frac{s+2}{s^3(s+1)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{s^3} + \frac{A_4}{s+1}$$

$$\Rightarrow \quad \frac{s+2}{s^3(s+1)} = \frac{A_1 s^2(s+1) + A_2 s(s+1) + A_3(s+1) + A_4(s^3)}{s^3(s+1)}$$

Now: $$s+2 = A_1 s^2(s+1) + A_2 s(s+1) + A_3(s+1) + A_4 s^3 \qquad (1)$$

First: $s=0$ in (1)

$$\Rightarrow \quad 0+2 = \underbrace{\cancel{A_1}(0)(1)}_{\to 0} + \underbrace{\cancel{A_2}(0)(1)}_{\to 0} + A_3(1) + \underbrace{\cancel{A_4}(0)}_{\to 0}$$

$$\boxed{2 = A_3}$$

Second: $s=-1$ in (1)

$$\Rightarrow \quad -1+2 = A_1(-1)^2\underbrace{\cancel{(-1+1)}}_{\to 0} + A_2(-1)\underbrace{\cancel{(-1+1)}}_{\to 0} + A_3\underbrace{\cancel{(-1+1)}}_{\to 0} + A_4(-1)^3$$

$$1 = A_4(-1) \quad \Rightarrow \quad \boxed{A_4 = -1}$$

Third: take s = any number except $s=0$, $s=-1$ (Poles of $H(s)$) and since we have 2 unknowns, ~~substitute~~ choose 2 different values of s then solve them

★ $s=1$ in (1)

$$1+2 = A_1(1)^2(1+1) + A_2(1)(1+1) + \overset{A_3}{2}(1+1) + \overset{A_4}{(-1)}(1)^3$$

$$\Rightarrow \quad 3 = 2A_1 + 2A_2 + 3 \quad \Rightarrow \quad 2A_1 + 2A_2 = 0 \quad \Rightarrow \quad A_1 + A_2 = 0 \quad \cdots (2)$$

★ $s=2$ in (1)

$$2+2 = A_1(2)^2(2+1) + A_2(2)(2+1) + 2(2+1) + (-1)(2)^3$$

$$4 = 12A_1 + 6A_2 - 2 \quad \Rightarrow \quad 6 = 12A_1 + 6A_2 \quad \Rightarrow \quad 2A_1 + A_2 = 1 \quad \cdots (3)$$

Solving (2) & (3) $\Rightarrow$ $A_1 = 1$, $A_2 = -1$

$$H(s) = \frac{1}{s} + \frac{-1}{s^2} + \frac{2}{s^3} + \frac{-1}{s+1}$$