## (A) Anti-derivatives

### Anti-derivatives

**Definition 1** *A function $F$ is called an anti-derivative of $f$ on an interval $I$ if*

$$F^{'}(x) = f(x) \text{ for every } x \in I \, .$$

**Example 1**

1. *Let* $F(x) = x^2 + 3x + 1$ *and* $f(x) = 2x + 3$.
   *Since* $F^{'}(x) = f(x)$, *the function* $F(x)$ *is an anti-derivative of* $f(x)$.
2. *Let* $G(x) = \sin(x) + x$ *and* $g(x) = \cos(x) + 1$.
   *We know that* $G^{'}(x) = \cos(x) + 1$ *and this means the function* $G(x)$ *is an anti-derivative of* $g(x)$.

Generally, if $F(x)$ is an anti-derivative of $f(x)$, then every function $F(x) + c$ is also anti-derivative of $f(x)$, where $c$ is a constant. The question that can be raised here is: Is the anti-derivative of a function $f$ unique? In other words, does the function $f(x)$ have any other anti-derivatives that are different from $F(x)+c$. The next theorem gives the answer to this question.

**Theorem 1** *If the functions $F(x)$ and $G(x)$ are anti-derivatives of a function $f(x)$ on the interval $I$, there exists a constant $c$ such that $G(x) = F(x) + c$.*

The last theorem means that any anti-derivative $G(x)$, which is different from the function $F(x)$ can be expressed as $F(x) + c$ where $c$ is an arbitrary constant. The following examples clarify this point.

**Example 2** *Let $f(x) = 2x$. The functions*
$F(x) = x^2 + 2$,
$G(x) = x^2 - \frac{1}{2}$,
$H(x) = x^2 - \sqrt[3]{2}$,
*and many other functions are anti-derivatives of a function $f(x)$. Generally, for the function $f(x) = 2x$, the function $F(x) = x^2 + c$ is the anti-derivative where $c$ is an arbitrary constant.*

## (B) Indefinite Integrals

### Indefinite Integrals

**Definition 2** *Let $f$ be a continuous function on an interval $I$. The Indefinite integral of $f(x)$ is the general anti-derivative of $f(x)$ on $I$ and symbolized by $\int f(x)\,dx$.*

**Remark 1** *If $F(x)$ is an anti-derivative of $f$, then*

$$\int f(x)\,dx = F(x) + c \, .$$

*The function $f(x)$ is called the integrand, the symbol $\int$ is the integral sign, $x$ is called the variable of integration and $c$ is the constant of integration.*

Now, by using the previous remark, the general anti-derivatives in Example 1 are

1. $\int 2x + 3\,dx = x^2 + 3x + c$.
2. $\int \cos(x) + 1\,dx = \sin(x) + x + c$.

The following table lists basic indefinite integrals.

| Derivative | Indefinite Integrals |
|---|---|
| $\frac{d}{dx}(x) = 1$ | $\int 1\,dx = x + c$ |
| $\frac{d}{dx}(\frac{x^{n+1}}{n+1}) = 1,\, n \neq 1$ | $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ |
| $\frac{d}{dx}(\sin\ x) = \cos\ x$ | $\int \cos\ dx = \sin\ x + c$ |
| $\frac{d}{dx}(-\cos\ x) = \sin\ x$ | $\int \sin\ x\,dx = -\cos\ x + c$ |
| $\frac{d}{dx}(\tan\ x) = \sec^2\ x$ | $\int \sec^2\ x\,dx = \tan\ x + c$ |
| $\frac{d}{dx}(-\cot\ x) = \csc^2\ x$ | $\int \csc^2\ x\,dx = -\cot\ x + c$ |
| $\frac{d}{dx}(\sec\ x) = \sec\ x\ \tan\ x$ | $\int \sec\ x\ \tan\ x\,dx = \sec\ x + c$ |
| $\frac{d}{dx}(-\csc\ x) = \csc\ x\ \cot\ x$ | $\int \csc\ x\ \cot\ x\,dx = -\csc\ x + c$ |

Table 1: The list of the basic integration rule.

**Example 3** *Evaluate the following integrals:*

1. $\int x^{-3}\ dx$
2. $\int \frac{1}{\cos^2 x}\ dx$

Solution:

In this section, we shall list main properties of indefinite integrals and use them to integrate some functions.

**Properties of Indefinite Integrals**

**Theorem 2** *Let $f$ and $g$ be integrable functions, then*

1. $\frac{d}{dx}\int f(x)\ dx = f(x)$ .
2. $\int \frac{d}{dx}(F(x))\ dx = F(x) + c$ .
3. $\int \big(f(x) \pm g(x)\big)\ dx = \int f(x) \pm \int g(x)\ dx$ .
4. $\int kf(x)\ dx = k\int f(x)\ dx$, *where $k$ is a constant*

In the following example, we use the previous properties and the table of the basic integrals to solve some indefinite integrals.

**Example 4** *Evaluate the following integrals:*

1. $\int 4x + 3\ dx$
2. $\int 2\sin x + 3\cos x\ dx$
3. $\int \sqrt{x} + \sec^2 x\ dx$
4. $\int \frac{d}{dx}(\sin x)\ dx$
5. $\frac{d}{dx}\int \sqrt{x+1}\ dx$

Solution:

**Example 5** *Solve the differential equation $f'(x) = 6x^2 + x - 5$ subject to the initial condition $f(0) = 2$.*

Solution:

$$\int f'(x)\ dx = \int (6x^2 + x - 5)\ dx$$
$$f(x) = 2x^3 + \frac{1}{2}x^2 - 5x + c\ .$$

Use the condition $f(0) = 2$ i.e., substitute $x = 0$ into the function $f(x)$. We have

$$f(0) = 0 + 0 - 0 + c = 2 \Rightarrow c = 2\ .$$

From this, the solution of the differential equation is $f(x) = 2x^3 + \frac{1}{2}x^2 - 5x + 2$ .

**Example 6** *Solve the differential equation $f''(x) = 5\cos x + 2\sin x$ subject to the initial condition $f(0) = 3$ and $f'(0) = 4$.*

Solution:

$$\int f''(x)\ dx = \int (5\cos x + 2\sin x)\ dx$$
$$f'(x) = 5\sin x - 2\cos x + c$$

The condition $f'(0) = 4$ yields

$$f'(0) = 0 - 2 + c = 4 \Rightarrow c = 6\ .$$

Thus, $f'(x) = 5\sin x - 2\cos x + 6$ . Now, again

$$\int f'(x)\ dx = \int (5\sin x - 2\cos x + 6)\ dx$$
$$f(x) = -5\cos x - 2\sin x + 6x + c$$

Use the condition $f(0) = 3$ by substituting $x = 0$ into $f(x)$. This yields

$$f(0) = -5 - 0 + 0 + c = 3 \Rightarrow c = 8\ .$$

Thus, the solution of the differential equation is $f(x) = -5\cos x - 2\sin x + 6x + 8$.

Note that, in the previous examples, we use $x$ as the variable of the integration. However, for this role, we can use any variable $y$, $z$, $t$, ... .

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The integration by substitution (known as u-substitution) is one technique for solving some complex integrals. The goal of changing the variable of the integration is to obtain a simple indefinite integral. In a sense that the substitution method turns the integral into a simpler integral involving the variable $u$ that can be solved by using either the table of the basic integrals or other techniques of integration. The following definition shows how the substitution technique works.

**Substitution Method**

**Theorem 3** *Let $g$ be a differentiable function on the interval $[a, b]$ where the derivative is continuous. Let $f$ be a continuous on an interval $I$ involves the range of the function $g$. If $F$ is an anti-derivative of the function $f$ on $I$, then*

$$\int f(g(x))g^{'}(x)\ dx = F(g(x)) + c,\ x \in [a, b]$$

### Steps of Integration by Substitution:

For simplicity, the substitution method can be summarized in the following steps:
*Step 1:* Choose a new variable $u$.
*Step 2:* Determine the value of $du$.
*Step 3:* Make the substitution i.e., eliminate all occurrences of $x$ in the integral by making the entire integral is in terms of $u$.
*Step 4:* Evaluate the new integral.
*Step 5:* Return the evaluation to the initial variable $x$.

**Example 7** *Evaluate the integral* $\int 2x(x^2 + 1)^3\ dx$.
Solution:

**Example 8** *Evaluate the integral* $\int \frac{x^2 - 1}{(x^3 - 3x + 1)^6}\ dx$.
Solution:

**Example 9** *Evaluate the integral* $\int x\sqrt{x - 1}\ dx$.
Solution:

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The upcoming corollary simplifies the process of the integration by substitution for some functions.

**Corollary 1** *If* $\int f(x)\ dx = F(x) + c$, *then for any* $a \neq 0$,

$$\int f(ax \pm b)\ dx = \frac{1}{a} F(ax \pm b) + c\ .$$

**Example 10** *Evaluate the following integrals:*

1. $\int \sqrt{2x-5}\ dx$
2. $\int \cos(3x+4)\ dx$

Solution:

Evaluate the following integrals:

1. $\int x\sqrt{1+x^2}\ dx$
2. $\int x^2\sqrt{x-1}\ dx$
3. $\int \frac{\tan x}{\cos^2 x}\ dx$
4. $\int \sin^5 x \cos x\ dx$
5. $\int \frac{x}{\sqrt{2x^2+1}}\ dx$
6. $\int \cos t\sqrt{1-\sin t}\ dt$
7. $\int \frac{\cos^3 x}{\csc x}\ dx$
8. $\int \cos(3x+4)\ dx$
9. $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\ dx$
10. $\int sec4x \tan 4x\ dx$
11. $\int \frac{\sqrt{\cot x}}{\sin^2 x}\ dx$
12. $\int (1+\frac{1}{t})t^{-2}\ dt$
13. $\int \frac{x}{\sqrt{2x-1}}\ dx$
14. $\int x^2(4x^3-6)^7\ dx$
15. $\int \sin^2(3x)\cos(3x)\ dx$

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## (A) Summation Notation

Summation is the addition of a sequence of numbers and the result is their sum or total.

**Summation Notation**

**Definition 3** *Let $\{a_1, a_2, ..., a_n\}$ be a set of numbers. The symbol $\sum_{k=1}^{n} a_k$ represents their sum:*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n \ .$$

**Example 11** *Evaluate the following sums:*

1. $\sum_{i=0}^{3}(i^3)$ .
2. $\sum_{j=1}^{4}(j^2+1)$ .
3. $\sum_{k=1}^{3}(k+1)k^2$ .

Solution:

## (B) Properties of Sum Notation

1. $\sum_{k=1}^{n} c = \underbrace{c + c + ... + c}_{\text{n-times}} = nc.$
2. $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k.$
3. $\sum_{k=1}^{n} c\ a_k = c\ \sum_{k=1}^{n} a_k$ for any $c \in \mathbb{R}$.

**Example 12** *Evaluate the following sums:*

1. $\sum_{k=1}^{10} 15$ .
2. $\sum_{k=1}^{4} k^2 + 2k$ .
3. $\sum_{k=1}^{3} 3(k+1)$ .

Solution:

In the following theorem, we present some summations of polynomial expressions. They will be used later in the Riemann sum to find the area under the graph of a function $f$.

**Theorem 4**

1. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ .
2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ .
3. $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$ .

**Example 13** *Evaluate the following sums:*

1. $\sum_{k=1}^{100} k$ .
2. $\sum_{k=1}^{10} k$ .
3. $\sum_{k=1}^{10} k$ .

Solution:

**Example 14** *Express the following sums in terms of $n$:*

1. $\sum_{k=1}^{n}(k+1)$ .
2. $\sum_{k=1}^{n}(2k^2 - k + 1)$ .

Solution:

## (C) Riemann Sum and Area

**Definition 4** *A set* $P = \{x_0, x_1, x_2, ..., x_n\}$ *is called a partition of a closed interval* $[a, b]$ *if*

$$a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b$$

*for any positive integer* $n$.

Note that,
1. the division of the interval $[a, b]$ by the partition $P$ generates $n$ sub-intervals: $[x_0, x_1], [x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n]$.
2. The length of each sub-interval $[x_{k-1}, x_k]$ is $\Delta x_k = x_k - x_{k-1}$.
3. The sub-intervals do not intersect and their union gives the main interval $[a, b]$.

**Definition 5** *The norm of the partition of* $P$ *is the largest length among* $\Delta x_0, \Delta x_1, \Delta x_2, ..., \Delta x_n$ *i.e.,*

$$\| P \| = max\{\Delta x_0, \Delta x_1, \Delta x_2, ..., \Delta x_n\} \ .$$

**Example 15** *If* $P = \{0, 1.2, 2.3, 3.6, 4\}$ *is a partition of the interval* $[0, 4]$, *find the norm of the partition* $P$.

Solution:

We need to find the sub-intervals and their lengths.

| Sub-interval $[x_{k-1}, x_k]$ | Length $\Delta x_k$ |
|---|---|
| $[0, 1.2]$ | $1.2 - 0 = 1.2$ |
| $[1.2, 2.3]$ | $2.3 - 1.2 = 1.1$ |
| $[2.3, 3.6]$ | $3.6 - 2.3 = 1.3$ |
| $[3.6, 4]$ | $4 - 3.6 = 0.4$ |

From the table, the norm is $\| P \| = 1.3$ .

**Remark 2**
1. *The partition* $P$ *of the interval* $[a, b]$ *is called regular if* $\Delta x_0 = \Delta x_1 = \Delta x_2 = ... = \Delta x_n = \Delta x$.
2. *For any positive integer* $n$, *if the partition* $P$ *is regular then*

$$\Delta x = \frac{b-a}{n} \quad and \quad x_k = a + k\,\frac{b-a}{n} \ .$$

To explain the previous result, let $P$ be a regular partition of the interval $[a, b]$. We know that $x_0 = a$ and $x_n = b$. Then,

$$x_1 = x_0 + \Delta x,$$
$$x_2 = x_1 + \Delta x = x_0 + 2\Delta x,$$
$$x_3 = x_2 + \Delta x = x_0 + 3\Delta x \ .$$

By continuing doing so, we have

$$x_k = x_0 + k\ \Delta x = x_0 + k\,\frac{b-a}{n} \ .$$

Note that, when $n \to \infty$, the norm $\| P \| \to 0$ .

**Example 16** *Define a regular partition* $P$ *that divides the interval* $[1, 4]$ *into* 4 *sub-intervals.*

Solution:

## Riemann Sum

**Definition 6** *Let $f$ be a defined function on the closed interval $[a,b]$ and let $P=\{x_0,x_1,...,x_n\}$ be a partition of $[a,b]$. Let $\omega_k\in[x_{k-1},x_k]$, $k=1,2,3,...,n$ be a mark on the partition $P$. Then, the Riemann sum of $f$ for $P$ is*

$$R_p=\sum_{k=1}^{n} f(\omega_k)\Delta x_k\ .$$

Consider Figure 1, we want to explain the definition of the Riemann sum of a function $f$ for the partition $P$. As shown in the figure, the amount $f(\omega_1)\Delta x_1$ is the area of the rectangular $A_1$, $f(\omega_2)\Delta x_2$ is the area of the rectangular $A_2$ and so on. The sum of these areas approximates the whole area under the graph of the function $f$. In other words, the area under $f$ bounded by $x=a$ and $x=b$ can be estimated by the Riemann sum where as the number of the sub-intervals increases (i.e., $n\to\infty$), the estimation becomes better. From this,

$$A=\lim_{n\to\infty} R_p=\lim_{n\to\infty}\sum_{k=1}^{n} f(\omega_k)\Delta x_k.$$

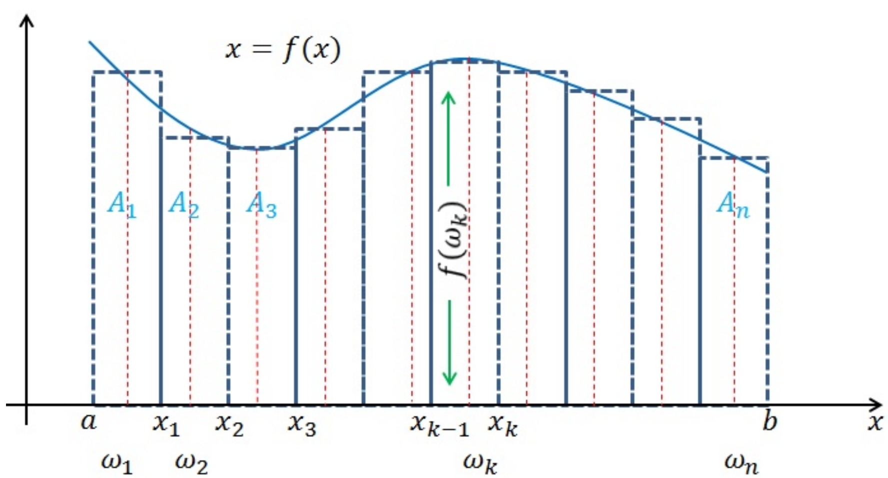


Figure 1: The Riemann sum of the function $f(x)$ for the partition $P$.

**Example 17** *Find the Riemann sum $R_p$ of the function $f(x)=2x-1$ for the partition $P=\{-2,0,1,3,4\}$ of the interval $[a,b]$ by choosing the mark as follows:*

1. *the left-hand end point,*
2. *the right-hand end point,*
3. *the mid point.*

Solution:

1. The left-hand end point.

| Sub-intervals | Length $\Delta x_k$ | $\omega_k$ | $f(\omega_k)$ | $f(\omega_k)\ \Delta x_k$ |
|---|---|---|---|---|
| $[-2,0]$ | $0-(-2)=2$ | $-2$ | $-5$ | $-10$ |
| $[0,1]$ | $1-0=1$ | $0$ | $-1$ | $-1$ |
| $[1,4]$ | $4-1=3$ | $1$ | $1$ | $3$ |
| $[4,6]$ | $6-4=2$ | $4$ | $7$ | $14$ |
| $R_p=\sum_{k=1}^{n} f(\omega_k)\Delta x_k$ | | | | $6$ |

2. The right-hand end point.

| Sub-intervals | Length $\Delta x_k$ | $\omega_k$ | $f(\omega_k)$ | $f(\omega_k)\ \Delta x_k$ |
|---|---|---|---|---|
| $[-2,0]$ | $0-(-2)=2$ | $0$ | $-1$ | $-2$ |
| $[0,1]$ | $1-0=1$ | $1$ | $0$ | $0$ |
| $[1,4]$ | $4-1=3$ | $4$ | $7$ | $21$ |
| $[4,6]$ | $6-4=2$ | $6$ | $11$ | $22$ |
| $R_p=\sum_{k=1}^{n} f(\omega_k)\Delta x_k$ | | | | $41$ |

3. The mid point.

| Sub-intervals | Length $\Delta x_k$ | $\omega_k$ | $f(\omega_k)$ | $f(\omega_k)\ \Delta x_k$ |
|---|---|---|---|---|
| $[-2,0]$ | $0-(-2)=2$ | $-1$ | $-3$ | $-6$ |
| $[0,1]$ | $1-0=1$ | $0.5$ | $0$ | $0$ |
| $[1,4]$ | $4-1=3$ | $1.5$ | $2$ | $6$ |
| $[4,6]$ | $6-4=2$ | $5$ | $9$ | $18$ |
| $R_p=\sum_{k=1}^{n} f(\omega_k)\Delta x_k$ | | | | $18$ |

**Example 18** *Let $A$ be the area under the graph of $f(x) = x + 1$ from $x = 1$ to $x = 3$. Find the area $A$ by taking the limit of the Riemann sum such that the partition $P$ is regular and the mark $\omega_k$ is the right end point of each sub-interval.*

Solution:

For regular partition, we have

1. $\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$, and
2. $x_k = x_0 + k\ \Delta x$ where $x_0 = 1$.

Since the mark $\omega_k$ is the right end point of the sub-intervals $[x_{k-1}, x_k]$, then $\omega_k = x_k = 1 + \frac{2k}{n}$. From this,

$$f(\omega_k) = \frac{2k}{n} + 2 = \frac{2}{n}(k+1) \ .$$

Now,

$$R_p = \sum_{k=1}^{n} f(w_k)\Delta x_k = \frac{4}{n^2}\sum_{k=1}^{n}(n+k) = \frac{4}{n^2}\left[n^2 + \frac{n(n+1)}{2}\right] = 4 + \frac{4n^2+n}{2n^2} \ .$$

Remember: $\sum_{k=1}^{n}(n + k) = \sum_{k=1}^{n} n + \sum_{k=1}^{n} k$ also $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Therefore, $\lim_{n\to\infty} R_p = 4 + 2 = 6$ .

## Exercises

**1 - 8** ■ If $P$ is a partition of the interval $[a, b]$, find the norm of the partition $P$:

1. $P = \{-1, 0, 1.3, 4, 4.1, 5\}$, $[-1, 5]$
2. $P = \{0, 0.5, 1, 2.5, 3.1, 4\}$, $[0, 4]$
3. $P = \{-3, 0, 2.3, 4.6, 4.8, 5.5, 6\}$, $[-3, 6]$
4. $P = \{-2, 0, 2.3, 3, 3.5, 4\}$, $[-2, 4]$
5. $P = \{3, 3.5, 3.6, 4, 4.9, 7\}$, $[3, 7]$
6. $P = \{-1, 0, 1.3, 4, 4.1, 5\}$, $[-1, 5]$
7. $P = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$, $[-1, 2]$
8. $P = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$, $[0, \pi]$

**1 - 4** ■ Define a regular partition $P$ that divides the interval $[a, b]$ into $n$ sub-intervals:

1. $[a, b] = [0, 3]$ $n = 5$
2. $[a, b] = [-1, 4]$ $n = 6$
3. $[a, b] = [-4, 4]$ $n = 8$
4. $[a, b] = [0, 1]$ $n = 4$

**5 - 7** ■ Find the Riemann sum $R_p$ of the function $f(x) = x^2 + 1$ for the partition $P = \{0, 1, 3, 4\}$ of the interval $[a, b]$ by choosing the mark as follows:

5. the left-hand end point,
6. the right-hand end point,
7. the mid point.

**8 - 11** ■ Let $A$ be the area under the graph of $f(x)$ from $a$ to $b$. Find the area $A$ by taking the limit of the Riemann sum such that the partition $P$ is regular and the mark $\omega_k$ is the right end point of each sub-intervals:

8. $f(x) = x/3$ $a = 1,$ $b = 2$
9. $f(x) = x - 1$ $a = 0,$ $b = 3$
10. $f(x) = 5 - x^2$ $a = -1,$ $b = 1$
11. $f(x) = x^3 - 1$ $a = 0,$ $b = 4$

### Definite Integrals

In this section, we are going to define the definite integral and how it is calculated. The following definition shows that the definite integral of a function $f$ on the interval $[a, b]$ is the Riemann sum when $\| P \| \to 0$.

**Definite Integrals**

**Definition 7** *Let $f$ be a function defined on a closed interval $[a, b]$. If $f$ is integrable on that interval, the definite integral of $f$ is*

$$\int_a^b f(x)\ dx = \lim_{n\to\infty}\sum_k f(\omega_k)\Delta x_k = A \ .$$

*The numbers $a$ and $b$ are called the limits of the integration.*

**Example 19** *Evaluate the following integral $\int_2^4 x + 2\ dx$.*

Solution:

The following remark simplifies the process of calculating the definite integrals.

**Remark 3** *To find the value of a definite integral* $\int_a^b f(x)\ dx$, *we first find the value of the indefinite integral* $\int f(x)\ dx = F(x) + c$ *as shown in Chapter* **??**. *Then, we substitute* $a$ *and* $b$ *into* $F(x)$ *as follows:*

$$\int_a^b f(x)\ dx = F(b) - F(a)\ .$$

**Example 20** *Evaluate the following integrals:*

1. $\int_{-1}^{2} 2x + 1\ dx$
2. $\int_0^3 x^2 + 1\ dx$
3. $\int_1^2 \frac{1}{\sqrt{x^3}}\ dx$
4. $\int_0^{\frac{\pi}{2}} \sin(x) + 1\ dx$
5. $\int_{\frac{\pi}{4}}^{\pi} \sec^2(x) - 4\ dx$
6. $\int_0^{\frac{\pi}{3}} \sec(x)\ tan(x) + x\ dx$

Solution:

One application of the definite integrals is to find the area under the graph of a function $f$ on the interval $[a, b]$. This is clear from Definition 7, if $f$ is integrable on the interval $[a, b]$, then

$$A = \int_a^b f(x)\ dx\ .$$

The application of the definite integrals will be discussed in details in Chapter **??**.

### Properties of Definite Integrals

**Theorem 5**

1. $\int_a^b c\ dx = c(b - a)$,
2. $\int_a^a f(x)\ dx = 0$.

**Reversed Interval of Definite Integrals**

**Theorem 6** *If* $f$ *is integrable on* $[a, b]$, *then*

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx\ .$$

**Linearity of Definite Integrals**

**Theorem 7**

1. *If* $f$ *and* $g$ *are integrable on* $[a, b]$, *then* $f + g$ *and* $f - g$ *are integrable on* $[a, b]$ *and*

$$\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x) \pm \int_a^b g(x)\ dx\ .$$

2. *If* $f$ *is integrable on* $[a, b]$ *and* $k \in \mathbb{R}$, *then* $k\ f$ *is integrable on* $[a, b]$ *and*

$$\int_a^b k\ f(x)\ dx = k\ \int_a^b f(x)\ dx\ .$$

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### Comparison of Definite Integrals

**Theorem 8**

1. *If $f$ and $g$ are integrable on $[a,b]$ and $f(x) \geq g(x)$ for all $x \in [a,b]$ then*

$$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx\ .$$

2. *If $f$ is integrable on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$ then*

$$\int_a^b f(x)\ dx \geq 0\ .$$

### Additive Interval of Definite Integrals

**Theorem 9** *If $f$ is integrable on the intervals $[a,c]$ and $[c,b]$, then $f(x)$ is integrable on $x \in [a,b]$ and*

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx\ .$$

**Example 21** *Evaluate the following integrals:*

1. $\displaystyle\int_0^2 3\ dx\ .$

2. $\displaystyle\int_2^2 x^2 + 4\ dx\ .$

Solution:

**Example 22** *If* $\displaystyle\int_0^2 f(x)\ dx = 4$ *and* $\displaystyle\int_0^2 g(x)\ dx = 2$, *then find* $\displaystyle\int_0^2 3f(x) - \frac{g(x)}{2}\ dx.$

Solution:

**Example 23** *Prove that* $\displaystyle\int_0^2 (x^3 + x^2 + 2)\ dx \geq \int_0^2 (x^2 + 1)\ dx$ *without evaluating the integrals.*

Solution:

Put $f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 + 1$. We find that $f(x) - g(x) = x^3 + 1 > 0$ for all $x \in [0,2]$. From Theorem 8, we have

$$\int_0^2 (x^3 + x^2 + 2)\ dx \geq \int_0^2 (x^2 + 1)\ dx\ .$$

**Example 24** *Evaluate the integral* $\displaystyle\int_0^2 |\ x - 1\ |\ \ dx$

Solution:

## (6) Mean Value Theorem for Integrals

**Mean Value Theorem for Integrals**

**Theorem 10** *If $f$ is continuous on the interval $[a, b]$, then there is at least one number $z \in (a, b)$ such that*

$$\int_a^b f(x)\ dx = f(z)(b-a)\ .$$

**Example 25** *Find the number $z$ that satisfies the conclusion of the mean value theorem for the function $f$ on the given interval $[a, b]$:*

1. $f(x) = 1 + x^2$, $[a, b] = [0, 2]$ .
2. $f(x) = \sqrt[3]{x}$, $[a, b] = [0, 1]$ .

Solution:

From the previous theorem, we define the average value of the function $f$ on the interval $[a, b]$.

**Average Value**

**Definition 8** *If $f$ is continuous on the interval $[a, b]$, then the average value $f_{av}$ of the function $f$ on that interval is*

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\ dx\ .$$

**Example 26** *Find the average value of the function $f$ on the given interval $[a, b]$:*

1. $f(x) = x^3 + x - 1$, $[a, b] = [0, 2]$ .
2. $f(x) = \sqrt{x}$, $[a, b] = [1, 3]$ .

Solution:

## (7) The Fundamental Theorem of Calculus

In this section, we formulate one of the most important results of calculus, the Fundamental Theorem. This result links together the notions of integrals and derivatives. Meaning that, the theorem aims at finding a function $F$ such that $F' = f$.

**The Fundamental Theorem of Calculus**

**Theorem 11** *Suppose $f$ is continuous on the closed interval $[a, b]$.*

1. *If $F(x) = \int_c^x f(t)\ dt$ for every $x \in [a, b]$, then $F(x)$ is an anti-derivative of $f$ on $[a, b]$.*
2. *If $F(x)$ is any anti-derivative of $f$ on $[a, b]$, then $\int_a^b f(x)\ dx = F(b) - F(a)$.*

From the previous theorem, if $f$ is continuous on $[a, b]$ and $F(x) = \int_c^x f(t)\ dt$ where $c \in [a, b]$, then

$$F'(x) = \frac{d}{dx}\Big[\int_a^x f(t)\ dt\Big] = f(x)\ \ \forall x \in [a, b]\ .$$

This result can be generalized as follows:

**Theorem 12** *Let $f$ be continuous on $[a, b]$. If $g(x)$ and $h(x)$ are differentiable, then*

$$\frac{d}{dx}\Big[\int_{g(x)}^{h(x)} f(t)\ dt\Big] = f(h(x))h'(x) - f(g(x))g'(x)\ \ \forall x \in [a, b]\ .$$

**Corollary 2** *Let $f$ be continuous on $[a,b]$. If $g(x)$ and $h(x)$ are differentiable, then*

1. $\frac{d}{dx}\left[\int_a^{h(x)} f(t)\,dt\right] = f(h(x))h'(x) \quad \forall x \in [a,b]$ ,
2. $\frac{d}{dx}\left[\int_{g(x)}^{a} f(t)\,dt\right] = -f(g(x))g'(x) \quad \forall x \in [a,b]$ .

**Example 27** *Find the following derivatives:*

1. $\frac{d}{dx}\int_1^{x} \sqrt{\cos t}\,dt$
2. $\frac{d}{dx}\int_1^{x^2} \frac{1}{t^3+1}\,dt$
3. $\frac{d}{dx}\int_{-x}^{x^2} \frac{1}{t^3+1}\,dt$
4. $\frac{d}{dx}\int_{\cos x}^{\sin x} \sqrt{1+t^4}\,dt$

Solution:

**1 - 6** Verify that the function $f$ satisfies the hypotheses of the Mean Value Theorem on the interval $[a,b]$. Then, find all numbers $z$ that satisfy the conclusion of the Mean value Theorem.

1. $f(x) = (x+1)^3, \quad [a,b] = [-1,1]$
2. $f(x) = 1 - x^3, \quad [a,b] = [-2,0]$
3. $f(x) = \sqrt{x}, \quad [a,b] = [1,4]$
4. $f(x) = \frac{2}{\sqrt{x}}, \quad [a,b] = [2,4]$
5. $f(x) = \sin x, \quad [a,b] = [0,\pi]$
6. $f(x) = \cos x, \quad [a,b] = [\frac{\pi}{4}, \frac{\pi}{2}]$

**7 - 10** Find the average value of the function $f(x)$ on the given interval $[a,b]$:

7. $f(x) = x^3 + x^2 - 1, \quad [a,b] = [0,2]$
8. $f(x) = \sqrt[3]{x}, \quad [a,b] = [-1,3]$
9. $f(x) = \frac{1}{x^3}, \quad [a,b] = [1,5]$
10. $f(x) = \sin x, \quad [a,b] = [0, \frac{\pi}{6}]$

**11 - 18** Find the following derivatives:

11. $\frac{d}{dx}\int_{\cos x}^{\sin x} \sqrt{t+1}\,dt$
12. $\frac{d}{dx}\int_{x}^{\sqrt{x}} \frac{1}{t^2+1}\,dt$
13. $\frac{d}{dx}\int_{1}^{x} (t-1)\,dt$
14. $\frac{d}{dx}\int_{x+1}^{3(x-1)} \frac{1}{t-1}\,dt$
15. $\frac{d}{dx}\int_{1}^{x} \sin x\sqrt{t}\,dt$
16. $\frac{d}{dx}\int_{-2x}^{x} \sin(t+1)\,dt$
17. $\frac{d}{dx}\int_{x^3}^{\pi} \frac{1}{t^4+1}\,dt$
18. $\frac{d}{dx}\int_{\tan x}^{secx} \sqrt{1+t^4}\,dt$

**19 - 22** If $F(x)$ is given, find the values:

19. $F(x) = \int_2^x \sqrt{3t^2+1}\,dt, \quad F(2),\ F'(2)$ and $F''(2)$.
20. $F(x) = \int_x^0 \frac{\sin t}{t+1}\,dt, \quad G(0),\ G'(0)$ and $G''(0)$.
21. $F(x) = \int_x^{x^2} \sqrt[5]{t+1}\,dt, \quad H'(2)$.
22. $F(x) = \sin x \int_0^x \big(1 + F'(t)\big)\,dt, \quad F(0)$ and $F'(0)$.

In this section, we will discuss two techniques of numerical integration to approximate definite integrals: Trapezoidal rule and Simpson's rule.

### (A) Trapezoidal Rule

As discussed before, a Riemann sum approximates the area underneath a curve of a function $f$ from $x = a$ to $x = b$ as follows.

**STEP 1:** We divide the interval $[a, b]$ by a regular partition $P$ to generate $n$ sub-intervals: $[x_0, x_1], [x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n]$.
**STEP 2:** We find the length of the sub-intervals: $\Delta x_k = \frac{b-a}{n}$.
**STEP 3:** We use the formula

$$\int_a^b f(x)\ dx \approx \frac{b-a}{2n}\Big[\sum_{k=1}^{n} f(\omega_{k-1}) + \sum_{k=1}^{n} f(\omega_k)\Big]$$

$$= \frac{b-a}{2n}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)\Big] .$$

**Error Estimation**

Although the numerical methods give an approximated value of a definite integral, there is a possibility that an error occurs.

**Theorem 13** *Suppose $f''$ is continuous on $[a, b]$ and $M$ is the maximum value for $f''$ over $[a, b]$. If $E_T$ is the error in calculating $\int_a^b f(x)\ dx$ under the trapezoidal rule, then*

$$| E_T |< \frac{M(b-a)^3}{12\ n^2} .$$

**Example 28** *By using the trapezoidal rule with $n = 4$, approximate the integral $\int_1^2 \frac{1}{x}\ dx$. Then, estimate the error.*

Solution:

(1) We approximate the integral $\int_1^2 \frac{1}{x}\ dx$ by the trapezoidal rule.

(a) Divide the interval $[a, b]$ into sub-intervals. The length of each sub-intervals is $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$. (b) Find the partition $P = \{x_0, x_1, x_2, ..., x_n\}$ where $x_k = x_0 + k(\Delta x) = x_0 + k\frac{(b-a)}{n}$.
The partition:
$x_0 = 1$,
$x_1 = 1 + \frac{1}{4} = 1\frac{1}{4}$,
$x_2 = 1 + 2\frac{1}{4} = 1\frac{1}{2}$,
$x_3 = 1 + 3\frac{1}{4} = 1\frac{3}{4}$, and
$x_4 = 1 + 4\frac{1}{4} = 2$ .
Thus $P = \{1, 1.25, 1.5, 1.75, 2\}$.
(c) Approximate the integral by using the following table:

| $k$ | $x_k$ | $f(x_k)$ | $m_k$ | $m_k f(x_k)$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 1.25 | 0.8 | 2 | 1.6 |
| 2 | 1.5 | 0.6667 | 2 | 1.3334 |
| 3 | 1.75 | 0.5714 | 2 | 1.1428 |
| 4 | 2 | 0.5 | 1 | 0.5 |
| Sum = $\sum_{k=1}^{4} m_k f(x_n)$ | | | | 5.5762 |

Thus, $\int_1^2 \frac{1}{x}\ dx \approx \frac{1}{8}\big[5.5762\big] = 0.6970$ .
(2) We estimate the error by using the theorem 13.

$$f(x) = \frac{1}{x} \Rightarrow f'(x) = \frac{-1}{x^2} \Rightarrow f''(x) = \frac{2}{x^3} .$$

Since $f''(x)$ is a decreasing function on the interval $[1, 2]$, then $f''(x)$ is maximized at $x = 1$. This means $M =| f''(1) |= 2$ and

$$| E_T |< \frac{2(2-1)^3}{12(4)^2} = \frac{1}{96} = 0.0104 .$$

**Remark 4** *By knowing the error amount, we can determine the number of the sub-intervals $n$ before starting approximating.*

**Example 29** *Find number of the sub-intervals to approximate the integral $\int_1^2 \frac{1}{x}\ dx$ such that the error is less than $10^{-3}$.*

Solution:

From the previous example, we know that $M = 2$. Thus, $| E_T |< \frac{2(2-1)^3}{12n^2} < 10^{-3}$. This implies that

$$n^2 > \frac{2(2-1)^3}{12}10^3 = \frac{10^3}{6} \Rightarrow n > \sqrt{\frac{500}{3}} = 12.91$$

This means we consider $n = 13$ .

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## (B) Simpson's Rule

Simpson's rule is another numerical method to approximate definite integrals.

**STEP 1:** We divide the interval $[a,b]$ by a regular partition $P$ to generate $n$ sub-intervals and $n$ **is an even number**: $[x_0,x_1],[x_1,x_2],[x_2,x_3],...,[x_{n-1},x_n]$.
**STEP 2:** We find the length of the sub-intervals: $\Delta x_k = \frac{b-a}{n}$.
**STEP 3:** We use the formula

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{3n}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+...+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big].$$

### Error Estimation

The estimation of the error under the Simpson's method is calculated by the following theorem.

**Theorem 14** *Suppose $f^{(4)}$ is continuous on $[a,b]$ and $M$ is the maximum value for $f^{(4)}$ on $[a,b]$. If $E_S$ is the error in calculating $\int_a^b f(x)\,dx$ under Simpson's rule, then*

$$| E_S | < \frac{M(b-a)^5}{180\ n^4}.$$

**Example 30** *By using the Simpson's rule with $n=4$, approximate the integral $\int_1^3 x^2+1\,dx$. Then, estimate the error.*

Solution:

(1) We approximate the integral $\int_1^3 x^2+1\,dx$ by the Simpson's rule.

(a) Divide the interval $[1,3]$ into sub-intervals. The length of each sub-intervals is $\Delta x = \frac{3-1}{4} = \frac{1}{2}$.
(b) Find the partition $P=\{x_0,x_1,x_2,...,x_n\}$ where $x_k = x_0 + k(\Delta x) = x_0 + k\,\frac{(b-a)}{n}$.
The partition:
$x_0 = 1$,
$x_1 = 1+\frac{1}{2} = 1\frac{1}{2}$,
$x_2 = 1+2\frac{1}{2} = 2$,
$x_3 = 1+3\frac{1}{2} = 2\frac{1}{2}$, and
$x_4 = 1+4\frac{1}{2} = 3$ .
Thus $P=\{1,1.5,2,2.5,3\}$.

(c) Approximate the integral by using the following table:

| $k$ | $x_k$ | $f(x_k)$ | $m_k$ | $m_k f(x_k)$ |
|---|---|---|---|---|
| 0 | 1 | 1.4142 | 1 | 1.4142 |
| 1 | 1.5 | 1.8028 | 4 | 7.2111 |
| 2 | 2 | 2.2361 | 2 | 4.4721 |
| 3 | 2.5 | 2.6926 | 4 | 10.7703 |
| 4 | 3 | 3.1623 | 1 | 3.1623 |
| | Sum $=\sum_{k=1}^4 m_k f(x_n)$ | | | 27.0301 |

Thus, $\int_1^2 \frac{1}{x}\,dx \approx \frac{2}{12}\big[27.0301\big] = 4.5050$ .
(2) We estimate the error by using the theorem 14.
Since $f^{(5)}(x) = -(15x(4x^2-3))/\sqrt{(x^2+1)^9}$, then $f^{(4)}(x)$ is a decreasing function on the interval $[1,3]$. Thus, $f^{(4)}(x)$ is maximized at $x=1$. This means $M = | f^{(4)}(1) | = 0.7955$ and

$$| E_s | < \frac{0.7955(3-1)^5}{180(4)^4} = 5.5243\times 10^{-4}.$$

**Example 31** *Find number of the sub-intervals to approximate the integral $\int_1^3 x^2+1\,dx$ such that the error is less than $10^{-2}$.*

Solution:

From the previous example, we know that $M=0.7955$. Thus, $| E_S | < \frac{2(3-1)^5}{180n^4} < 10^{-2}$. This implies that

$$n^4 > \frac{0.7955(2-1)^5}{180}10^2 \Rightarrow n > 3.7606$$

This means we consider take $n=4$.

**1 - 2** ■ By using trapezoidal rule and Simpson's rule, approximate the definite integral for the given $n$, then estimate the error:

1. $\int_{-1}^1 \sqrt{x^2+1}\,dx$, $n=4$
2. $\int_0^\pi \sin x\,dx$, $n=6$

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## (A) Natural Logarithmic Function

As mentioned in Chapter 5, the integral $\int x^r\, dx = \frac{x^{r+1}}{r+1} + c$ if $r \neq -1$. This means, the previous formula cannot be used when $r = -1$ because the denominator will become zero. The task in this section is to find the general anti-derivative of the function $\frac{1}{x}$ i. e., we are looking for a function $F(x)$ such that $F'(x) = \frac{1}{x}$.
Consider the function $f(t) = \frac{1}{t}$. It is continuous on the interval $(0, +\infty)$ and this implies the function is integrable on the interval $[1, x]$. The area under the graph of the function $f(t) = \frac{1}{t}$ bounded from $t = 1$ to $t = x$ as shown in the Figure 2 is

$$f(x) = \int_1^x \frac{1}{t}\, dx$$

**Definition 9** *The natural logarithmic function is defined as follows:*

$$ln : (0, \infty) \to \mathbb{R} ,$$

$$ln(x) = \int_1^x \frac{1}{t}\, dt$$

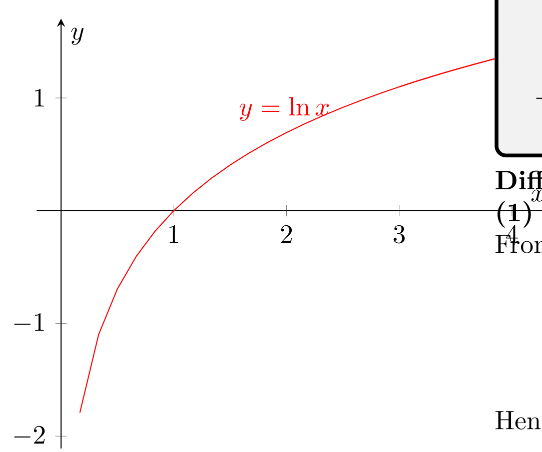


Figure 2: The graph of the function $y = \ln x$.

### Properties of the Natural Logarithmic Function

1. The domain of the function $\ln(x)$ is $(0, \infty)$.
2. The range of the function $\ln(x)$ is $\mathbb{R}$ as follows:

$$\ln(x) = \begin{cases} y > 0 & : x > 1 \\ y = 0 & : x = 1 \\ y < 0 & : 0 < x < 1 \end{cases}$$

3. The function $\ln(x)$ is differentiable and continuous on the domain. From the fundamental theorem of calculus, we have

$$\frac{d}{dx}(\ln(x)) = \frac{d}{dx}\int_1^x \frac{1}{t}\, dt = \frac{1}{x}, \forall x > 0 .$$

From this, the function $\ln(x)$ is increasing on the interval $(0, \infty)$.
4. The second derivative $\frac{d^2}{dx^2}(\ln(x)) = \frac{-1}{x^2} < 0$ for all $x \in (0, \infty)$. Hence, the function $\ln(x)$ is concave downward on the interval $(0, \infty)$.
5. $\lim_{x\to 0^+} \ln(x) = -\infty$ and $\lim_{x\to\infty} \ln(x) = +\infty$.

**Theorem 15** *For every $a, b > 0$ and $r \in \mathbb{Q}$ [a], then*
*1.* $\ln(a\ b) = \ln(a) + \ln(b)$ .
*2.* $\ln(\frac{a}{b}) = \ln(a) - \ln(b)$ .
*3.* $\ln(a^r) = r \ln(a)$ .

[a] $Q$ is a set of rational numbers.

### Differentiating and Integrating Natural Logarithm Function

#### (1) Differentiating Natural Logarithm Function

From our discussion above, we know that

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Hence,

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x} .$$

From this, we have

$$\frac{d}{dx}\ln(|\ x\ |) = \frac{1}{x}\ \forall x \neq 0 .$$

Generally, if $u = g(x)$ is differentiable and $u \neq 0$ for every $x$ in an interval $I$ , then

$$\frac{d}{dx}\ln(|\ u\ |) = \frac{1}{u}\frac{du}{dx}, \forall x \in I$$

**Example 32** *Find the derivative of the following functions:*

1. $f(x) = \ln(x+1)$
2. $g(x) = \ln(x^3 + 2x - 1)$
3. $h(x) = \ln(\sqrt{x^2+1})$
4. $y = \sqrt{\ln(x)}$
5. $f(x) = \ln(\cos x)$
6. $g(x) = \sqrt{x}\ln(x)$
7. $h(x) = \sin(\ln(x))$
8. $y(x) = \ln(x + \ln x)$

Solution:

**Example 33** *Find the derivative of the function:* $y = \sqrt[5]{\frac{x-1}{x+1}}$ .

Solution:

**Example 34** *Find the derivative of the function:* $y = \dfrac{\sqrt{x}\cos x}{(x+1)\sin x}$ .

**(2) Integrating Natural Logarithm Function**

$$\int \frac{u'}{u}\, dx = \ln|u| + c$$

and

$$\int \frac{1}{x}\, dx = \ln|x| + c$$

**Example 35** *Evaluate the following integrals*

1. $\displaystyle\int \frac{2x}{x^2+1}\, dx$
2. $\displaystyle\int \frac{6x^2+1}{4x^3+2x+1}\, dx$
3. $\displaystyle\int_2^e \frac{dx}{x\ln x}$
4. $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}(1+\sqrt{x})}$
5. $\displaystyle\int \tan x\, dx$
6. $\displaystyle\int \cot x\, dx$
7. $\displaystyle\int \sec x\, dx$
8. $\displaystyle\int \csc x\, dx$

Solution:

## (B) Natural Exponential Function

The natural logarithm function $\ln : (0,\infty) \longrightarrow \mathbb{R}$ is one-to-one since it is a strictly increasing function (see Figure 3). The function ln is also onto and this implies that the natural logarithm function has an inverse function. This function is called the natural exponential.

**Definition 10** *The natural exponential function is defined as follows:*

$$exp : \mathbb{R} \longrightarrow (0,\infty) \ ,$$

$$y = exp(x) \Leftrightarrow \ln y = x$$

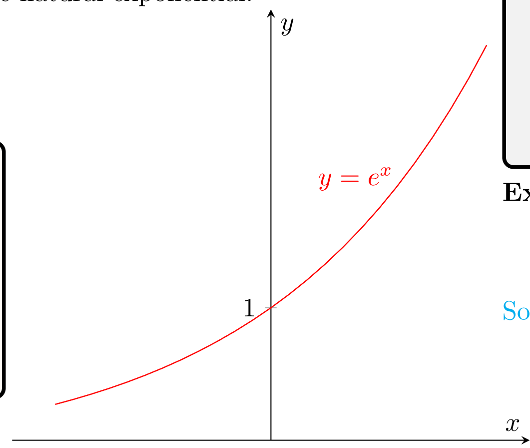


Figure 3: The graph of the function $y = e^x$.

**Theorem 16** *For every $a, b > 0$ and $r \in \mathbb{Q}$ , then*
1. $e^a e^b = e^{a+b}$ .
2. $\dfrac{e^a}{e^b} = e^{a-b}$ .
3. $(e^a)^r = e^{ar}$ .

**Example 36** *Find value of $x$:*
1. $\ln x = 2$
2. $\ln(\ln x) = 0$
3. $(x-1)e^{-\ln\frac{1}{x}} = 2$
4. $xe^{2\ln x} = 8$

Solution:

**Example 37** *Simplify the following:*
1. $\ln(e^{\sqrt{x}})$
2. $e^{\frac{1}{3}\ln x}$
3. $(x+1)\ln(e^{x-1})$
4. $e^{(\sqrt{x}+2\ln x)}$

Solution:

### Properties of the Natural Exponential Function

1. The domain of the function $exp(x)$ is $\mathbb{R}$.
2. The range of the function $exp(x)$ is $(0,\infty)$ as follows:

$$exp(x) = \begin{cases} y > 1 & : x > 0 \\ y = 1 & : x = 0 \\ y < 1 & : x < 0 \end{cases}$$

3. Usually, the symbol $exp(x)$ is written as $e^x$. Thus, $exp(1) = e$ and from the definition of the natural exponential function, we have $\ln(e) = 1$. Also, $\ln(e^r) = r \ln e = r \ \forall \ r \in \mathbb{Q}$.
4. The function $e^x$ is differentiable and continuous on the domain

$$\frac{d}{dx}(e^x) = e^x, \forall x \in \mathbb{R} \ .$$

From this, the function $e^x$ is increasing on $\mathbb{R}$.

5. The second derivative $\frac{d^2}{dx^2}(e^x) = e^x > 0$ for all $x \in \mathbb{R}$. Hence, the function $e^x$ is concave upward on the domain $\mathbb{R}$.
6. $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.
7. Since $e^x$ and $\ln x$ are inverse functions, then

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### Differentiating and Integrating Natural Exponential Function

### (A) Differentiating Natural Exponential Function

$$\frac{d}{dx}e^x = e^x$$

Generally, if $u = g(x)$ is differentiable on the interval $I$, then

$$\frac{d}{dx}e^u = e^u u', \ \forall x \in I$$

**Example 38** *Find the derivative of the following functions:*

1. $y = e^{\sqrt[3]{x+1}}$
2. $y = e^{-5x^2}$
3. $y = e^{3\cos x - 4x^2}$
4. $y = e^{\frac{1}{x}} - \frac{1}{e^x}$
5. $y = e^{\ln \sin x}$
6. $y = \ln(e^{2x} + \sqrt{1 - e^x})$

Solution:

### (B) Integrating Natural Exponential Function

Recall, $\frac{d}{dx}e^u = e^u u'$ where $u = g(x)$ is a differentiable function. By integrating both sides, we have

$$\begin{aligned}\int e^u u' &= \int \frac{d}{dx}e^u \ dx + c \\ &= e^u + c\end{aligned}$$

This can be stated as follows

$$\int e^u u' = e^u + c$$

and

$$\int e^x \ dx = e^x + c$$

**Example 39** *Evaluate the following integrals:*

1. $\int xe^{-x^2} \ dx$
2. $\int_0^{ln5} e^x(3 - 4e^x) \ dx$
3. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} \ dx$
4. $\int \frac{e^{\tan x}}{\cos^2 x} \ dx$

Solution:

## General Exponential and Logarithmic Functions

### (A) General Exponential Function

We defined the natural exponential function $a^x$ when $a = e$. In the following, we define the general case where we always assume that $a > 0$.

**Definition 11** *The general logarithmic function is defined as follows:*

$$\log_a : \mathbb{R} \to (0, \infty) \ ,$$
$$a^x = e^{x \ln a} \ .$$

The function $a^x$ is called the general exponential function for the base $a$.

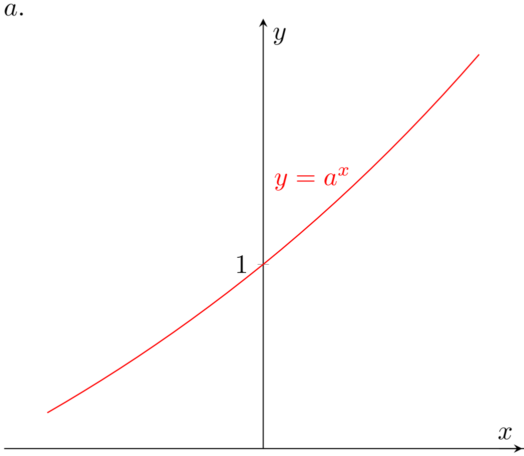


Figure 4: The function $y = a^x$ for $a > 1$.

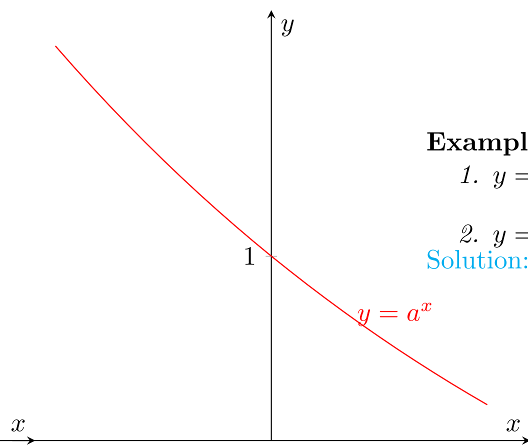


Figure 5: The function $y = a^x$ for $a < 1$.

### Properties of the General Exponential Function

Let $f(x) = a^x \ \forall x \in \mathbb{R}$.

1. The domain of $f(x)$ is $\mathbb{R}$ and the range is $(0, \infty)$.

2. If $a > 1$, $\ln a > 0$ and this implies that $x \ln a$ is an increasing function with $x$. This indicates that $f(x)$ is an increasing function. However, if $a < 1$, $\ln a < 0$ and this implies that $x \ln a$ and $f(x)$ are decreasing functions. Figure 5 shows this remark for $a > 1$ and $a < 1$.

3. $\ln a^x = x \ln a$ for all $x \in \mathbb{R}$.

**Theorem 17** *For every $x, y > 0$ and $a, b \in \mathbb{R}$ , then*

1. $x^a x^b = x^{a+b}$ .
2. $\frac{x^a}{x^b} = x^{a-b}$ .
3. $(x^a)^b = x^{a\ b}$ .
4. $(xy)^a = x^a y^a$ .

### Differentiating and Integrating General Exponential Function

If $u = g(x)$ is a differentiable function, then

$$\frac{d}{dx}a^u = a^u.u'.\ln a \ \Rightarrow \int a^u.u' \, dx = \frac{1}{\ln a}.a^u + c$$

**Example 40** *Find the derivative of the following functions:*

1. $y = 2^{\sqrt{x}}$
2. $y = 3^{x^2 \sin x}$
3. $\ln(\tan 5^x)$
4. $y = (10^x + 10^{-x})^{10}$

Solution:

**Example 41** *Find the derivative of the following function $y = (\sin x)^x$.*

Solution:

Take ln for both sides. This implies $\ln y = x \ln(\sin x)$.

Now, differentiating both sides yields

$$\frac{y'}{y} = \ln(\sin x) + \frac{x \cos x}{\sin x}$$
$$\Rightarrow y' = \big(\ln(\sin x) + \frac{x \cos x}{\sin x}\big)(\sin x)^x \ .$$

**Example 42** *Evaluate the following integrals:*

1. $\int x3^{-x^2}\, dx$
2. $\int 5^x\sqrt{5^x+1}\, dx$
3. $\int 3^x \sin 3^x\, dx$
4. $\int \frac{2^x}{2^x+1}\, dx$

Solution:

### (B) General Logarithmic Function

**Definition 12** *The general logarithmic function is defined as follows:*

$$\log_a : (0,\infty) \to \mathbb{R}\ ,$$

$$x = a^y \Leftrightarrow y = \log_a(x)\ .$$

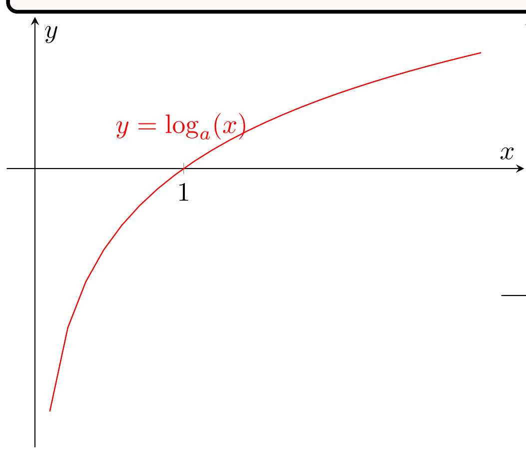


Figure 6: The function $y = \log_a(x)$ for $a > 1$.

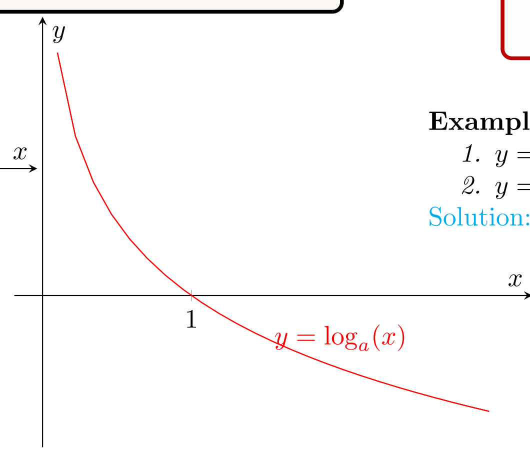


Figure 7: The function $y = \log_a(x)$ for $a < 1$.

### Properties of the General Logarithm Function

1. The function $\log_a x = \frac{\ln x}{\ln a}$.
   To see this, let $y = \log_a x \Rightarrow x = a^y$. Take ln for both sides,
   $$\ln x = \ln a^y = y \ln a \Rightarrow y = \frac{\ln x}{\ln a}\ .$$
2. If $a > 1$, the function $\log_a(x)$ is increasing function, but if $0 < a < 1$, the function $\log_a(x)$ is decreasing function (see Figure 7).
3. The function ln is $\log_e$.
4. The function $\log_{10}$ is log.
5. $\log_a(a) = 1$.

**Theorem 18** *For every $x, y > 0$ and $r \in \mathbb{R}$, then*

1. $\log_a(xy) = \log_a(x) + \log_a(y)$ .
2. $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$ .
3. $\log_a(x^r) = r\log_a(x)$ .

### Differentiating and Integrating General Logarithmic Function

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a}.u' \quad \text{and} \quad \int \frac{1}{u\ln a}.u'\, dx = \log_a(u) + c$$

**Example 43** *Find the derivatives of the following functions:*

1. $y = \log_3 \sin x$ .
2. $y = \log\sqrt{x}$ .

Solution:

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### (A) Inverse Trigonometric Functions

The inverse trigonometric functions are the inverse functions of the trigonometric functions: the sine, cosine, tangent, cotangent, secant, and cosecant functions. The inverse trigonometric functions give angles from any of the angle's trigonometric ratios. The most common notations to name the inverse trigonometric functions are $arcsin(x)$, $arccos(x)$, $arctan(x)$, etc. However, the notations $\sin^{-1}(x), \cos^{-1}(x), \tan^{-1}(x)$, etc., are often used as well.

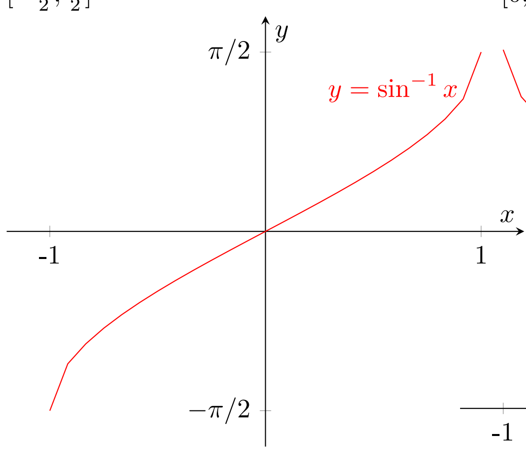
**Note:** To find the inverse of any function, we need to show bijection of that function (i.e., it is one-to-one and onto).

Common mistake:

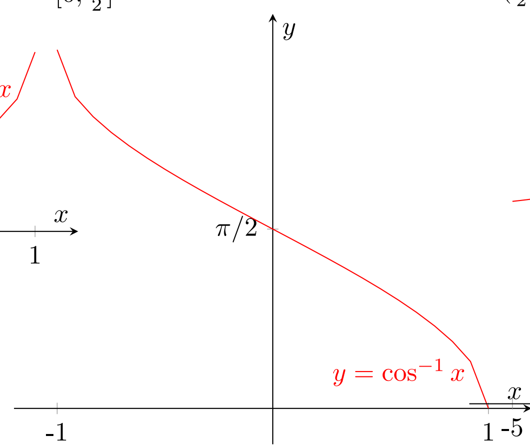
$\sin^{-1}(x) = (\sin(x))^{-1} = \dfrac{1}{\sin(x)}$

which is not true.

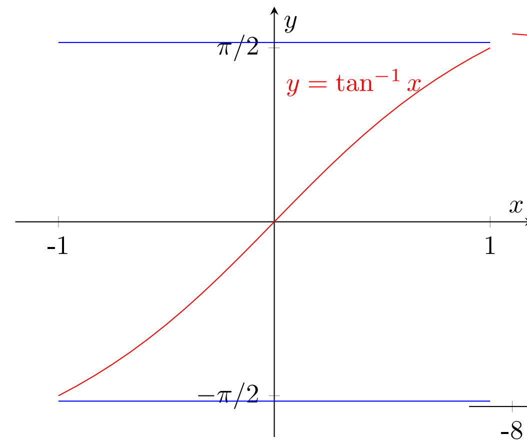
(1) The inverse sine
$\sin y = x \Leftrightarrow y = \sin^{-1} x$
Domain: $[-1, 1]$ Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$



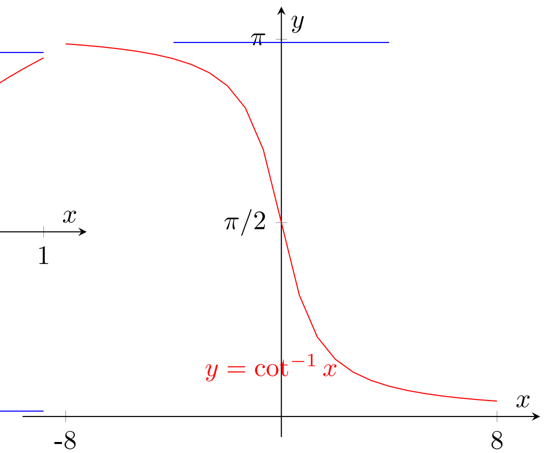

(2) The inverse cosine
$\cos y = x \Leftrightarrow y = \cos^{-1} x$
Domain: $[-1, 1]$ Range: $[0, \frac{\pi}{2}]$



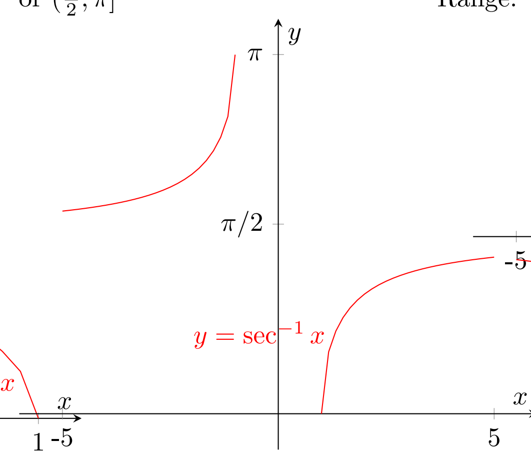

(3) The inverse tangent
$\tan y = x \Leftrightarrow y = \tan^{-1} x$
Domain: $\mathbb{R}$
Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$



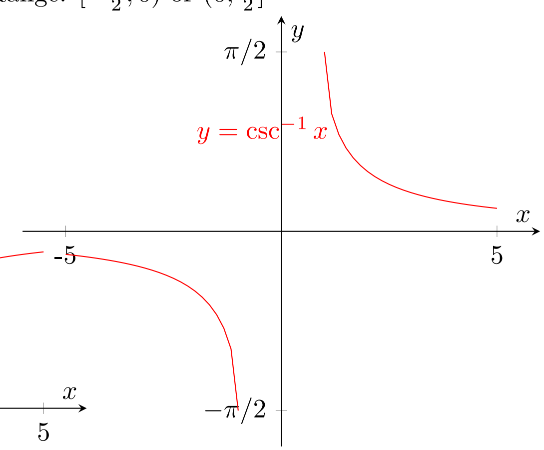

(4) The inverse cotangent
$\cot y = x \Leftrightarrow y = \cot^{-1} x$
Domain: $\mathbb{R}$ Range: $(0, \pi)$




(5) The inverse secant
$sec\ y = x \Leftrightarrow y = sec^{-1}\ x$
Domain: $\mathbb{R} \setminus (-1, 1)$ Range: $[0, \frac{\pi}{2})$ or $(\frac{\pi}{2}, \pi]$




(6) The inverse cosecant
$csc\ y = x \Leftrightarrow y = csc^{-1}\ x$
Domain: $\mathbb{R} \setminus (-1, 1)$
Range: $[-\frac{\pi}{2}, 0)$ or $(0, \frac{\pi}{2}]$



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## Differentiating and Integrating Inverse Trigonometric Functions

In the following, we list the derivatives of the inverse trigonometric functions. Then, we list the integral rules.

In general, if $u = g(x)$ is differentiable function, then

1. $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}u'$
2. $\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}u'$
3. $\frac{d}{dx}\tan^{-1} u = \frac{1}{u^2+1}u'$
4. $\frac{d}{dx}\cot^{-1} u = \frac{-1}{u^2+1}u'$
5. $\frac{d}{dx}\sec^{-1} u = \frac{1}{u\sqrt{u^2-1}}u'$
6. $\frac{d}{dx}\csc^{-1} u = \frac{-1}{u\sqrt{u^2-1}}u'$

**Example 44** *Find the derivatives of the following functions:*

1. $y = \sin^{-1}(5x)$
2. $y = \tan^{-1}(e^x)$
3. $y = \sec^{-1}(2x)$
4. $y = \sin^{-1}(x-1)$

Solution:

From the list of the derivatives of the inverse trigonometric functions, we have the following integral rules:

1. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}(\frac{x}{a}) + c$ .
2. $\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$ .
3. $\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = -\frac{1}{a}sec^{-1}(\frac{x}{a}) + c$ .

**Example 45** *Evaluate the following integrals:*

1. $\int \frac{1}{\sqrt{4-25x^2}}\, dx$ .
2. $\int \frac{1}{x\sqrt{x^6-4}}\, dx$ .
3. $\int \frac{1}{9x^2+5}\, dx$ .
4. $\int \frac{1}{\sqrt{e^{2x}-1}}\, dx$ .

Solution:

## (B) Hyperbolic Functions

In this section, we present hyperbolic functions that are based on the natural exponential function. They are analogs of the ordinary trigonometric functions. In the sense that, the hyperbolic functions share many properties with the corresponding trigonometric functions.

**Definition 13** *The hyperbolic sine* (sinh) *and the hyperbolic cosine* (cosh) *are defined as follows:*

$$\sinh x = \frac{e^x - e^{-x}}{2}, \ \forall x \in \mathbb{R} \ ,$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \ \forall x \in \mathbb{R} \ .$$

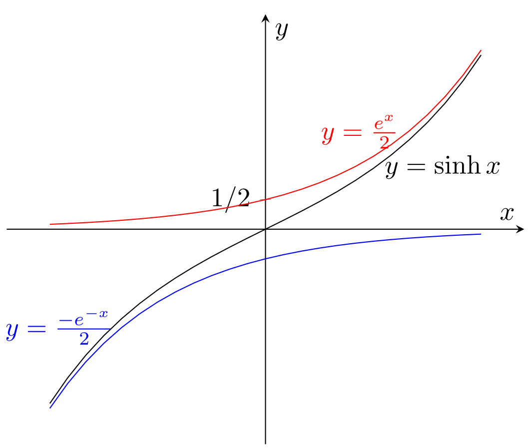
The remaining hyperbolic functions can be defined from the hyperbolic sine and the hyperbolic cosine as follows:

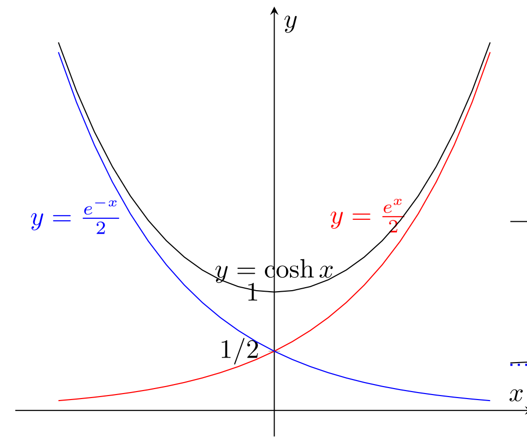
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \forall x \in \mathbb{R}$$

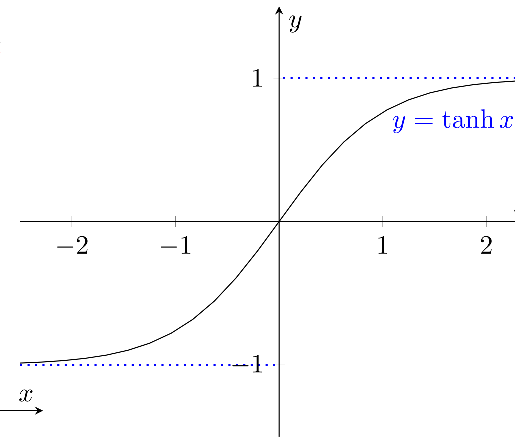
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad \forall x \neq 0$$

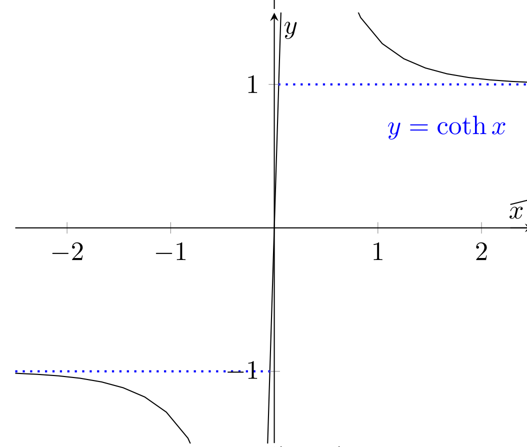
$$sech x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad \forall x \in \mathbb{R}$$

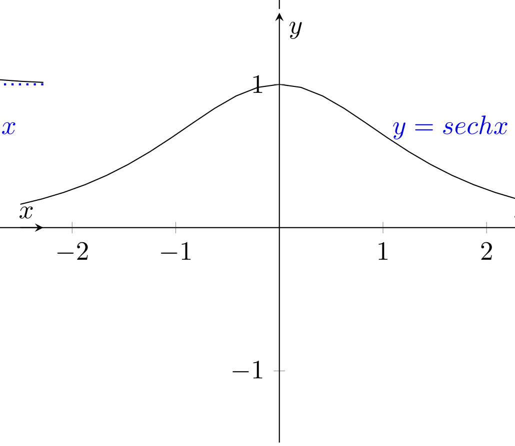
$$csch x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad \forall x \neq 0$$

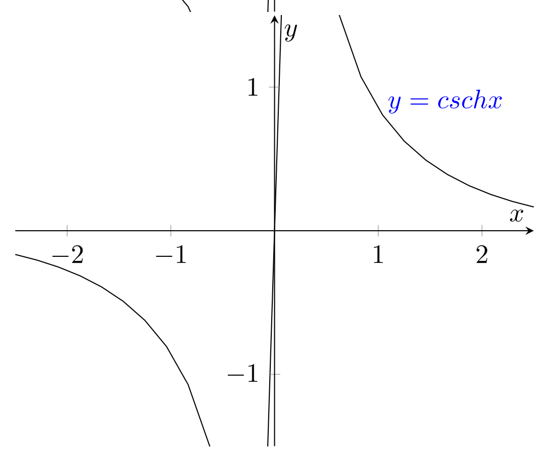


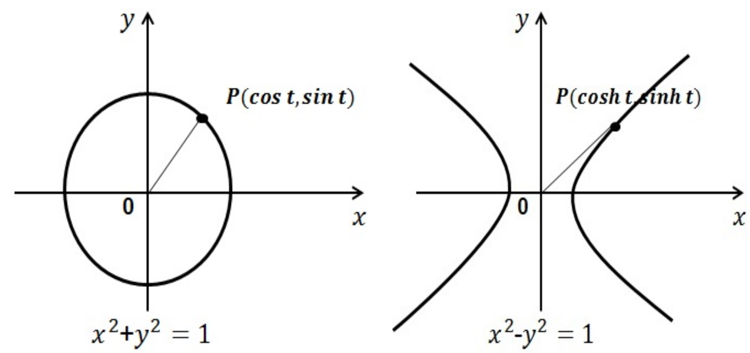



## Properties of Hyperbolic Functions

We provide the main characteristics of the hyperbolic functions.

1. The graph of the hyperbolic sine (sinh) and the hyperbolic cosine depends on the natural exponential functions $e^x$ and $e^{-x}$ (as shown in Figure **??**).
2. From Figure **??**, the range of sinh is $\mathbb{R}$ and the range of cosh is $[1, \infty)$.
3. The hyperbolic sine is an odd function (i.e., $\sinh(-x) = -\sinh x$); whereas the hyperbolic cosine is an even function (i.e., $\cosh(-x) = \cosh x$). Hence, the functions tanh, coth and *csch* are odd functions and the function *sech* is even. This means that the graphs of the functions sinh, tanh, coth and *csch* are symmetric around the original point; whereas the graph of the functions cosh and *sech* are symmetric around the y-axis.
4. $\cosh^2 x - \sinh^2 x = 1, \ \forall x \in \mathbb{R}$ .
   To see this, let $x \in \mathbb{R}$. We can find that
   $$\cosh x - \sinh x = e^{-x} \text{and} \cosh x + \sinh x = e^x \ .$$
   Therefore,
   $$(\cosh x - \sinh x)(\cosh x + \sinh x) = \cosh^2 x - \sinh^2 x = e^{-x}e^x = e^0 = 1 \ .$$
5. Since $\cos^2 t + \sin^2 t = 1$ for any $t \in \mathbb{R}$, then the point $P(\cos t, \sin t)$ located on the unit circle $x^2 + y^2 = 1$. However, for any $t \in \mathbb{R}$, the point $P(\cosh t, \sinh t)$ located on the hyperbola $x^2 - y^2 = 1$. Figure illustrates this item.




The inverse hyperbolic functions can be formalized as natural logarithmic functions. This because is the hyperbolic functions depend on the natural exponential function. The following theorem shows how the inverse hyperbolic functions can be expressed in terms of the natural logarithm.

**Theorem 19**

1. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ .
2. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$ .
3. $\sinh(2x) = 2 \sinh x \cosh x$ .
4. $\cosh(2x) = 2\cosh^2 x - 1 = 2\sinh^2 x + 1 = \cosh^2 x + \sinh^2 x$ .
5. $1 - \tanh^2 x = sech^2 x$ .
6. $\coth^2 x - 1 = csch^2 x$ .
7. $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$ .
8. $\tanh(2x) = \dfrac{2 \tanh x}{1 + \tanh^2 x}$ .

**Theorem 20**

1. $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \ \forall x \in \mathbb{R}$ .
2. $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \ \forall x \in [1, \infty)$ .
3. $\tanh^{-1} x = \frac{1}{2} \ln \mid \frac{1+x}{1-x} \mid, \ \forall x \in (-1, 1)$ .
4. $\coth^{-1} x = \frac{1}{2} \ln \mid \frac{x+1}{x-1} \mid, \ \forall x \in \mathbb{R} \setminus [-1, 1]$ .
5. $sech^{-1} x = \ln(\frac{1+\sqrt{1-x^2}}{x}), \ \forall x \in \mathbb{R} \setminus [-1, 1]$ .
6. $csch^{-1} x = \ln(\frac{1}{x} + \sqrt{1 + (\frac{1}{x})^2}, \ \forall x \in \mathbb{R} \setminus \{0\}$ .

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**Differentiating and Integrating Hyperbolic Functions**

The derivations of the hyperbolic functions are listed in Theorem 21 .

**Theorem 21**

1. $\frac{d}{dx}\sinh x = \cosh x$
2. $\frac{d}{dx}\cosh x = \sinh x$
3. $\frac{d}{dx}\tan x = sech^2 x$
4. $\frac{d}{dx}\coth x = -csch^2 x$
5. $\frac{d}{dx}sechx = -sechx \tanh x$
6. $\frac{d}{dx}cschx = -cschx \coth x$

**Example 46** *Find the derivative of the following functions:*

1. $y = \sinh x^2$
2. $y = \sqrt{x}\cosh x$
3. $y = e^{\sinh x}$
4. $y = (x+1)\tanh^2 x^3$

Solution:

From the list of the derivation given in Theorem 21, we have the following list of integrals:

- $\int \sinh x \, dx = \cosh x + c$
- $\int \cosh x \, dx = \sinh x + c$
- $\int sech^2 x \, dx = \tanh x + c$
- $\int csch^2 x \, dx = -\coth x + c$
- $\int sechx \tanh x \, dx = -sechx + c$
- $\int cschx \coth x \, dx = -cschx + c$

**Example 47** *Evaluate the following integrals:*

1. $\int \sinh^2 x \cosh x \, dx$
2. $\int e^{\cosh x} \sinh x \, dx$
3. $\int \tanh x \, dx$
4. $\int e^x \, sechx \, dx$

Solution:

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## (C) Inverse Hyperbolic Functions

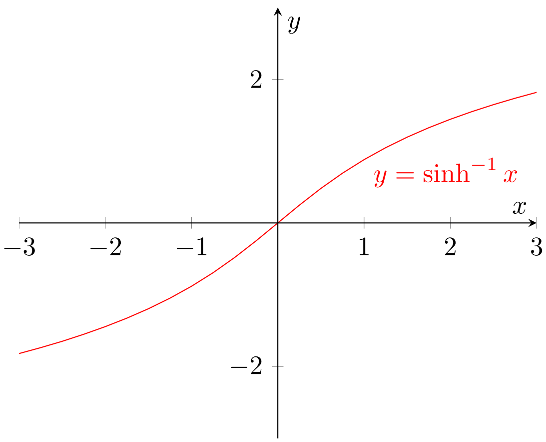
In the first section of this chapter, we defined the inverse trigonometric functions. In analogical way, we define the inverse hyperbolic functions.

**Definition and Properties**

(1) The function $\sinh : \mathbb{R} \to \mathbb{R}$ is bijective (i.e., it is one-to-one and onto), so it has an inverse function

$$\sinh^{-1} : \mathbb{R} \to \mathbb{R}$$

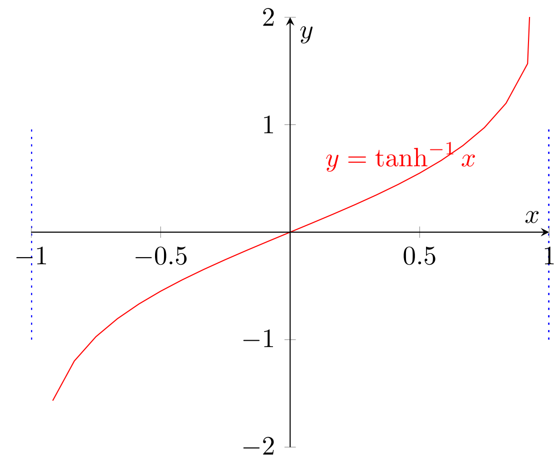
$$\sinh y = x \Leftrightarrow y = \sinh^{-1} x$$




(4) The function $\coth : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus [-1, 1]$ is bijective, so it has an inverse function

$$\coth^{-1} : \mathbb{R}\backslash[-1,1] \to \mathbb{R}\backslash\{0\}$$

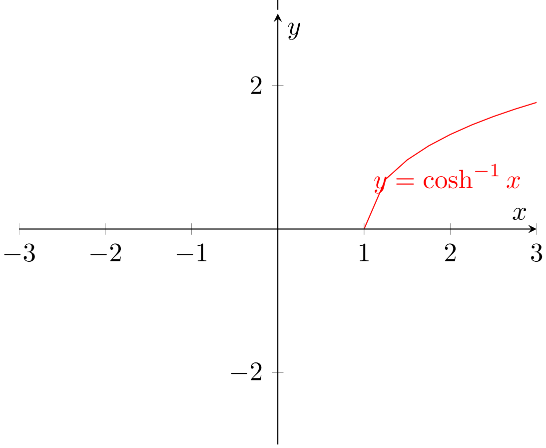
$$\coth y = x \Leftrightarrow y = \coth^{-1} x$$




(2) The function cosh is injective on $[0, \infty)$, so $\cosh : [0, \infty) \to [1, \infty)$ is bijective on $[0, \infty)$. It has an inverse function

$$\cosh^{-1} : [1, \infty) \to [0, \infty)$$

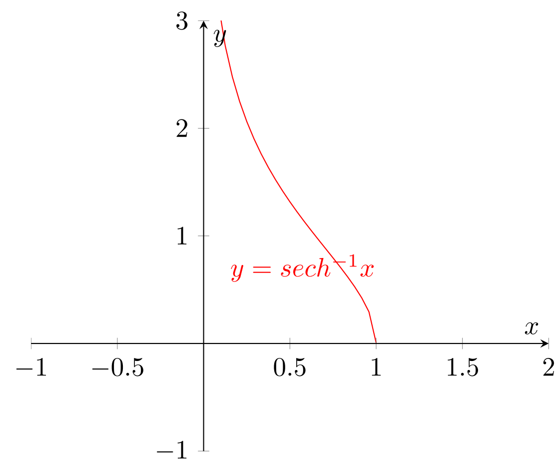
$$\cosh y = x \Leftrightarrow y = \cosh^{-1} x$$




(5) The function : $(0, 1] \to [0, \infty)$ is bijective and the inverse function is

$$sech^{-1} : [0, \infty) \to (0, 1]$$

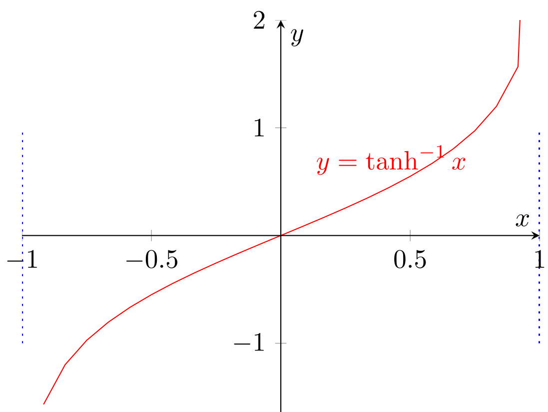
$$sechy = x \Leftrightarrow y = sech^{-1}x$$




(3) The function $\tanh : \mathbb{R} \to (-1, 1)$ is bijective, so it has an inverse function

$$\tanh^{-1} : (-1, 1) \to \mathbb{R}$$

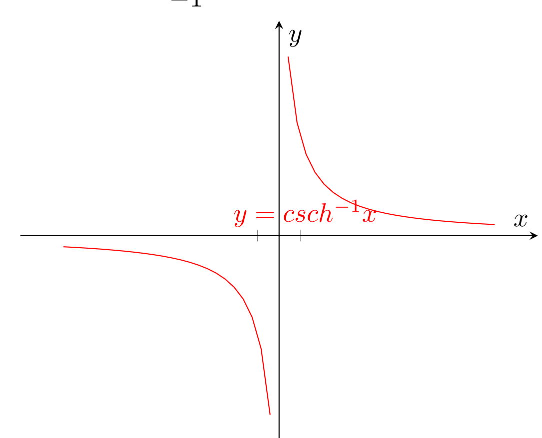
$$\tanh y = x \Leftrightarrow y = \tanh^{-1} x$$




(6) The function : $\mathbb{R}\setminus\{0\} \to \mathbb{R}\backslash\{0\}$ is bijective and the inverse function is

$$csch^{-1} : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$$

$$cschy = x \Leftrightarrow y = csch^{-1}x$$



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## Differentiation and Integration

**Theorem 22** *If $u = g(x)$ is differentiable function, then*

1. $\frac{d}{dx}\sinh^{-1}(u) = \frac{1}{\sqrt{u^2+1}}$ .
2. $\frac{d}{dx}\cosh^{-1}(u) = \frac{1}{\sqrt{u^2-1}}, \ \forall u \in (1, \infty)$ .
3. $\frac{d}{dx}\tanh^{-1}(u) = \frac{1}{1-u^2}, \ \forall u \in (-1, 1)$ .
4. $\frac{d}{dx}\coth^{-1}(u) = \frac{1}{1-u^2}, \ \forall u \in \mathbb{R} \setminus [-1, 1]$ .
5. $\frac{d}{dx}sech^{-1}(u) = \frac{-1}{u\sqrt{1-u^2}}, \ \forall u \in (0, 1)$ .
6. $\frac{d}{dx}csch^{-1}(u) = \frac{-1}{|u|\sqrt{u^2+1}}, \ \forall u \in \mathbb{R} \setminus \{0\}$ .

**Example 48** *Find the derivative of the following functions:*

1. $y = \sinh^{-1}(\sqrt{x})$
2. $y = \tanh^{-1}(e^x)$
3. $y = \cosh^{-1}(\sin^2 x)$
4. $y = \ln(\sinh^{-1} x)$
5. $y = csch^{-1}(4x)$
6. $y = x \ \tanh^{-1}(\frac{1}{x})$

Solution:

From the list of the derivatives given in Theorem 22, we have the following list of integrals:

- $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \sinh^{-1}\frac{x}{a} + c.$
- $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \cosh^{-1}\frac{x}{a} + c, \ x > a.$
- $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{a}\tanh^{-1}\frac{x}{a} + c, \ |\, x \,| < a.$
- $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{a}\coth^{-1}\frac{x}{a} + c, \ |\, x \,| > a.$
- $\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = -\frac{1}{a}sech^{-1}\frac{|x|}{a} + c, \ |\, x \,| < a.$
- $\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = -\frac{1}{a}csch^{-1}\frac{|x|}{a} + c, \ |\, x \,| > a.$

**Example 49** *Evaluate the following integral:* $\int \frac{1}{\sqrt{x^2-4}}\, dx$

Solution:

===

**Example 50** *Evaluate the following integrals:*

Solution:

1. $\int \frac{1}{\sqrt{4x^2+9}}\,dx$

$$\int_0^1 \frac{1}{16-x^2}\,dx$$

2. $\int \frac{1}{\sqrt{e^{2x}+9}}\,dx$

$$\int_5^7 \frac{1}{16-x^2}\,dx$$

3. $\int \frac{1}{x\sqrt{1-x^6}}\,dx$

## (1) Integration by Parts

Integration by parts is a method to transfer the original integral to an easier integral that can be evaluated. Practically, the integration by parts divides the original integral into two parts $u$ and $dv$. Then, we try to find the $du$ by deriving $u$ and $v$ by integrating $dv$.

**Integration by Parts**

**Theorem 23** *If $u = f(x)$ and $v = g(x)$ such that $f'(x)$ and $g'(x)$ are continuous, then*

$$\int u\ dv = uv - \int v\ du\ .$$

**Proof.** We know that $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$. Thus, $f(x)g'(x) = \frac{d}{dx}(f(x)g(x)) - f'(x)g(x)$.
By integrating the both sides, we have

$$\int f(x)g'(x)\ dx = \int \frac{d}{dx}(f(x)g(x))\ dx - \int f'(x)g(x)\ dx$$
$$= f(x)g(x) - \int f'(x)g(x)\ dx\ .$$

Since $u = f'(x)$ and $dv = g'(x)$, then

$$\int u\ dv = uv - \int v\ du\ . \blacksquare$$

Theorem 23 shows that the integration by parts transfers the integral $\int u\ dv$ into the integral $\int v\ du$ that should be easier than the original integral. The question here is, what we choose as $u(x)$ and what we choose as $dv = v'(x)\ dx$. It is useful to choose $u$ as a function that simplifies when differentiated, and to choose $v'$ as a function that simplifies when integrated. This is explained in a clearer sight through the following examples.

**Example 51** *Evaluate the following integral $\int x\ \cos x\ dx$.*

Solution:

Let $I = \int x\ \cos x\ dx$. Put $u = x$ and $dv = \cos x\ dx$. Then,

$$u = x \Rightarrow du = dx\ ,$$

$$dv = \cos x \Rightarrow v = \int \cos x\ dx = \sin x\ .$$

Hence, $I = x\ \sin x - \int \sin x\ dx = x\ \sin x + \cos x + c$.

Try to choose $u = \cos x$ and $dv = x$ Do you have the same result?

**Example 52** *Evaluate the following integral $\int x\ e^x\ dx$.*

Solution:

Let $I = \int x\ e^x\ dx$. Put $u = x$ and $dv = e^x\ dx$. Then,

$$u = x \Rightarrow du = dx\ ,$$

$$dv = e^x \Rightarrow v = \int e^x\ dx = e^x\ .$$

This implies

$$I = x\ e^x - \int e^x\ dx = x\ e^x - e^x + c\ .$$

Try to choose $u = e^x$ and $dv = x$

$$u = e^x \Rightarrow du = e^x\ ,$$

$$dv = x \Rightarrow v = \int x\ dx = x^2/2\ .$$

This implies $I = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x\ dx$.

However, the integral $\int \frac{x^2}{2}e^x\ dx$ is more difficult than the original one $\int xe^x\ dx$.

**Remark 5**

1. *Remember that when we consider the integration by parts, we want to have an easier integral. As we saw in Example 52, if we choose $u = e^x$ and $dv = x\ dx$ we have $\int \frac{x^2}{2}e^x\ dx$ which is more difficult than the original one.*

2. *When considering the integration by parts, we have to choose $dv$ a function that can be integrated (see Example 53).*

3. *Sometimes we need to use the integration by parts two times as in Examples 54 and 55.*

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**Example 53** *Evaluate the following integral* $\int \ln x \, dx$.

Solution:

**Example 54** *Evaluate the following integral* $\int e^x \cos x \, dx$.

Solution:

**Example 55** *Evaluate the following integral* $\int x^2 e^x \, dx$.

Solution:

**Example 56** *Evaluate the following integral* $\int_0^1 \tan^{-1} x \, dx$.

Solution:

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### (2) Trigonometric Functions
### (A) Integration of Power of Trigonometric Functions

In this section, we evaluate integrals of forms $\int \sin^n x \ \cos^m x \ dx$, $\int \tan^n x \ sec^m x \ dx$ and $\int \cot^n x \ csc^m x \ dx$.

**Form 1:** $\int \sin^n x \ \cos^m x \ dx$ .

This form of integrals is treated as follows:

1. If $n$ is odd, we write $\sin^n x \ \cos^m x = \sin^{n-1} x \ \cos^m x \ \sin x$. Then, we use $\cos^2 x + \sin^2 x = 1$ and the substitution $u = \cos x$.
2. If $m$ is odd, we write $\cos^m x \ \sin^n x = \cos^{m-1} x \ \sin^m x \ \cos x$. Then, we use $\cos^2 x + \sin^2 x = 1$ and the substitution $u = \sin x$.
3. If $m$ and $n$ are even, we use the formulas $\cos^2 x = \frac{1+cos2x}{2}$ and $\sin^2 x = \frac{1-cos2x}{2}$.

**Example 57** *Evaluate the following integrals:*

*1.* $\int \sin^3 x \ dx$ *3.* $\int \sin^5 x \cos^4 x \ dx$

*2.* $\int \cos^4 x \ dx$ *4.* $\int \sin^2 x \cos^2 x \ dx$

Solution:

1. $\int \sin^3 x \ dx = \int \sin^2 x \sin x \ dx = \int (1 - \cos^2 x) \sin x \ dx$ .

We write $\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x$ .
This implies

$$\int \sin^3 x \ dx = \int (1 - \cos^2 x) \sin x \ dx \ .$$

Put $u = \cos x$, this implies $du = - \ sinx \ dx$. By substitution, we have

$$\int (1 - u^2) \ du = u - \frac{u^3}{3} + c \ .$$

$$\int \sin^3 x \ dx = \cos x - \frac{\cos^3 x}{3} + c \ .$$

2. $\int \cos^4 x \ dx$ .

We write $\cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2$ .
This implies

$$\begin{aligned} \int \cos^4 x \ dx &= \int (\frac{1+\cos 2x}{2})^2 \ dx \\ &= \frac{1}{4} \int 1 + 2\cos 2x + \cos^2 2x \ dx \\ &= \frac{1}{4} \int 1 \ dx + \frac{1}{4} \int 2\cos 2x \ dx + \frac{1}{4} \cos^2 2x \ dx \\ &= \frac{1}{4} x + \frac{1}{4} \sin 2x + \frac{1}{8} \int 1 + \cos 4x \ dx \\ &= \frac{1}{4} x + \frac{1}{4} \sin 2x + \frac{1}{8} (x + \frac{\sin 4x}{4}) + c \ . \end{aligned}$$

3. $\int \sin^5 x \cos^4 x \ dx$

4. $\int \sin^2 x \cos^2 x \ dx$

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**Form 2:** $\int \tan^n x \;\; sec^m x \, dx$ .

This form is treated as follows:

1. If $n = 0$ and
   (a) $m$ is odd, we write $sec^m x = sec^{m-2} x \;\; sec^2 x$, then we use the integration by parts.
   (b) $m$ is even, we write $sec^m x = sec^{m-2} x \;\; sec^2 x$, then we use $sec^2 x = 1 + \tan^2 x$ and the substitution $u = \tan x$.
2. If $m = 0$ and $n$ is odd or even, we write $\tan^n x = \tan^{n-2} x \; tan^2 x$, then we use $\tan^2 x = sec^2 x - 1$ and the substitution $u = \tan x$.
3. If $n$ is even and $m$ is odd, then we use $\tan^2 x = sec^2 x - 1$ to change the integral to $\int sec^r x \, dx$.
4. $m \geq 2$ is even, we write $\tan^n x \; sec^m x = \tan^n x \; sec^{m-2} x \; sec^2 x$, then we use $sec^2 x = 1 + \tan^2 x$ and the substitution $u = \tan x$.

**Example 58** *Evaluate the following integrals:*

1. $\int \tan^5 x \, dx$

Solution:
Write $\tan^5 x = \tan^3 x \tan^2 x = \tan^3 x \; (sec^2 x - 1)$. Thus,

$$\begin{aligned}
\int \tan^5 x \, dx &= \int \tan^3 x \; (sec^2 x - 1) \, dx \\
&= \int \tan^3 x \; sec^2 x \, dx - \int \tan^3 x \\
&= \frac{\tan^4 x}{4} - \int \tan x \; (sec^2 x - 1) \, dx \\
&= \frac{\tan^4 x}{4} - \int \tan x \; sec^2 x \, dx + \int \tan x \, dx \\
&= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln | \; sec \; x \; | + c \; .
\end{aligned}$$

2. $\int \tan^6 x \, dx$

Solution:
Write $\tan^6 x = \tan^4 x \tan^2 x = \tan^4 x \; (sec^2 x - 1)$. From this, the integral becomes

$$\begin{aligned}
\int \tan^6 x \, dx &= \int \tan^4 x \; (sec^2 x - 1) \, dx \\
&= \int \tan^4 x \; sec^2 x \, dx - \int \tan^4 x \\
&= \frac{\tan^5 x}{5} - \int \tan^2 x \; (sec^2 x - 1) \, dx \\
&= \frac{\tan^5 x}{5} - \int \tan^2 x \; sec^2 x \, dx + \int \tan^2 x \, dx \\
&= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \int sec^2 - 1 \, dx \\
&= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x + x + c \; .
\end{aligned}$$

3. $\int sec^3 x \, dx$

Solution:
Write $sec^3 x = secx \; sec^2 x$ and let $I = \int secx \; sec^2 x \, dx$.
We use the integration by part to evaluate the integral as follows:

$$u = secx \Rightarrow du = secx \;\; \tan x \, dx \; ,$$

$$dv = sec^2 x \, dx \Rightarrow v = \int sec^2 x \, dx = \tan x \; .$$

Hence,

$$\begin{aligned}
I &= secx \;\; \tan x - \int secx \;\; \tan^2 x \, dx \\
&= secx \;\; \tan x - \int sec^3 x - secx \\
&= secx \;\; \tan x - I - \ln | \; secx + \tan x \; | \\
&= \frac{secx \;\; \tan x - \ln | \; secx + \tan x \; |}{2} + c \; .
\end{aligned}$$

4. $\int \tan^5 x \; sec^4 x \, dx$

5. $\int \tan^4 x \; sec^4 x \, dx$

**Form 3:** $\int \cot^n x \, csc^m x \, dx$ .

This form of integrals is treated as the integrals $\int \tan^n x \, sec^m x \, dx$ where we use $csc^2 x = 1 + \cot^2 x$.

**Example 59** *Evaluate the following integrals:*

1. $\int \cot^3 x \, dx$
2. $\int \cot^4 x \, dx$
3. $\int \cot^5 x csc^4 x \, dx$

Solution:

1. Write $\cot^3 x = \cot x \, (\csc^2 x - 1)$. Then,

$$\begin{aligned}\int \cot^3 x \, dx &= \int \cot x \, (\csc^2 x - 1) \, dx \\ &= (\cot x csc^2 x - \cot x) \, dx \\ &= \frac{-1}{2} \cot^2 x - \ln|\sin x| + c .\end{aligned}$$

2. The integrand can be expressed as $\cot^4 x = \cot^2 x(\csc^2 x - 1)$. Thus,

$$\begin{aligned}\int \cot^4 x \, dx &= \int \cot^2 x \, (\csc^2 x - 1) \, dx \\ &= \cot^2 x \, csc^2 x - \int \cot^2 x \, dx \\ &= \frac{-\cot^3 x}{3} + \cot x + x + c .\end{aligned}$$

3. $\int \cot^5 x csc^4 x \, dx$

**(B)Integration of Forms sin u x cos v x, sin u x sin v x and cos u x cos v x**

We deal with these integrals by using the following formulas:

$$\begin{aligned}\sin ux \; \cos vx &= \frac{1}{2}\big(\sin(u-v)\,x + \sin(u+v)\,x\big), \\ \sin ux \; \sin vx &= \frac{1}{2}\big(\cos(u-v)\,x - \cos(u+v)\,x\big), \\ \cos ux \; \cos vx &= \frac{1}{2}\big(\cos(u-v)\,x + \cos(u+v)\,x\big) .\end{aligned}$$

**Example 60** *Evaluate the following integrals:*

1. $\int \sin 5x \; \sin 3x \, dx$
2. $\int \sin 7x \; \cos 2x \, dx$
3. $\int \cos 5x \; \sin 2x \, dx$
4. $\int \cos 4x \; \sin 6x \, dx$

Solution:

1. $\int \sin 5x \; \sin 3x \, dx$ .

From the above formulas, we have

$$\sin 5x \; \sin 3x = \frac{1}{2}\big(\cos(2)x - \cos(8)x\big) .$$

Then,

$$\begin{aligned}\int \sin 5x \; \sin 3x \, dx &= \frac{1}{2}\int (\cos 2x - \cos 8x) \, dx \\ &= \frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + c .\end{aligned}$$

2. $\int \sin 7x \; \cos 2x \, dx$ .

Since

$$\sin 7x \; \cos 2x = \frac{1}{2}\big(\sin 5x + \sin 9x\big) .$$

Then,

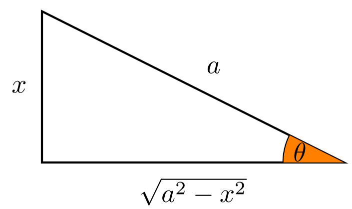
$$\begin{aligned}\int \sin 7x \; \cos 2x \, dx &= \frac{1}{2}\int (\sin 5x + \sin 9x) \, dx \\ &= -\frac{1}{10}\cos 5x - \frac{1}{18}\cos 9x + c .\end{aligned}$$

### (3) Trigonometric Substitutions

We are going to study integrals consist of the following expressions $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$ where $a>0$. To get rid of the square root, we use substitutions rely on trigonometric functions. The result is integrals evaluated by methods in the previous section. The conversion of the previous square roots is explained as follows:

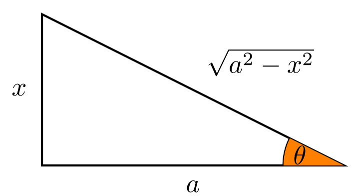
1 $\sqrt{a^2-x^2} = a\ \cos\theta$ if $x = a\ \sin\theta$.
If $x = a\ \sin\theta$ where $\theta \in (-\pi/2, \pi/2)$, then

$$\begin{aligned}\sqrt{a^2-x^2} &= \sqrt{a^2-a^2\sin^2\theta}\\ &= \sqrt{a^2(1-\sin^2\theta)}\\ &= \sqrt{a^2\cos^2\theta}\\ &= a\ \cos\theta\ .\end{aligned}$$



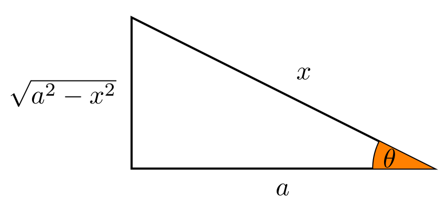

2 $\sqrt{a^2+x^2} = a\ \sec\theta$ if $x = a\ \tan\theta$.
If $x = a\ \tan\theta$ where $\theta \in (-\pi/2, \pi/2)$, then

$$\begin{aligned}\sqrt{a^2+x^2} &= \sqrt{a^2+a^2\tan^2\theta}\\ &= \sqrt{a^2(1+\tan^2\theta)}\\ &= \sqrt{a^2\sec^2\theta}\\ &= a\ \sec\theta\ .\end{aligned}$$




3 $\sqrt{x^2-a^2} = a\ \tan\theta$ if $x = a\sec\theta$.
If $x = a\ sec\theta$ where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$, then

$$\begin{aligned}\sqrt{x^2-a^2} &= \sqrt{a^2\sec^2\theta - a^2}\\ &= \sqrt{a^2(\sec^2\theta-1)}\\ &= \sqrt{a^2\tan^2\theta}\\ &= a\ \tan\theta\ .\end{aligned}$$




**Example 61** *Evaluate the following integrals:*

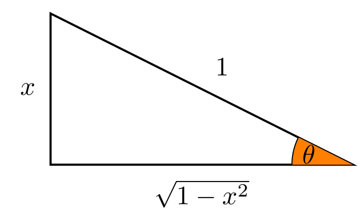
*1.* $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ *2.* $\displaystyle\int_5^6 \frac{\sqrt{x^2-25}}{x^4}\,dx$ *3.* $\displaystyle\int \sqrt{x^2+9}\,dx$

Solution:

1. $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

Let $x = \sin\theta$ where $\theta \in (-\pi/2, \pi/2)$, thus $dx = \cos\theta\ d\theta$. By substitution, we have

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\ \cos\theta\ d\theta\\ &= \int \frac{\sin^2\theta\cos\theta}{\cos\theta}\ d\theta\\ &= \int \sin^2\theta\ d\theta\\ &= \frac{1}{2}\int 1-\cos 2\theta\ d\theta\\ &= \frac{1}{2}(\theta - \frac{1}{2}\sin 2\theta) + c\\ &= \frac{1}{2}(\theta - \sin\theta\ \cos\theta) + c\ .\end{aligned}$$



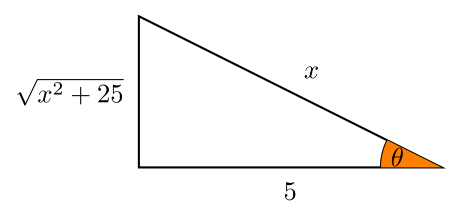

Now, we must return to the original variable $x$:

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}(\sin^{-1}\theta - x\sqrt{1-x^2}) + c\ .$$

2. $\displaystyle\int_5^6 \frac{\sqrt{x^2-25}}{x^4}\,dx$

Let $x = 5\ sec\theta$ where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$, thus $dx = 5\ sec\theta\ \tan\theta\ d\theta$. After substitution, the integral becomes

$$\begin{aligned}&\int \frac{\sqrt{25\ sec^2\theta - 25}}{625\ sec^4\theta}\ 5sec\theta\ \tan\theta\ d\theta\\ &= \frac{1}{25}\int \frac{\tan^2\theta}{sec^3\theta}\ d\theta\\ &= \frac{1}{25}\int \sin^2\theta\ \cos\theta\ d\theta\\ &= \frac{1}{75}\sin^3\theta + c\ .\end{aligned}$$




We must return to the original variable $x$:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{x^2-25}}{x^4}\,dx = \frac{1}{75}\Big[\frac{(x^2-25)^{3/2}}{x^3}\Big]_5^6 = \frac{1}{600}\ .$$

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### (4) Integrals of Rational Functions

In this section, we study rational functions of the form $q(x)=\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials. The previous techniques like integration by parts is not enough to evaluate the integral of the rational functions. Therefore, we need to know a new technique to help us to evaluate the integral of the rational functions. This technique is called decomposition of the rational functions into a sum of partial fractions.

The practical steps of integrals of rational functions can be summarized as follows:

Step 1: If the degree of $g(x)$ is less than the degree of $f(x)$, we do polynomial long-division, otherwise we move to step 2.
From the long division shown on the right side, we have

$$q(x)=\frac{f(x)}{g(x)}=h(x)+\frac{r(x)}{g(x)} ,$$

$$\begin{array}{r|l} & \text{h(x)} \\ \hline g(x)\ ) & \text{f(x)} \\ & \dots \\ \hline & \text{r(x)} \end{array}$$

where $h(x)$ is called the quotient and $r(x)$ is called the remainder.

Step 2: Factor the denominator $g(x)$ into irreducible polynomials where the result is either linear or irreducible quadratic polynomials.

Step 3: Find the partial fraction decomposition. This step depends on step 2 where if degree of $f(x)$ is less than the degree of $g(x)$, then the fraction $\frac{f(x)}{g(x)}$ can be written as a sum of partial fractions:

$$q(x)=P_1(x)+P_2(x)+P_3(x)+...+P_n(x) ,$$

where each $P_i(x)=\frac{A}{(ax+b)^m}, m\in\mathbb{N}$ or $P_i(x)=\frac{Ax+B}{(ax^2+bx+c)^m}$ if $b^2-4ac<0$. The constants $A,B,...$ are computed later.

Step 4: Integrate the result of step 3.

**Example 62** *Evaluate the following integral* $\int \frac{x+1}{x^2-2x-8}\,dx$.

Solution:
Factor the denominator $g(x)$ into irreducible polynomials

$$g(x)=x^2-2x-8=(x+2)(x-4) .$$

We need to find constants $A$ and $B$ such that

$$\frac{x+1}{x^2-2x-8}=\frac{A}{x+2}+\frac{B}{x-4}=\frac{Ax-4A+Bx+2B}{(x+2)(x-4)} .$$

Coefficients of the numerators:

$$A+B=1$$
$$-4A+2B=1$$

Students may find $B$ from first equation and substitute the result into the second equation. OR multiply the first equation by 4 and add the result with the second equation to find $B$ and then $A$.

We have $A=\frac{1}{6}$ and $B=\frac{5}{6}$, thus

$$\int \frac{x+1}{x^2-2x-8}\,dx=\int\frac{1/6}{x+2}\,dx+\int\frac{5/6}{x-4}\,dx=\frac{1}{6}\ln|x+2|+\frac{5}{6}\ln|x-4|+c .$$

**Example 63** *Evaluate the integral* $\int \frac{2x^3-4x^2-15x+5}{x^2+3x+2}\,dx$.

Solution:
Since the degree of the denominator $g(x)$ is less than the degree of the numerator $f(x)$, we do polynomial long-division:

$$\begin{array}{r|rrrr} & 2x & -10 & & \\ \hline x^2+3x+2\ ) & 2x^3 & -4x^2 & -15x & +15 \\ & 2x^3 & +6x^2 & +4x & \\ \hline & & -10x^2 & -19x & +15 \\ & & -10x^2 & -30x & -20 \\ \hline & & & 11x & +35 \end{array}$$

From the long division given on the right side, we have

$$q(x)=(2x-10)+\frac{11x+35}{x^2+3x+2} .$$

Now, factor the denominator $g(x)$ into irreducible polynomials

$$g(x)=x^2+3x+2=(x+1)(x+2) .$$

Thus,

$$q(x) = (2x-10) + \frac{11x+35}{x^2+3x+2} = (2x-10) + \frac{A}{x+1} + \frac{B}{x+2} = \frac{Ax+2A+Bx+B}{(x+1)(x+2)},$$

and we need to find the constants $A$ and $B$.
Coefficients of the numerators:

$$A + B = 11$$
$$2A + B = 35$$

Students may subtract the equations to have the value of $A$. Then, by substitution, we have the value of $B$.

We have $A = 26$ and $B = 15$. Hence,

$$\int q(x)\,dx = \int (2x-10)\,dx + \int \frac{26}{x+1}\,dx + \int \frac{15}{x+2}\,dx$$
$$= x^2 - 10x + 26\ln|x+1| + 15\ln|x+2| + c\,.$$

**Remark 6**

1. *The number of constants $A, B, C, \ldots$ is equal to the degree of the denominator $g(x)$. Therefore, in the case of repeated factors of the denominator, we have to check the number of the constants and the degree of $g(x)$.*

2. *If the denominator $g(x)$ contains on irreducible quadratic factors, the numerators of fraction decomposition should be polynomials of degree one.*

**Example 64** *Evaluate the following integral* $\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}dx$.

Solution:
Since the denominator $g(x)$ has repeated factors, then

$$\frac{2x^2-25x-33}{(x+1)^2(x-5)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-5}$$
$$= \frac{A(x^2-4x-5) + B(x-5) + C(x^2+2x+1)}{(x+1)^2(x-5)}$$

Coefficients of the numerators:

$$A + C = 2$$
$$-4A + B + 2C = -25$$
$$-5A - 5B + C = -33$$

From the second and third equations, a student can find a new equation contains on $A$ or $C$. Then, solve a new system consists of the first and new equations.

By solving the system of equations, we have $A = 5$, $B = 1$ and $C = -3$.
Hence,

$$\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}\,dx = \int \frac{5}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx + \int \frac{-3}{x-5}\,dx$$
$$= 5\ln|x+1| + \int (x+1)^{-2}\,dx - 3\ln|x-5|$$
$$= 5\ln|x+1| - \frac{1}{(x+1)} - 3\ln|x-5| + c\,.$$

**Example 65** *Evaluate the following integral* $\int \frac{x+1}{x(x^2+1)}dx$.

Solution:
The denominator $g(x)$ is factored into irreducible polynomials, so

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}\,.$$

Coefficients of the numerators:

$$A + B = 0$$
$$C = 1$$
$$A = 1$$

We have $A = 1$, $B = -1$ and $C = 1$. Hence,

$$\int \frac{x+1}{x(x^2+1)}\,dx = \int \frac{1}{x}\,dx + \int \frac{-x+1}{x^2+1}\,dx$$
$$= \ln|x| - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$
$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \tan^{-1}x + c\,.$$

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**Example 66** *Evaluate the integral* $\int_0^{\pi/4} \frac{sec^2 x}{\tan^2 x + 3\tan x + 2}\, dx.$

Solution:

**1 - 12** ■ Evaluate the following integrals:

1. $\int \frac{1}{x(x-1)}\, dx$
2. $\int_1^2 \frac{1}{x^2+4x+3}\, dx$
3. $\int \frac{x+1}{x^2+8x+12}\, dx$
4. $\int \frac{x}{x^2+7x+12}\, dx$
5. $\int_1^5 \frac{x^2-1}{x^2+3x-4}\, dx$
6. $\int \frac{x^3}{x^2-25}\, dx$
7. $\int \frac{x^3+2x+1}{x^2-3x-10}\, dx$
8. $\int_0^{\pi/2} \frac{\sin x}{\cos^2 x - \cos x - 1}\, dx$
9. $\int \frac{2-x}{x^3+x^2}\, dx$
10. $\int_0^1 \frac{1}{1+e^x}\, dx$
11. $\int \frac{e^x}{e^{2x}-2e^x-15}\, dx$
12. $\int \frac{1}{x^4-x^2}\, dx$

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### (5) Integrals of Quadratic Forms

In this section, we provide a new technique for integrals that contain on irreducible quadratic expressions $ax^2+bx+c$ where $b \neq 0$. This technique is completing square method: $a^2 \pm 2ab + b^2 = (a \pm b)^2$. Before starting presenting this method, we provide an example to remind the reader on how to complete the square.

**Example 67** *The quadratic expression $x^2-6x+13$ is irreducible. To complete the square, we find $(\frac{b}{2})^2$, then add and substrate it as follows:*

$$x^2-6x+13 = \underbrace{x^2-6x+9}_{=(x+3)^2} - \underbrace{9+13}_{=4}$$

*If a quadratic polynomial has roots, it called reducible; otherwise it is called irreducible.*

*Hence, $x^2-6x+13=(x+3)^2+4$.*

In the following examples, we use the previous idea to evaluate the integrals.

**Example 68** *Evaluate the following integral* $\int \frac{1}{x^2-6x+13}dx$.

Solution:

The quadratic expression $x^2-6x+13$ is irreducible. By using the complete the square, we have

$$\int \frac{1}{x^2-6x+13}dx = \int \frac{1}{(x+3)^2+4}dx \ .$$

Let $u = x+3 \Rightarrow du = dx$. By substitution,

$$\int \frac{1}{u^2+4}du = \frac{1}{2}\tan^{-1}(\frac{u}{2}) + c = \frac{1}{2}\tan^{-1}(\frac{x+3}{2}) + c \ .$$

**Example 69** *Evaluate the following integral* $\int \frac{x}{x^2-4x+8}dx$.

Solution:

Since the quadratic expression $x^2-4x+8$ is irreducible, we use the complete the square as follows:

$$\int \frac{x}{x^2-4x+8}\,dx = \int \frac{x}{(x-2)^2+4}\,dx$$

Let $u = x-2 \Rightarrow du = dx$. By substitution,

$$\int \frac{u+2}{u^2+4}\,du = \int \frac{u}{u^2+4}\,du + \int \frac{2}{u^2+4}\,du$$
$$= \frac{1}{2}\ln|\,u^2+4\,| + \tan^{-1}(\frac{u}{2})$$
$$= \frac{1}{2}\ln|\,(x-2)^2+4\,| + \tan^{-1}(\frac{x-2}{2}) + c$$

**Example 70** *Evaluate the following integral* $\int \frac{1}{\sqrt{2x-x^2}}dx$.

Solution:

By completing the square, we have $2x-x^2 = 2x-x^2-1+1 = 1-(x-1)^2$. Thus,

$$\int \frac{1}{\sqrt{2x-x^2}}dx = \int \frac{1}{\sqrt{1-(x-1)^2}}dx \ .$$

Let $u = x-1$, then $du = dx$ and by substitution, the integral becomes $\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + c = \sin^{-1}(x-1) + c$ .

**1 - 12** ■ Evaluate the following integrals:

1. $\int_0^1 \frac{1}{x^2+4x+5}\,dx$
2. $\int \frac{1}{x^2-6x+1}\,dx$
3. $\int \frac{2x+3}{x^2+2x-3}\,dx$
4. $\int \frac{x^2-2x+5}{2x-x^2}\,dx$
5. $\int_{-1}^0 \frac{1}{\sqrt{8+2x-x^2}}\,dx$
6. $\int \frac{1}{x^2+8x-9}\,dx$
7. $\int \frac{5}{\sqrt{1-4x-x^2}}\,dx$
8. $\int \frac{e^x}{e^{2x}+2e^x-1}\,dx$
9. $\int \frac{1}{\sqrt{6-6x-2x^2}}\,dx$
10. $\int \sqrt{x(2-x)}\,dx$
11. $\int \frac{sec^2 x}{\tan^2 x - 6\tan x + 12}\,dx$
12. $\int \sqrt{8-2x-x^2}\,dx$

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### (6) Miscellaneous Substitutions

We study in this section other two important substitutions needed in some cases. The first substitution is for integrals consisted of rational expressions in $\sin x$ and $\cos x$. The second substitution is for integrals consisted of fractional powers.

### (A) Fractional Functions in sin x and cos x

The integrals that consist of rational expressions in $\sin x$ and $\cos x$ are treated by using the substitution $u = \tan(x/2)$, $-\pi < x < \pi$. This implies that $du = \frac{sec^2(x/2)}{2}\ dx$. Since $sec^2 x = \tan^2 x + 1$, then $du = \frac{u^2+1}{2}\ dx$. Also,

$$\begin{aligned}\sin(x) = \sin 2(\frac{x}{2}) = 2\ \sin(\frac{x}{2})\ \cos(\frac{x}{2}) &= 2\frac{\sin(\frac{x}{2})}{\cos(\frac{x}{2})}\cos(\frac{x}{2})\ \cos(\frac{x}{2})\\ &= 2\tan(\frac{x}{2})\ \cos^2(\frac{x}{2})\\ &= 4 + \frac{4n^2+n}{2n^2}\\ &= 2\frac{\tan(\frac{x}{2})}{\sec^2(\frac{x}{2})}\\ &= \frac{2u}{u^2+1}\ .\end{aligned}$$

For $\cos x$, we can find that

$$\cos(\frac{x}{2}) = \frac{1}{\sqrt{u^2+1}} \text{ and } \sin(\frac{x}{2})) = \frac{u}{\sqrt{u^2+1}}\ .$$

Use the formula: $sec^2(\frac{x}{2}) = \tan^2(\frac{x}{2}) + 1$. Then, use the formula $\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2}) = 1$

Thus, $\cos(x) = \cos 2(\frac{x}{2}) = \cos^2(\frac{x}{2})) - \sin^2(\frac{x}{2}))$. From this,

$$\cos(x) = \frac{1-u^2}{1+u^2}\ .$$

The previous discussion can be summarized as follows:

For the integrals that contain on rational expressions in $\sin x$ and $\cos x$, we assume

$$u = \tan(x/2),\quad du = \frac{u^2+1}{2}\ dx,\quad \sin(x) = \frac{2u}{u^2+1},\quad \cos(x) = \frac{1-u^2}{1+u^2}\ .$$

**Example 71** *Evaluate the following integrals:*

*1.* $\displaystyle\int \frac{1}{1+\sin x}\ dx$

*2.* $\displaystyle\int \frac{1}{2+\cos x}\ dx$

*3.* $\displaystyle\int \frac{1}{1+\sin(x)+\cos(x)}\ dx$

Solution:

1. $\displaystyle\int \frac{1}{1+\sin x}\ dx$

Let $u = \tan\frac{x}{2}$, this implies $du = \frac{1+u^2}{2}$ and $\sin x = \frac{2u}{1+u^2}$. By substituting the results into the integral, we have

$$\begin{aligned}\int \frac{1}{1+\frac{2u}{1+u^2}}\frac{2}{1+u^2}\ du = 2\int \frac{1}{u^2+2u+1}\ du &= 2\int (u+1)^{-2}\ du\\ &= \frac{-2}{u+1} + c\\ &= \frac{-2}{\tan\frac{x}{2}+1} + c\ .\end{aligned}$$

2. $\displaystyle\int \frac{1}{2+\cos x}\ dx$

Let $u = \tan\frac{x}{2}$, this implies $du = \frac{1+u^2}{2}$ and $\cos x = \frac{1-u^2}{1+u^2}$. By substitution, we have

$$\begin{aligned}\int \frac{1}{2+\frac{1-u^2}{1+u^2}}\frac{2}{1+u^2}\ du = 2\int \frac{1}{u^2+3}\ du &= \frac{2}{\sqrt{3}}\tan^{-1}(\frac{u}{\sqrt{3}}) + c\\ &= \frac{2}{\sqrt{3}}\tan^{-1}(\frac{\tan\frac{x}{2}}{\sqrt{3}}) + c\ .\end{aligned}$$

3. $\displaystyle\int \frac{1}{1+\sin(x)+\cos(x)}\ dx$

Assume $u = \tan\frac{x}{2}$, this implies $du = \frac{1+u^2}{2}$ and $\cos x = \frac{1-u^2}{1+u^2}$. By substitution, we have

$$\begin{aligned}\int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\frac{2}{1+u^2}\ du = \int \frac{2}{2+2u}\ du &= \int \frac{1}{1+u}\\ &= \ln|1+u| + c\\ &= \ln|1+\tan\frac{x}{2}| + c\ .\end{aligned}$$

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### (B) Integrals of Fractional Powers

In the case of integrands that consist of fractional powers, it is better to use the substitution $u = x^{\frac{1}{n}}$ where $n$ is the least common multiple of the denominators of the powers. To see this, we provide some examples.

**Example 72** *Evaluate the following integral* $\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\ dx$ .

Solution:
Put $u = x^{\frac{1}{4}}$, we find $x = u^4$ and $dx = 4u^3 du$. By substitution, we have

$$\begin{aligned}\int \frac{1}{u^2+u} 4u^3\ du &= 4\int \frac{u^2}{u+1}\ du \\ &= 4\int u-1\ du + 4\int \frac{1}{1+u}\ du \\ &= 2u^2 - 4u + 4\ln|\ u+1\ | + c\ .\end{aligned}$$

### (C) Integrals of $\sqrt[n]{f(x)}$

Here, we assume that the integrand is a function of form $\sqrt[n]{f(x)}$. To solve such integrals, it is useful to assume $u = \sqrt[n]{f(x)}$. This case differs from that given in the substitution method in Chapter **??** i. e., $\sqrt[n]{f(x)}\ f'(x)$ where the difference lies on existence of the derivatives of $f(x)$.

**Example 73** *Evaluate the following integral* $\int \sqrt{e^x+1}\ dx$

Solution:
Assume $u = \sqrt{e^x+1}$, this implies $du = \frac{e^x}{2\sqrt{e^x+1}}\ dx$ and $u^2 = e^x + 1$. By substitution,

$$\begin{aligned}\int \frac{2u^2}{u^2-1}\ du &= \int 2\ du + 2\int \frac{1}{u^2-1}\ du \\ &= 2u + 2\int \frac{1}{u-1}\ du + 2\int \frac{1}{u-1} \\ &= 2u + 2\ln|\ u-1\ | + 2\ln|\ u+1\ | + c\ .\end{aligned}$$

**1 - 12** ■ Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\ dx$
2. $\int \frac{x^{1/2}}{1+x^{3/5}}\ dx$
3. $\int \frac{1}{\sqrt{\cos x+1}}\ dx$
4. $\int \frac{\sqrt{x}}{\sqrt{x}+4}\ dx$
5. $\int \frac{1}{1+3\sin x}\ dx$
6. $\int \frac{\sin x}{3-\sin x}\ dx$
7. $\int \frac{1}{\sqrt{x}+\sqrt[5]{x}}\ dx$
8. $\int \frac{x^{1/3}}{1+x^{1/4}}\ dx$
9. $\int \frac{1}{\sqrt{e^{2x}+1}}\ dx$
10. $\int \frac{1}{x^{1/2}-x^{3/5}}\ dx$
11. $\int \frac{1}{1-2\cos x}\ dx$
12. $\int \frac{1}{\sin x+\cos x}\ dx$

## Limit Rules

The limit is defined as the value that the function as the variable approaches to $t$ value. A few examples are given below:

In general, the limit is equal to the value of the function at $x$.

**Example 74**

1. $\lim_{x\to 2} 3 = 3$ . 2. $\lim_{x\to 1} x = 1$ .
3. $\lim_{x\to -\infty} \sin x = -\infty$ . 4. $\lim_{x\to 8} \sqrt{x} = 2\sqrt{2}$ .

As you noted, the functions in the previous example are continuous. Meaning that, the limit is equal to the value of the function if it is continuous. Before discussing this issue deeply, let's see some of the general rules for limits.

If $\lim_{x\longrightarrow c} f(x) = L$ and $\lim_{x\longrightarrow c} g(x) = M$, then

1. **Sum Rule**

$$\lim_{x\longrightarrow c} \big(f(x)+g(x)\big) = \lim_{x\longrightarrow c} f(x) + \lim_{x\longrightarrow c} g(x) = L + M$$

2. **Difference Rule**

$$\lim_{x\longrightarrow c} \big(f(x)-g(x)\big) = \lim_{x\longrightarrow c} f(x) - \lim_{x\longrightarrow c} g(x) = L - M$$

3. **Product Rule**

$$\lim_{x\longrightarrow c} \big(f(x).g(x)\big) = \lim_{x\longrightarrow c} f(x) \times \lim_{x\longrightarrow c} g(x) = L \times M$$

4. **Constant Multiple Rule**

$$\lim_{x\longrightarrow c} \big(k\ f(x)\big) = k\ \lim_{x\longrightarrow c} f(x) = k\ L$$

5. **Quotient Rule**

$$\lim_{x\longrightarrow c} \Big(\frac{f(x)}{g(x)}\Big) = \frac{\lim_{x\longrightarrow c} f(x)}{\lim_{x\longrightarrow c} g(x)} = \frac{L}{M}$$

**Example 75**

1. $\lim_{x\to 0}(x^2 - 2x + 1) = \lim_{x\to 0} x^2 - 2\lim_{x\to 0} x + \lim_{x\to 0} 1 = 0 - 0 + 1 = 1$ .
2. $\lim_{x\to \pi} \sin x \cos x = 0$ .
3. $\lim_{x\to 3} \frac{1}{(x-3)} = \frac{\lim_{x\to 3} 1}{\lim_{x\to 3}(x-3)} = \infty$ .
4. $\lim_{x\to 1} \frac{x}{(x^2+1)} = \frac{\lim_{x\to 1} x}{\lim_{x\to 1}(x^2+1)} = \frac{1}{2}$ .

### (1) Indeterminate Forms

An indeterminate form is a case occurs when we do not have an exact value for the limit i.e., the value of the limit cannot be determined. A few examples are given below:

**Example 76**

1. $\lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0}$
2. $\lim_{x\to \infty} \frac{e^x}{x} = \frac{\infty}{\infty}$
3. $\lim_{x\to 0^+} x^2 \ln x = 0.\infty$
4. $\lim_{x\to 1} \big(\frac{1}{x-1} - \frac{1}{\ln x}\big) = \infty - \infty$

In the following table, we categorize the indeterminate forms:

| Form | Indeterminate Forms |
|---|---|
| Quotient | $\frac{0}{0}$ and $\frac{\infty}{\infty}$ |
| Product | $0.\infty$ and $0.(-\infty)$ |
| Sum & Difference | $(-\infty) + \infty$ and $\infty - \infty$ |
| Exponential | $0^0$, $1^\infty$, $1^{-\infty}$ and $\infty^0$ |

To treat such limits, students were multiplying the function by a conjugate or using factoring method. In this course, we present a new method that is called L'Hopital Rule. Usually, this method is used for a fractional function where we calculate derivatives of the numerator and denominator.

**L'Hopital Rule**

**Theorem 24** *Suppose $f(x)$ and $g(x)$ are differentiable on an interval $I$ and $c \in I$ where $f$ and $g$ may not be differentiable at $c$. If $\frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = c$ and $g'(x) \neq 0$ for $x \neq c$, then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

*such that $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists or equals to $\infty$ or $-\infty$.*

**Remark 7**

1. *L'Hopital rule works if $c = \pm\infty$ or when $x \to c^+$ or $x \to c^-$.*
2. *Sometimes we need to apply L'Hopital rule twice to evaluate limits.*

**Example 77** *Use L'Hopital rule to find the following limits:*

1. $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25}$ .

Solution:
$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25} = \frac{0}{0}$ and this is an indeterminate form. By applying L'Hopital rule,

$$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25} = \lim_{x\to 5} \frac{1}{4x\sqrt{x-1}} = \frac{1}{40} .$$

2. $\lim_{x\to 0} \frac{\sin x}{x}$ .

Solution:
$\lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0}$. To treat this indeterminate form, we apply L'Hopital rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1 .$$

3. $\lim_{x\to 0^+} x^2 \ln x$

Solution: The limit is of the form $0.\infty$, so we cannot use the L'Hopital rule. However, if we rearrange the expression, we may able to use the L'Hopital rule. Meaning that, we need to rewrite the expression in a way to apply the L'Hopital rule. Note that

$$x^2 \ln x = \frac{\ln x}{\frac{1}{x^2}} .$$

The limit of the new expression ($\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}}$) is of form $\frac{\infty}{\infty}$. Therefore, we can apply the L'Hopital rule:

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}}$$
$$= \lim_{x\to 0^+} \frac{x^2}{-2} = 0 . \qquad \text{L'Hopital rule}$$

4. $\lim_{x\to \frac{\pi}{4}} (1-\tan x)\ \sec(2x)$

Solution: The limit is of the form $0.\infty$ so we try to rewrite the function to apply the L'Hopital rule. We know that $secx = 1/\cos x$, thus

$$(1-\tan x)\sec(2x) = \frac{(1-\tan x)}{\cos(2x)} .$$

Now, the limit of the new expression is of form $\frac{0}{0}$. From the L'Hopital rule, we have

$$\lim_{x\to \frac{\pi}{4}} (1-\tan x)\sec(2x) = \lim_{x\to \frac{\pi}{4}} \frac{(1-\tan x)}{\cos(2x)} = \lim_{x\to \frac{\pi}{4}} \frac{\sec^2 x}{2\sin 2x} = \frac{(\frac{1}{\sqrt{2}})^2}{2} = \frac{1}{4} .$$

5. $\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$

Solution: By substituting 1 into the function, we have the indeterminate form $\infty - \infty$. To treat this form, we write the expression as a single fraction

$$\frac{1}{x-1} - \frac{1}{\ln x} = \frac{\ln x - x + 1}{(x-1)\ln x} .$$

The new form takes the indeterminate form $\frac{0}{0}$. From the L'Hopital rule,

$$\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1} \frac{1-x}{x\ \ln x + x - 1}$$

which is of form $\frac{0}{0}$. Therefore, we apply the L'Hopital rule again. This implies

$$\lim_{x\to 1} \frac{\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1} \frac{-1}{\ln x + 2} = \frac{-1}{2} .$$

6. $\lim_{x\to 0} (1+x)^{\frac{1}{x}}$

Solution: This limit is of the form $1^\infty$. To treat this form, we assume that $y = (1+x)^{\frac{1}{x}}$. By taking ln for both sides, we have

$$\ln y = \frac{1}{x} \ln(1+x)$$
$$\Rightarrow \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{1}{x} \ln(1+x)$$
$$= \lim_{x\to 0} \frac{\ln(1+x)}{x} = \frac{0}{0} .$$

By applying the L'Hopital rule, we have

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1 .$$

Thus,

$$\lim_{x\to 0} \ln y = 1 \Rightarrow e^{\lim_{x\to 0} \ln y} = e^1$$
$$\Rightarrow \lim_{x\to 0} e^{(\ln y)} = e$$
$$\Rightarrow \lim_{x\to 0} y = e$$
$$\Rightarrow \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e .$$

## (2) Improper Integrals

**Definition 14** *The integral* $\int_a^b f(x)\,dx$ *is proper if*

1. *the interval* $[a,b]$ *is finite, and*
2. $f(x)$ *is continuous on* $[a,b]$.

If condition 1 or 2 is not satisfied, the integral is called **improper**. From this, we have two cases of the improper integrals.

### (A) Integrals with Infinite Limits of Integration

In this section, we study integrals of forms

$$\int_a^{\infty} f(x)\,dx,\ \int_{-\infty}^{a} f(x)\,dx,\ \int_{\infty}^{-\infty} f(x)\,dx\ .$$

**Definition 15** *1. Let $f$ be a continuous function on $[a,\infty)$. The improper integral* $\int_a^{\infty} f(x)\,dx$ *is defined as follows:*

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx\ .$$

*2. Let $f$ be a continuous function on $(-\infty,b]$. The improper integral* $\int_{-\infty}^{b} f(x)\,dx$ *is defined as follows:*

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx\ .$$

*The integrals are convergent (or to converge) if the limit exists as a finite number However, if the limit does not exist or equal to $\pm\infty$, the integral is called divergent (or to diverge).*
*3. Let $f$ be a continuous function on $\mathbb{R}$ and $a\in\mathbb{R}$. The improper integral* $\int_{-\infty}^{\infty} f(x)\,dx$ *is defined as follows:*

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx\ .$$

*The integral is convergent if both integrals on the right side are convergent; otherwise the integral is divergent.*

**Example 78** *Determine whether the integral converges or diverges:*

1. $\int_0^{\infty} \frac{1}{(x-2)^2}\,dx$
2. $\int_0^{\infty} \frac{x}{1+x^2}\,dx$
3. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Solution:

1. $\int_0^{\infty} \frac{1}{(x-2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{(x-2)^2}\,dx.$
The integral

$$\int_0^t \frac{1}{(x-2)^2}\,dx = \int_0^t (x-2)^{-2}\,dx = \left[\frac{-1}{x-2}\right]_0^t = -(\frac{1}{t-2}+\frac{1}{2})\ .$$

Thus,

$$\lim_{t\to\infty}\int_0^t \frac{1}{(x-2)^2}\,dx = -\lim_{t\to\infty}(\frac{1}{t-2}+\frac{1}{2}) = -(0+\frac{1}{2}) = -\frac{1}{2}\ .$$

Therefore, the integral converges.

2. $\int_0^{\infty} \frac{x}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx.$
The integral

$$\int_0^t \frac{x}{1+x^2}\,dx = \left[\ln(1+x^2)\right]_0^t = \ln(1+t^2)-\ln(1) = \ln(1+t^2)\ .$$

Thus,

$$\lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx = \lim_{t\to\infty}\ln(1+t^2) = \infty\ .$$

Therefore, the integral diverges.

3. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx.$
We know that, $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$, so

$$\begin{aligned}\lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2} &= \lim_{t\to-\infty}\left[0-\tan^{-1}(t)\right] + \lim_{t\to\infty}\left[\tan^{-1}(t)-0\right]\\ &= -\lim_{t\to-\infty}\tan^{-1}(t) + \lim_{t\to\infty}\tan^{-1}(t)\\ &= -(-\frac{\pi}{2})+\frac{\pi}{2}\\ &= \pi\ .\end{aligned}$$

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## (B) Integrals with Discontinuous Integrands

### Discontinuous Integrands

**Definition 16**

1. *If $f$ is continuous on $[a,b)$ and has an infinite discontinuity at $b$ i.e., $\lim_{x\to b^-} f(x)=\pm\infty$, then*

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx\ .$$

2. *If $f$ is continuous on $(a,b]$ and has an infinite discontinuity at $a$ i.e., $\lim_{x\to a^+} f(x)=\pm\infty$, then*

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^a f(x)\,dx\ .$$

*In items 1 and 2, the integral is convergent if the limit exists as a finite number; otherwise the integral is divergent.*

3. *If $f$ is continuous on $[a,b]$ except at $c\in(a,b)$ such that $\lim_{x\to c^\pm} f(x)=\pm\infty$, the improper integral $\int_a^b f(x)\,dx$ is defined as follows:*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\ .$$

*The integral is convergent if the limit of the integrals on the right exists as a finite number.*

**Example 79** *Determine whether the integral converges or diverges:*

*1.* $\displaystyle\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\,dx$ *2.* $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\,dx$ *3.* $\displaystyle\int_{-3}^1 \frac{1}{x^2}\,dx$

Solution:

1. Since $\lim_{x\to 4^-}\frac{1}{(4-x)^{\frac{3}{2}}}=\infty$ and the integrand is continuous on $[0,4)$, then

$$\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\,dx = \lim_{t\to 4^-}\int_0^t (4-x)^{-\frac{3}{2}}\,dx = \lim_{t\to 4^-}\left[\frac{-2}{\sqrt{4-x}}\right]_0^t$$
$$= \lim_{t\to 4^-}\left[\frac{-2}{\sqrt{4-t}}+1\right] = -\infty\ .$$

Thus, the integral is diverges.

2. The limit $\lim_{x\to 0^+}\frac{\cos x}{\sqrt{\sin x}}=\infty$ and the integrand is continuous on $(0,\frac{\pi}{4}]$, then

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\,dx &= \lim_{t\to 0^+}\int_t^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}} \\ &= \lim_{t\to 0^+}\left[\sqrt{\sin x}\right]_t^{\frac{\pi}{4}} \\ &= \lim_{t\to 0^+}\left(\frac{1}{\sqrt[4]{2}}-\sqrt{\sin t}\right) \\ &= \frac{1}{\sqrt[4]{2}}\ .\end{aligned}$$

The integral is converges.

3. Since $\lim_{x\to 0^-}\frac{1}{x^2}=\lim_{x\to 0^+}\frac{1}{x^2}=\infty$ and the integrand is continuous on $[-3,0)\cup(0,1]$, then

$$\begin{aligned}\int_{-3}^1 \frac{1}{x^2}\,dx &= \lim_{t\to 0^-}\int_{-3}^{\frac{\pi}{t}} \frac{1}{x^2} + \lim_{t\to 0^+}\int_0^{\frac{\pi}{1}} \frac{1}{x^2} \\ &= \lim_{t\to 0^-}\left[\frac{-1}{x}\right]_{-3}^{\frac{\pi}{t}} - \lim_{t\to 0^+}\left[\frac{-1}{x}\right]_t^{\frac{\pi}{1}} \\ &= -\lim_{t\to 0^-}\left[\frac{1}{t}+\frac{1}{3}\right] + \lim_{t\to 0^+}\left[-1+\frac{1}{t}\right] \\ &= \infty\ .\end{aligned}$$

The integral is diverges.

**1 - 10** ■ Determine whether the integral converges or diverges:

1. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
2. $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$
3. $\displaystyle\int_4^\infty \frac{1}{\sqrt{x}}\,dx$
4. $\displaystyle\int_{-\infty}^0 e^x\,dx$
5. $\displaystyle\int_0^\infty e^x\,dx$
6. $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$
7. $\displaystyle\int_0^\infty (1-x)e^{-x}\,dx$
8. $\displaystyle\int_0^\infty \frac{dx}{x^2+4}$
9. $\displaystyle\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\,dx$
10. $\displaystyle\int_0^{\pi/2} \tan x\,dx$

===

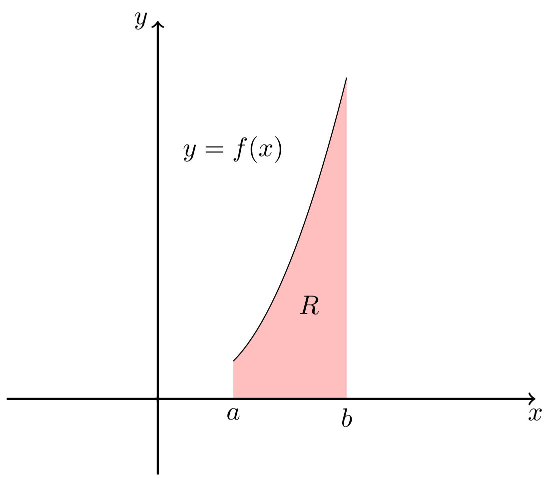
### (1) Areas

Integration can be used to calculate areas under bounded graphs. In general, we consider the following cases:

1. the area between a curve, the x-axis or y-axis, and two given ordinates,
2. the area between a curve, the x-axis or y-axis, and two ordinates given by crossing the curve the axis,
3. the area between two curves.

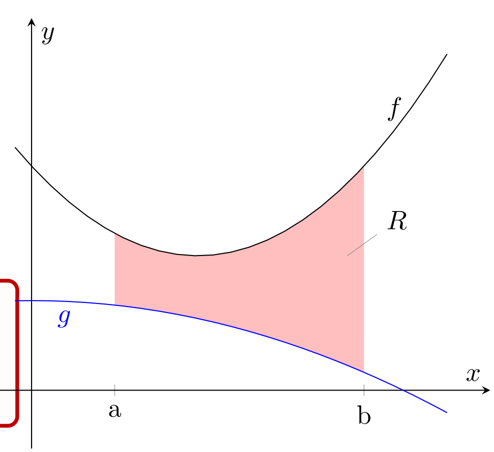
1. If $y = f(x)$ is continuous on $[a, b]$ and $f(x) \geq 0\ \forall x \in [a, b]$, the area of the region under the graph of $f(x)$ from $x = a$ to $x = b$ is given by the integral:

$$A = \int_a^b f(x)\ dx$$



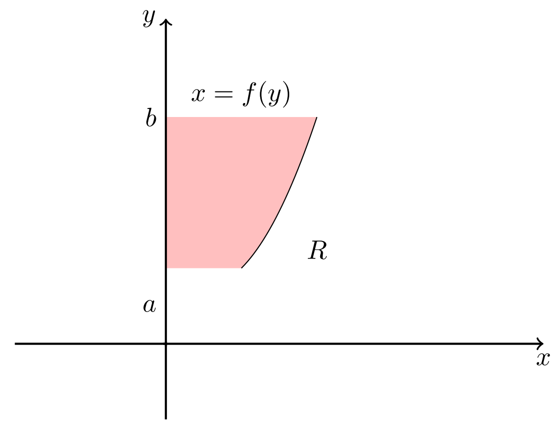

If $f(x)$ and $g(x)$ are continuous and $f(x) \geq g(x)$ for every $x \in [a, b]$, then the area $A$ of the region bounded by the graphs of $f$ and $g$ is given by the integral:

$$A = \int_a^b \big(f(x) - g(x)\big)\ dx$$



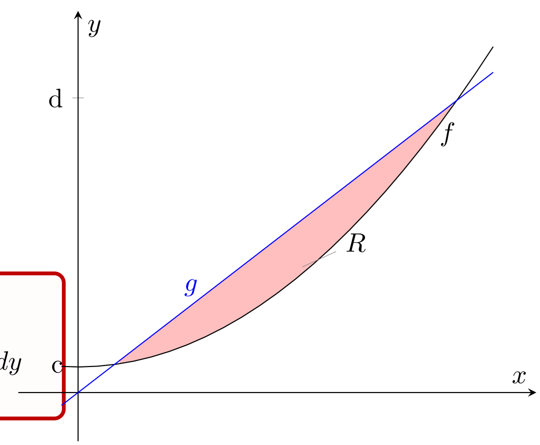

3. If $x = f(y)$ is continuous on $[c, d]$ and $f(y) \geq 0\ \forall y \in [c, d]$, the area of the region under the graph of $f(y)$ from $y = c$ to $y = d$ is given by the integral:

$$A = \int_c^d f(y)\ dy$$




4. If $f(y)$ and $g(y)$ are continuous and $f(y) \geq g(y)$ for every $x \in [c, d]$, then the area $A$ of the region bounded by the graphs of $f$ and $g$ is given by the integral:

$$A = \int_c^d \big(f(y) - g(y)\big)\ dy$$




**Example 80** *Sketch the region by the graph of* $y = x$ *on the interval* $[0, 3]$*, then find its area.*

Solution:

==================================================================

**Example 81** *Sketch the region by the graph of* $y = x^2$, *then find its area.*

Solution:

**Example 82** *Sketch the region by the graphs of* $y = x^3$ *and* $y = x$, *then find its area.*

Solution:

**Example 83** *Sketch the region by the graphs of* $x = \sqrt{y}$ *from* $y = 0$ *and* $y = 1$, *then find its area.*

Solution:

**Example 84** *Sketch the region by the graphs of* $y = \sin x$, $y = \cos x$, $x = 0$ *and* $x = \frac{\pi}{4}$, *then find its area.*

Solution:

## (2) Volume of Solids of Revolution

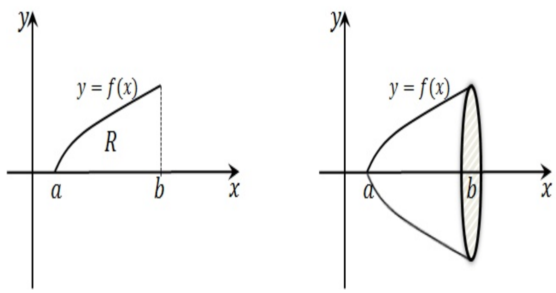
### Solids of Revolution

In this section, we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is.

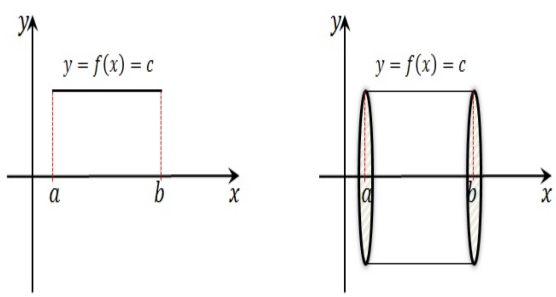
**Solid of Revolution**

**Definition 17** *The solid of revolution (S) is a solid generated from rotating a region R about a line in the same plane where the line is called the axis of revolution.*

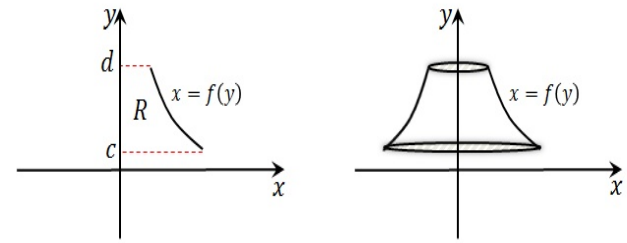
**Example 85** *Let $f(x) \geq 0$ be continuous for every $x \in [a, b]$. Let $R$ be a region bounded by the graph of $f$ and x-axis form $x = a$ to $x = b$. Rotating the region $R$ about x-axis generates a solid given in Figure 85.*




**Example 86** *Let $f(x)$ be a constant function, as in Figure 86. The region $R$ is a rectangle and rotating it about x-axis generates a circular cylinder.*




**Example 87** *Consider the region $R$ bounded by the graph of $f(y)$ from $y = c$ to $y = d$ as in Figure . Revolution of $R$ about y-axis generates a solid of revolution.*




### Volumes of Solid of Revolution

One of the simplest applications of integration is to determine a volume of solid of revolution. In this section, we will study three methods to evaluate volumes of revolution known as the disk method, the washer method and the method of cylindrical shells.

### (A) Disk Method

Let $f$ be continuous on $[a, b]$ and let $R$ be a region bounded by the graphs, x-axis and the points $x = a,\ x = b$. Let $S$ be a solid generated by revolving $R$ about x-axis. Assume $P$ is a partition of $[a, b]$ and $w_k \in [x_{k-1}, x_k]$ is a marker. For each $[x_{k-1}, x_k]$, we form a rectangle, its high is $f(w_k)$ and its width is $\Delta x_k$.

The revolution of the rectangle about x-axis generates a circular disk as shown in Figure **??**. Its radius and high are

$$r = f(w_k)\ , \quad h = \Delta x_k\ .$$

From Figure **??**, the volume of each circular disk is

$$V_k = \pi(f(w_k))^2 \Delta x_k\ .$$

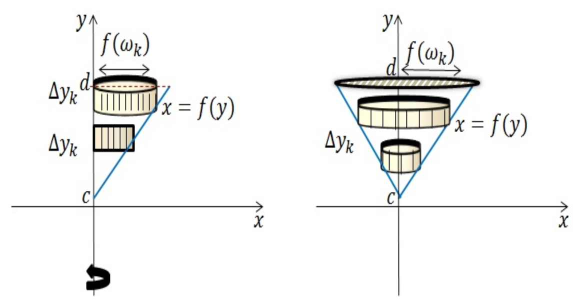
The sum of volumes of the circular disks approximately gives the volume of the solid of revolution:

$$V = \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} \pi\ (f(w_k))^2\ \Delta x_k = \pi \int_a^b \left[f(x)\right]^2\, dx\ .$$

$$V = \pi \int_a^b \left[f(x)\right]^2\, dx\ .$$

Similarly, we find the volume of the solid of revolution about y-axis. Let $f$ be continuous on $[c,d]$ and let $R$ be a region bounded by the graphs, y-axis and the points $y=c$, $y=d$. Let $S$ be a solid generated by revolving $R$ about y-axis. Assume $P$ is a partition of $[c,d]$ and $w_k \in [y_{k-1}, y_k]$. For each $[y_{k-1}, y_k]$, we form a rectangle, its high is $f(w_k)$ and its width is $\Delta y_k$.
Revolution of each rectangle about y-axis generates a circular disk as shown in . Its radius and high are

$$r = f(w_k)\ , \quad h = \Delta y_k\ .$$

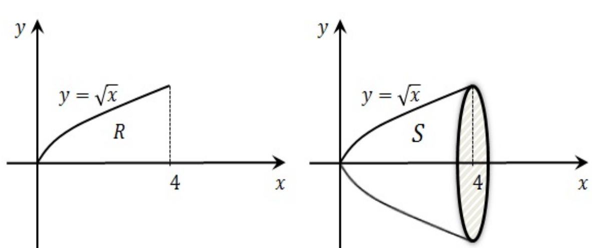



The volume of the solid of revolution given in is the sum of volumes of circular disks gives:

$$V = \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} \pi (f(w_k))^2 \Delta y_k = \pi \int_c^d \left[f(y)\right]^2 \, dy\ .$$

$$V = \pi \int_c^d \left[f(y)\right]^2 \, dy\ .$$

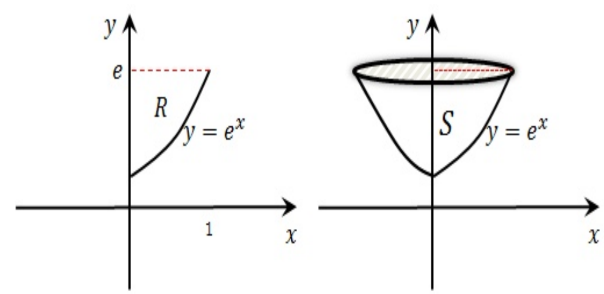
**Example 88** *Sketch the region $R$ bounded by the graphs of the equations $y=\sqrt{x}$, $x=4$, $y=0$. Then, find the volume of the solid generated if $R$ is revolved about x-axis.*
Solution:




The previous figure shows the solid generated from revolving the region $R$. Since the rotation is about x-axis, we have a vertical disk with radius $y=\sqrt{x}$ and thickness $dx$.
Thus, the volume of the solid $S$ is

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx$$
$$= \frac{\pi}{2}\left[x^2\right]_0^4$$
$$= \frac{\pi}{2}\left[16-0\right] = 8\pi\ .$$

**Example 89** *Sketch the region $R$ bounded by the graphs of the equations $y=e^x$ and $y=e$. Then, find the volume of the solid generated if $R$ is revolved about y-axis.*
Solution:




The previous figure shows the region $R$ and the solid generated from revolving that region. Since the revolution of $R$ is about y-axis, then we need to rewrite the function to become $x=f(y)$.

$$y = e^x \Rightarrow \ln y = \ln e^x \Rightarrow x = \ln y = f(y)$$

thus,

$$dy = e^x \, dx \Rightarrow dy = y \, dx \Rightarrow \frac{dy}{y} = dx\ .$$

Now, we have a horizontal disk with radius $x=\ln y$ and thickness $dy$.
Hence, the volume of the solid $S$ is

$$V = \pi \int_1^e \frac{(\ln y)^2}{y} \, dy = \frac{\pi}{3}\left[(\ln y)^3\right]_1^e = \frac{\pi}{3}\left[(\ln e)^3 - (\ln 1)^3\right] = \frac{\pi}{3}\ .$$

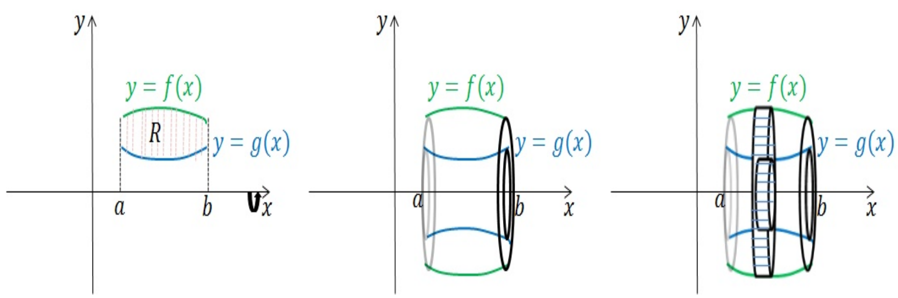
==========================================================

## (B) Washer Method

The washer method is the generalization of the disk method for a region between two functions $f(x)$ and $g(x)$. Let $R$ be a region bounded by the graphs of $f(x)$ and $g(x)$ such that $f(x) > g(x)$ from $x = a$ to, $x = b$. The volume of the solid $S$ generated from rotating the area bounded by the graphs of $f(x)$ and $g(x)$ around x-axis is

$$V = \int_a^b [f(x)]^2\, dx - \int_a^b [g(x)]^2\, dx,$$

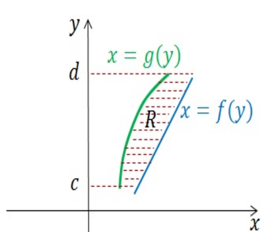
$$V = \int_a^b \left([f(x)]^2 - [g(x)]^2\right)\, dx\ .$$

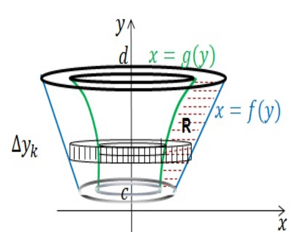



Similarly, let $R$ be a region bounded by the graphs of $f(y)$ and $g(y)$ such that $f(y) > g(y)$ from $y = c$ to, $y = d$. The volume of the solid $S$ generated from rotating the area bounded by the graphs of $f$ and $g$ around y-axis is

$$V = \int_c^d [f(y)]^2\, dy - \int_c^d [g(y)]^2\, dy,$$

$$V = \int_c^d \left([f(y)]^2 - [g(y)]^2\right)\, dy\ .$$



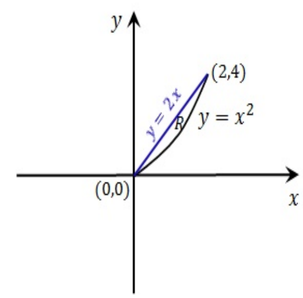



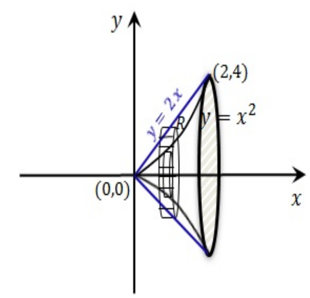

**Example 90** *Evaluate the volume of the solid generated by revolution of the bounded region by graphs of the following two functions $y = x^2$ and $y = 2x$ about x-axis.*

Solution:

Let $f(x) = x^2$ and $g(x) = 2x$. First, we find the intersection points:

$$\begin{aligned} f(x) = g(x) &\Rightarrow x^2 = 2x \\ &\Rightarrow x^2 - 2x = 0 \\ &\Rightarrow x(x-2) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 2\ . \end{aligned}$$




Substitute $x = 0$ into $f(x)$ or $g(x)$ gives the same value $y = 2$. Similarly, substitute $x = 2$, we have $y = 2$. Hence, the two curves $f$ and $g$ intersect in two points $(0, 0)$ and $(2, 4)$.

The figure shows the region $R$ and the solid generated from revolving that region about x-axis. A vertical rectangle generates a washer where

the outer radius: $y_2 = 2x$,

the inner radius: $y_2 = x^2$ and

thickness: $dx$ .

The volume of the washer is

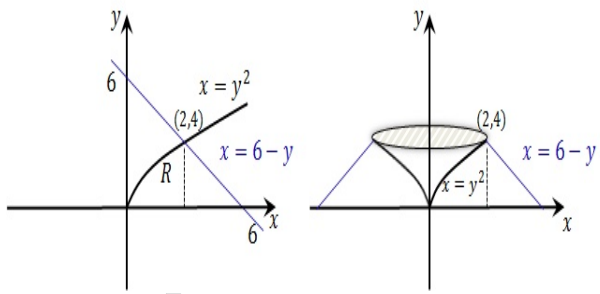
$$dV = \pi\left[x^2 - 2x\right]\, dx\ .$$

Thus, the volume of the solid over the interval $[0, 2]$ is

$$\begin{aligned} V = \pi \int_0^2 \left[(2x)^2 - (2x)^2\right]\, dx = \pi \int_0^2 4x^2 - x^4\, dx \\ = \pi\left[\frac{4}{3}x^3 - \frac{x^5}{5}\right]_0^2 \\ = \pi\left[\frac{32}{3} - \frac{32}{5}\right] \\ = \frac{64}{15}\pi\ . \end{aligned}$$

**Example 91** *Consider a region $R$ bounded by the graphs $y = \sqrt{x}$, $y = 6 - x$ and $x$-axis. Rotate this region about $y$-axis and find the volume of the generated solid.*

Solution:




The two curves $y = \sqrt{x}$ and $y = 6 - x$ intersect in one point $(4, 2)$. The region $R$ is shown in the figure. The revolution of that region generates a solid $S$. Since the rotation is about y-axis, first, we need to rewrite the functions as $x = f(y)$ and $x = g(y)$. Thus, $x = y^2$ and $x = 6 - y$. Second, a horizontal rectangle generates a washer where

$y = \sqrt{x} \Rightarrow x = y^2 = f(y)$ and $y = 6 - x$
$\Rightarrow x = 6 - y = g(y)$
$\Rightarrow x = 6 - y = g(y)$

the outer radius: $x_1 = 6 - y$,
the inner radius: $x_2 = y^2$ and
thickness: $dy$ .
The volume of the washer is

$$dV = \pi\left[(6-y)^2 - (y^2)^2\right]\, dy\ .$$

The volume of the solid over the interval $[0, 2]$ is

$$\begin{aligned} V = \pi \int_0^2 \left[(6-y)^2 - (y^2)^2\right]\, dy &= \pi\Big[ -\frac{(6-y)^3}{3} - \frac{y^5}{5}\Big]_0^2 \\ &= \pi\Big[\big(-\frac{64}{3} - \frac{32}{5}\big) - \big(-\frac{216}{3} - 0\big)\Big] \\ &= \frac{664}{15}\pi\ . \end{aligned}$$

**Example 92** *Reconsider the same region as in Example 91 enclosed by the curves $y = \sqrt{x}$, $y = 6 - x$ and $x$-axis. Now rotate this region about the $x$-axis instead and find the resulting volume.*

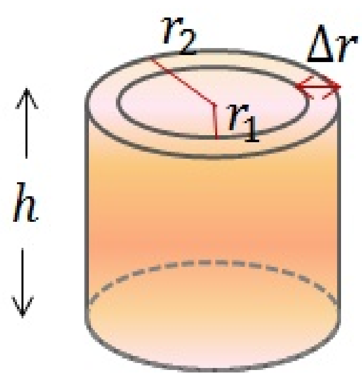
Solution:

## (C) Method of Cylindrical Shells

The method of cylindrical shells sometimes easier than the washer method. This is because solving equations for one variable in terms of another is not sometimes simple (i. e., solving $x$ in terms of $y$ and versa visa). For example, the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis. By the washer method, we would have to solve the cubic equation for $x$ in terms of $y$ and this is not simple.

In the washer method, we assume that the rectangle from each sub-interval is vertical to the axis of the revolution, but in the method of cylindrical shells, the rectangle is parallel to the axis of revolution.

As shown in the next figure, let
$r_1$ be the inner radius of the shell,
$r_2$ be the outer radius of the shell,
$h$ be high of the shell,
$\Delta r = r_2 - r_1$ be the thickness of the shell,
$r = \frac{r_1+r_2}{2}$ be the average radius of the shell.



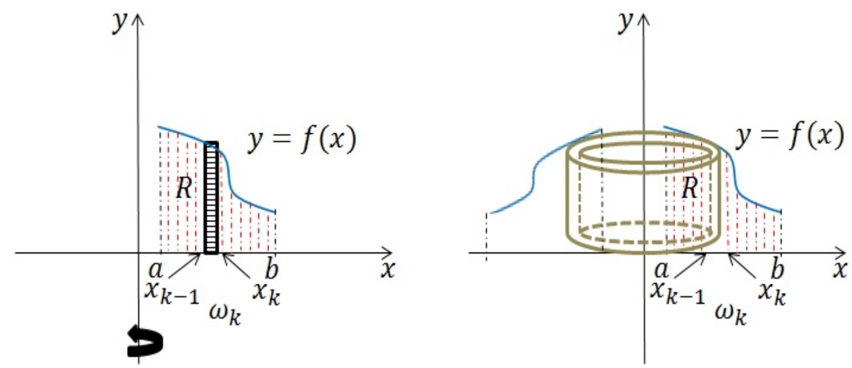

The volume of the cylindrical shell is

$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi(\frac{r_2 + r_1}{2})h(r_2 - r_1) \\ &= 2\pi r h \Delta r \ . \end{aligned}$$

Now, consider the graph given in the figure below. The revolution of the region $R$ about y-axis generates a solid given in the same figure. Let $P$ be a partition of the interval $[a, b]$ and let $w_k$ be the midpoint of $[x_{k-1}, x_k]$.

The revolution of the rectangle about y-axis generates a cylindrical shell where

the high $= f(w_k)$,
the average radius $= w_k$ and
the thickness $= \Delta x_k$ .




Hence, the volume of the cylindrical shell

$$V_k = 2\pi w_k f(w_k)\Delta x_k \ .$$

To evaluate the volume of the whole solid, we sum the volume of all cylindrical shells. This means

$$V = \sum_{k=1}^{n} V_k = 2\pi \sum_{k=1}^{n} w_k f(w_k)\Delta x_k \ .$$

From Riemann sum

$$\sum_{k=1}^{n} w_k f(w_k)\Delta x_k = \int_a^b x f(x)\ dx$$

and this implies

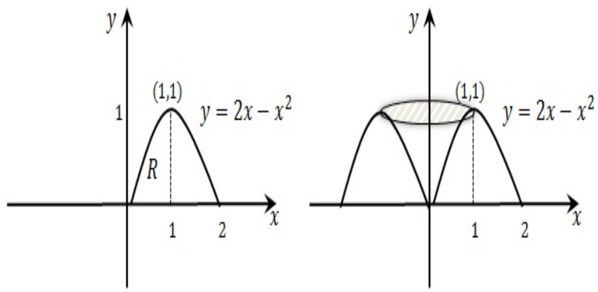
$$V = 2\pi \int_a^b x f(x)\ dx \ .$$

Similarly, if the revolution of the region about x-axis, the volume of the solid of revolution is

$$V = 2\pi \int_c^d y f(y)\ dy \ .$$

**Example 93** *Sketch the region R bounded by the graphs of the equations $y = 2x - x^2$ and $x = 0$. Then, by the method of cylindrical shells, find the volume of the solid generated if R is revolved about y-axis.*

Solution:




Since the revolution is about y-axis, the rectangle is vertical where
the high: $y = 2x - x^2$,
the average radius: $x$,
the thickness: $dx$ .
The volume of a cylindrical shell

$$V = 2\pi x(2x - x^2)\ dx\ .$$

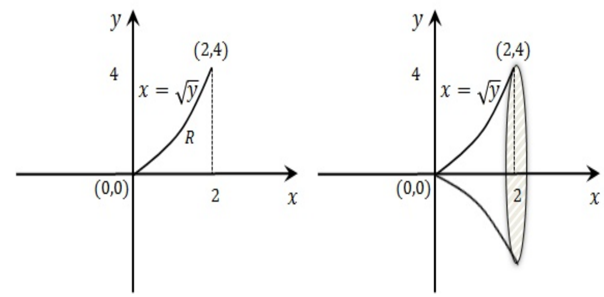
Thus, the volume of the solid is

$$\begin{aligned} V = 2\pi \int_0^2 x(2x - x^2)\ dx &= 2\pi \int_0^2 2x^2 - x^3\ dx \\ &= 2\pi \Big[\frac{2x^3}{3} - \frac{x^4}{4}\Big]_0^2 \\ &= 2\pi (\frac{16}{3} - \frac{16}{4}) \\ &= \frac{8\pi}{3}\ . \end{aligned}$$

**Example 94** *Sketch the region R bounded by the graphs of the equations $x = \sqrt{y}$ and $x = 2$. Then, find the volume of the solid generated if R is revolved about x-axis.*

Solution:

Since the revolution is about x-axis, the rectangle is horizontal where
the high: $x = \sqrt{y}$,
the average radius: $y$ and
the thickness: $dy$ .




The volume of the cylindrical shell

$$V = 2\pi y\sqrt{y}\ dy\ .$$

Thus, the volume of the solid is

$$\begin{aligned} V = 2\pi \int_0^2 y\sqrt{y}\ dy &= 2\pi \int_0^2 y^{\frac{3}{2}}\ dx \\ &= \frac{4\pi}{5}\Big[y^{\frac{5}{2}}\Big]_0^2 \\ &= \frac{4\pi}{5}\Big[\sqrt{32} - 0\Big] \\ &= \frac{16\sqrt{2}\pi}{5}\ . \end{aligned}$$

**1 - 9** ■ Set up evaluate an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis:

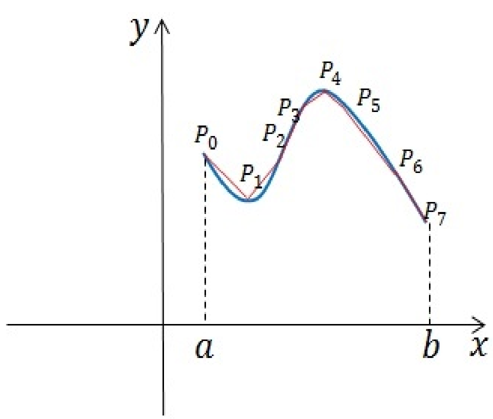
1. $y = x^2, y = 1$, about $x = 1$
2. $y = x^2, y = 1$, about x-axis
3. $y = x^2, x = y^2$ about y-axis
4. $y = \sqrt{x - 1}, y = 0, x = 5$ about $x = 5$
5. $y = x^2, x = 0, y = 1, y = 4$ about $y = 1$
6. $y = x - x^2, y = 0$ about $x = 2$
7. $y = x^2, y = 0, x = 1, x = 2$ about x-axis
8. $y = x^2, y = 0, x = 1, x = 2$ about $x = 4$
9. $y = x^4, y = \sin(\frac{\pi x}{2})$ about $x = -1$

## (3) Arc Length and Surfaces of Revolution

### (A) Arc Length

Let $y = f(x)$ be a smooth function on $[a, b]$. Assume $P = \{x_0, x_1, ..., x_n\}$ is a regular partition of the interval $[a, b]$ and let $y_0, y_1, ..., y_n$ be the points on the curve as shown in the following figure.
The distance between any two points $(x_{k-1}, y_{k-1})$ and $(x_k, y_k)$ in the curve is

$$\begin{aligned} d(y_{k-1}, y_k) &= \sqrt{(\triangle x_k)^2 + (\triangle y_k)^2} \\ &= \sqrt{(\triangle x_k)^2 + (f(x_k) - f(x_{k-1}))^2} \\ &= (\triangle x_k)\sqrt{1 + \frac{(f(x_k) - f(x_{k-1}))^2}{(\triangle x_k)^2}} \\ &= \frac{b-a}{n}\sqrt{1 + \left[\frac{f(x_k) - f(x_{k-1})}{\triangle x_k}\right]^2} \end{aligned}$$




From the conditions of the mean value theorem of differential calculus for the function $f$ on $[x_{k-1}, x_k]$, we have

$$f'(c_i) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

for some $c_i \in (x_{k-1}, x_k)$. Thus, the distance between $(x_{k-1}, y_{k-1})$ and $(x_k, y_k)$ is

$$d(y_{k-1}, y_k) = \frac{b-a}{n}\sqrt{1 + \left[f'(c_i)\right]^2}\ .$$

The sum of the distances is

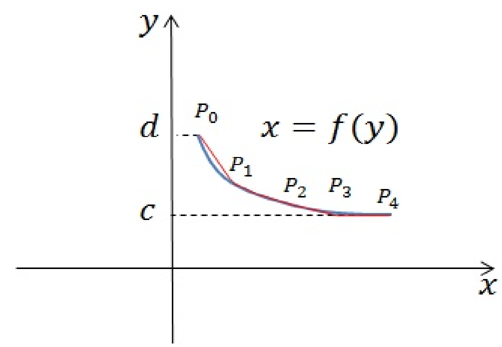
$$\frac{b-a}{n}\left[\sqrt{1 + \left[f'(c_1)\right]^2} + \sqrt{1 + \left[f'(c_2)\right]^2} + ... + \sqrt{1 + \left[f'(c_n)\right]^2}\right] .$$

The previous sum is a Riemann sum for the function $\sqrt{1 + \left[f'(x_i)\right]^2}$ from $a$ to $b$ where for a better approximation, we let $n$ be large enough.
From this, the arc length is

$$L(f) = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx$$

Similarly, let $x = g(y)$ be a smooth function on $[c, d]$. The length of the arc of $g$ from $y = c$ to $y = d$ is

$$L(g) = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\ dy\ .$$




**Example 95** *Find the arc length of the graph of the given equation from A to B:*

1. $y = 5 - \sqrt{x^3}$; $A(0, 4)$, $B(4, -3)$
2. $x = 4y$; $A(0, 0)$, $B(1, 4)$

Solution:

**Example 96** *Find the arc length of the graph of the given equation over the indicated intervals:*

1. $y = \cosh x$; $0 \le x \le 2$
2. $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$; $-2 \le y \le -1$

Solution:

## (2) Surfaces of Revolution

**Surfaces of Revolution**

**Definition 18** *The surface of revolution is generated by rotating the curve of a continuous function about an axis.*

**Surface Area of Revolution**

**Theorem 25**

1. *Let $y = f(x)$ be a smooth function on $[a, b]$.*
   - *If the rotating is about x-axis,*

$$S.A = 2\pi \int_a^b |\, y \,| \sqrt{1 + \left(f'(x)\right)^2}\; dx \; .$$

   - *If the rotating is about y-axis,*

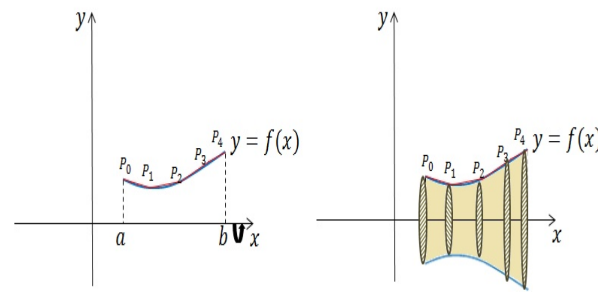
$$S.A = 2\pi \int_a^b |\, x \,| \sqrt{1 + \left(f'(x)\right)^2}\; dx \; .$$

2. *Let $x = g(y)$ be a smooth function on $[c, d]$. The surface area of revolution about y-axis is*
   - *If the rotating is about y-axis,*

$$S.A = 2\pi \int_c^d |\, x \,| \sqrt{1 + \left(g'(y)\right)^2}\; dy \; .$$

   - *If the rotating is about x-axis,*

$$S.A = 2\pi \int_c^d |\, y \,| \sqrt{1 + \left(g'(y)\right)^2}\; dy \; .$$




**Example 97** *Find the surface area generated by revolving the curve of the function $\sqrt{4 - x^2}$, $-2 \le x \le 2$ around x-axis.*

Solution:

We use the formula $S.A = 2\pi \int_a^b |\, f(x) \,| \sqrt{1 + (f'(x))^2}\; dx$.

$$\text{If } y = \sqrt{4 - x^2} \Rightarrow f'(x) = \frac{-2x}{2\sqrt{4 - x^2}}$$
$$\Rightarrow (f'(x))^2 = \frac{x^2}{4 - x^2}$$
$$\Rightarrow 1 + (f'(x))^2 = \frac{4}{4 - x^2}$$
$$\Rightarrow \sqrt{1 + (f'(x))^2} = \frac{2}{\sqrt{4 - x^2}} \; .$$

The area of the revolution surface is $S.A = 2\pi \int_{-2}^{2} \sqrt{4 - x^2} \frac{2}{\sqrt{4 - x^2}}\; dx = 4\pi\left[2 + 2\right] = 16\pi$ .

**Example 98** *Find the surface area generated by revolving the curve of the function $x = y^3$ on the interval $[0, 1]$ around y-axis.*

Solution:

We use the formula $S.A = 2\pi \int_c^d |\, f(y) \,| \sqrt{1 + (f'(y))^2}\; dy$.

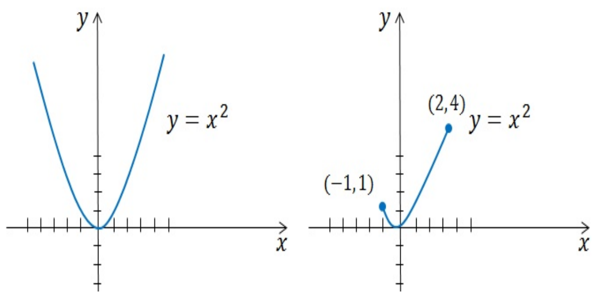
$$\text{If } x = y^3 \Rightarrow g'(y) = 3y^2$$
$$\Rightarrow (g'(y))^2 = 9y^4$$
$$\Rightarrow 1 + (g'(y))^2 = 1 + 9y^4$$
$$\Rightarrow \sqrt{1 + (g'(y))^2} = \sqrt{1 + 9y^4} \; .$$

The area of the revolution surface is $S.A = 2\pi \int_0^1 y^3 \sqrt{1 + 9y^4}\; dy = \frac{\pi}{27}\left[(1 + 9y^4)^{\frac{3}{2}}\right]_0^1 = \frac{\pi}{27}\left[10\sqrt{10} - 1\right]$ .

## (1) Parametric Equations of Plane Curves

In this section, we rather than considering only functions $y = f(x)$, it is sometimes convenient to view both $x$ and $y$ as functions of a third variable $t$ (called a parameter). The resulting equations $x = f(t)$ and $y = g(t)$ are called parametric equations. Each value of $t$ determines a point $(x, y)$, which we can plot in a coordinate plane. As $t$ varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve $C$, which we call a parametric curve.

**Example 99** *Let $y = f(x) = x^2$. The function is continuous and its graph given in the following figure (left):*




*If we consider the interval $-1 \le x \le 2$, then we have the second plot (right) Now, let $x = t$ and $y = t^2$ for $-2 \le t \le t$. We have the same graph where the last equations are called parametric equations for the curve $C$.*

**Remark 8**
1. *Parametric equations give the same graph of $y = f(x)$.*
2. *Parametric equations give the orientation of $C$.*
3. *To find the parametric equations, we introduce a third variable $t$. Then, we rewrite $x$ and $y$ as functions of $t$. The result is the parametric equations:*

$$x = f(t) \text{ parametric equation for } x,$$
$$y = g(t) \text{ parametric equation for } y.$$

**Example 100** *Write the curve given by $x(t) = 2t+1$ and $y(t) = 4t^2-9$ as $y = f(x)$.*

Solution:
Since $x = 2t + 1$, then $t = (x - 1)/2$. This implies

$$y(t) = 4t^2 - 9 = 4\left(\frac{x-1}{2}\right)^2 - 9 \Rightarrow y = x^2 - 2x - 8 .$$

**Example 101** *Sketch and identify the curve defined by the parametric equations*

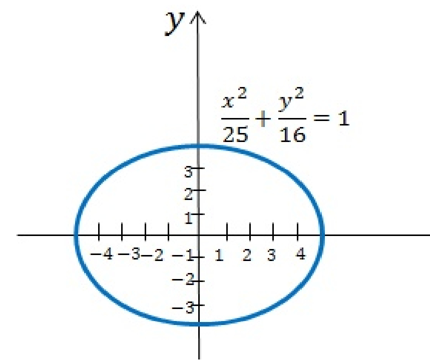
$$x = 5\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi .$$

Solution:
Let's first find the equation in $x$ and $y$. Since $x = 5\cos t$ and $y = 2\sin t$, then $\cos t = x/5$ and $\sin t = y/2$.

We know that

$$\cos^2 t + \sin^2 t = 1$$
$$\Rightarrow \frac{x^2}{25} + \frac{y^2}{4} = 1$$

Thus, the graph of the parametric equations is an ellipse.



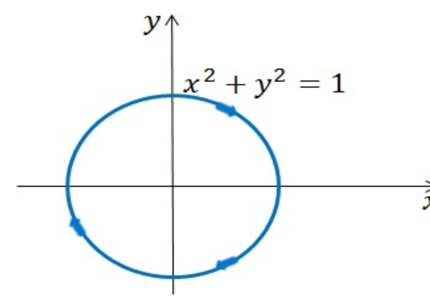

**Example 102** *For the following curve $x = \sin t$, $y = \cos t$, $0 \le t \le \pi$,*
1. *find an equation in $x$ and $y$ whose graph contains the points on the curve,*
2. *sketch the graph of $C$,*
3. *indicate the orientation.*

Solution:
1. We know that $\cos^2 t + \sin^2 t = 1$.
This implies

$$x^2 + y^2 = 1 .$$

Therefore, the graph of the parametric equations is a circle.

2.




3. The orientation can be indicated as follows:

| $t$ | $0$ | $\frac{\pi}{2}$ | $\pi$ |
|---|---|---|---|
| $x$ | $0$ | $1$ | $0$ |
| $y$ | $1$ | $0$ | $-1$ |
| $(x, y)$ | $(0, 1)$ | $(1, 0)$ | $(0, -1)$ |

## (A) Slope of the Tangent Line

Suppose $f$ and $g$ are differentiable functions. We want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$ where $y$ is also a differentiable function of $x$. From the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} .$$

If $dx/dt \neq 0$, we can solve for $dy/dx$:

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ if } \frac{dx}{dt} \neq 0$$

**Remark 9**

- *If $dy/dt = 0$ such that $dx/dt \neq 0$, the curve has a horizontal tangent.*
- *If $dx/dt = 0$ such that $dy/dt \neq 0$, the curve has a vertical tangent.*

**Example 103** *Find the slope of the tangent line to the curve at the indicated value:*

1. $x = t + 1$, $y = t^2 + 3t$; *at* $t = -1$
2. $x = t^3 - 3t$, $y = t^2 - 5t - 1$; *at* $t = 2$

Solution:

1. $y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+3}{1} = 2t + 3.$

   The slope of the tangent line at $t = -1$ is $\frac{dy}{dx} = 1$.

**Example 104** *Find the equations of the tangent line and the vertical line at $t = 2$ to the curve $x = 2t, \quad y = t^2 - 1$.*

Solution:

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t .$$

The slope of the tangent line at $t = 2$ is $m = 2$. Thus, the slope of the vertical line is $\frac{-1}{m} = \frac{-1}{2}$.

At $t = 2$, we find $(x_0, y_0) = (4, 3)$. Therefore, the tangent line is $y - 3 = 2(x - 4)$ and the vertical line is $y - 3 = -\frac{1}{2}(x - 4)$ .

## (B) Second Derivative in a Parametric Form

$$\frac{d^2y}{dx^2} = \frac{d(y')}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

**Example 105** *Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the indicated value $x = t$, $y = t^2 - 1$ at $t = 1$.*

Solution:

$\frac{dy}{dt} = 2t$ and $\frac{dx}{dt} = 1$. Thus, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2t$. At $t = 1$, $\frac{dy}{dx} = 2(1) = 2$.

The second derivative $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = 2$ .

## (C) Arc Length and Surface Area of Revolution

**Arc Length and Surface Area of Revolution**

**Theorem 26** *Let the curve $C$ has the parametric equations $x = f(t)$, $y = g(t)$ and $a \leq t \leq b$ such that $f'$ and $g'$ are continuous.*

1. *Arc length:*

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt .$$

2. *Surface area of revolution:*
   - *if the revolution is about x-axis,*

$$S.A = 2\pi \int_a^b g(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt .$$

   - *if the revolution is about y-axis,*

$$S.A = 2\pi \int_a^b f(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt .$$

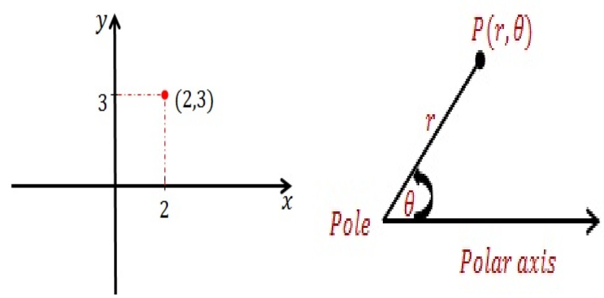
**Example 106** *Find the arc length of the curve $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \leq t \leq \frac{\pi}{2}$. Then, find the surface area of revolution of the curve around x-axis.*

Solution:

## (2) Polar Coordinates System

Previously, we used Cartesian coordinate to determine points $(x, y)$ as shown in Figure (left). In this section, we are going to study a new coordinate system called a polar coordinate.

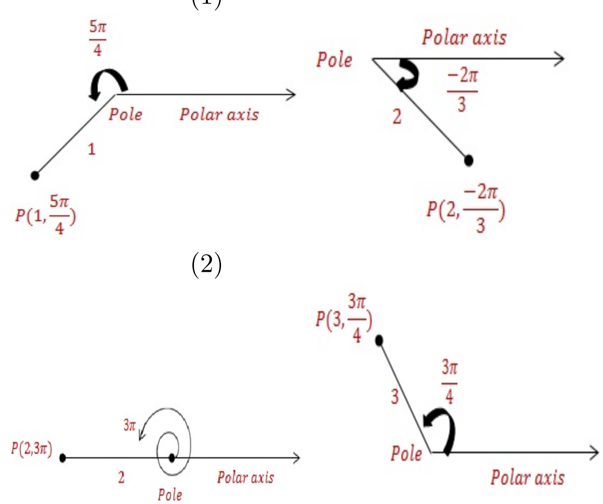
**Definition 19** *The polar coordinate system is a two-dimensional coordinate system in which each point $P$ on a plane is determined by a distance $r$ from a fixed point $O$ that is called the pole (or origin) and an angle $\theta$ from a fixed direction.*




**Example 107** *Plot the points whose polar coordinates are given:*

1. $(1, 5\pi/4)$
2. $(2, 3\pi)$
3. $(2, -2\pi/3)$
4. $(3, 3\pi/4)$

Solution:




The following remark is important in the study of the polar coordinate system.

**Remark 10**

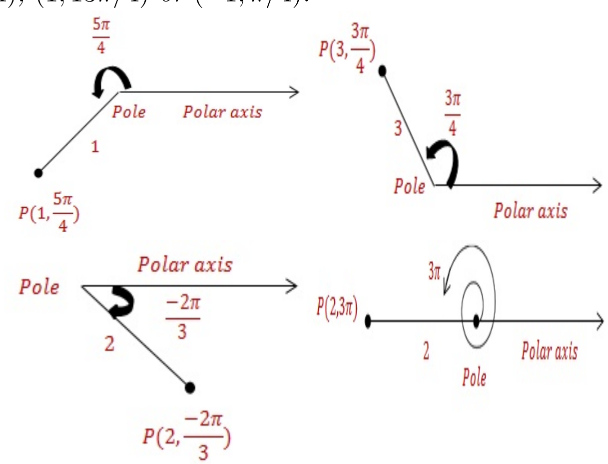
1. *From the definition, the point $P$ in the polar coordinate system is represented by the ordered pair $(r, \theta)$ where $r$, $\theta$ are called polar coordinates.*

2. *In the polar coordinates $(r, \theta)$, if $r > 0$, the point $(r, \theta)$ lies in the same quadrant as $\theta$; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Meaning that, the polar coordinates $(r, \theta)$ and $(-r, \theta)$ lie in the same line through the pole $O$ and at the same distance $| r |$ from $O$, but on opposite sides of $O$.*

3. *In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. The following formula gives all representations of each point $P(r, \theta)$ in the polar coordinate system*

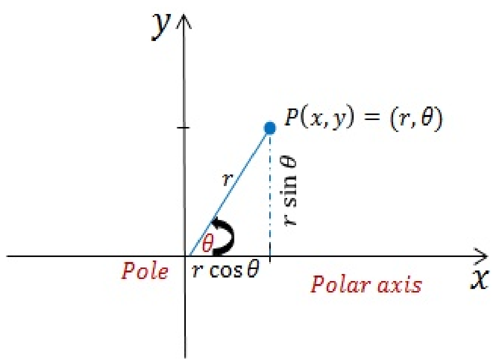
$$(r, \theta + 2n\pi) = (r, \theta) = (-r, \theta + (2n + 1)\pi) \quad n \in \mathbb{Z} .$$

**Example 108** *In Example 107, the point $(1, 5\pi/4)$ could be written as $(1, -3\pi/4)$, $(1, 13\pi/4)$ or $(-1, \pi/4)$:*



## (A) Relationship between Rectangular and Polar Coordinates

Let $(x, y)$ be a rectangular coordinate and $(r, \theta)$ be a polar coordinate. Let the pole at the origin point and polar axis on x-axis, and the line $\theta = \frac{\pi}{2}$ on y-axis as shown in Figure .




From the triangle OAP

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta \ ,$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta \ .$$

Thus,

$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2,$$
$$= r^2(\cos^2\theta + \sin^2\theta)\ dy \ .$$

Then, $x^2 + y^2 = r^2$ and $\tan\theta = \frac{y}{x}$.
The previous relationships can be summarized as follows:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$

**Example 109** *Convert the points from the polar coordinates to the rectangular coordinates:*

*1.* $(1, \pi/4)$ *2.* $(2, \pi)$

Solution:

1. $r = 1$ and $\theta = \frac{\pi}{4}$.

$$x = r\cos\theta = (1)\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and}$$

$$y = r\sin\theta = (1)\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \ .$$

Hence, $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

**Example 110** *Convert the points from the rectangular coordinates to polar coordinates:*

*1.* $(5, 0)$ *2.* $(2\sqrt{3}, -2)$

Solution:

1. $x = 5$ and $y = 0$

$$\Rightarrow r^2 = x^2 + y^2 = 5^2 + 0^2$$
$$\Rightarrow r = 5 \ .$$

Also,

$$\tan\theta = \frac{y}{x} = \frac{0}{5} = 0 \Rightarrow \theta = 0 \ .$$

Thus, $(r, \theta) = (5, 0)$.

**Example 111** *Convert the rectangular equations to polar forms and verse versa:*

*1.* $x^2 + y^2 = 4$ *2.* $r = \sin\theta$

Solution:

1.

$$x^2 + y^2 = 4 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 4$$
$$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 4$$
$$\Rightarrow r^2 = 4$$
$$\Rightarrow r = 2 \ .$$

2. $r = \sin\theta \Rightarrow r = \frac{y}{r} \Rightarrow r^2 = y \Rightarrow x^2 + y^2 = y \Rightarrow x^2 + y^2 - y = 0.$

### (B) Tangent Line to a Polar Curve

Let $r = f(\theta)$ be a polar curve where $f'$ is continuous at $(r_0, \theta_0)$. Then,

$$x = f(\theta)\cos\theta\ ,$$

$$y = f(\theta)\sin\theta\ .$$

From chain rule, we have

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta = -r\sin\theta + \frac{dr}{d\theta}\cos\theta\ ,$$

$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta = r\cos\theta + \frac{dr}{d\theta}\sin\theta\ .$$

If $\frac{dx}{d\theta} \neq 0$ at $\theta = \theta_0$, the slope of the tangent line to the graph of $r = f(\theta)$ at $(r_0, \theta_0)$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r_0\cos\theta_0 + \sin\theta_0(dr/d\theta)}{r_0\sin\theta_0 + \cos\theta_0(dr/d\theta)}\ .$$

**Remark 11**

1. *If $\frac{dy}{d\theta} = 0$ such that $\frac{dx}{d\theta} \neq 0$, the curve has a horizontal tangent line.*
2. *If $\frac{dx}{d\theta} = 0$ such that $\frac{dy}{d\theta} \neq 0$, the curve has a vertical tangent line.*

**Example 112** *Find the slope tangent of the curve $r = \sin\theta$ at $\theta = \frac{\pi}{4}$.*
Solution:

$$x = r\cos\theta \Rightarrow x = \sin\theta\cos\theta \Rightarrow \frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta\ ,$$

$$y = r\sin\theta \Rightarrow x = \sin^2\theta \Rightarrow \frac{dy}{d\theta} = 2\sin\theta\cos\theta\ .$$

$$\frac{dy}{dx} = \frac{2\sin\theta - \cos\theta}{\cos^2\theta - \sin^2\theta}$$

At $\theta = \frac{\pi}{4}$, $\frac{dy}{d\theta} = 1$ and $\frac{dx}{d\theta} = 0$. Thus, the curve has a vertical tangent line.

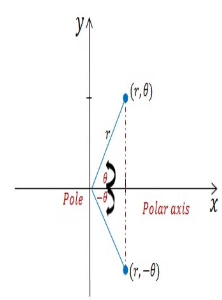
### (C) Graphs in Polar Coordinates

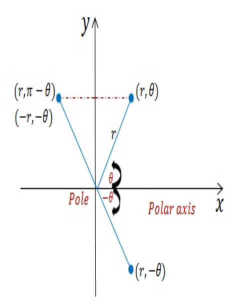
Before starting sketching polar curves, it is important to know symmetry of these curves about the polar axis, the vertical line $\theta = \frac{\pi}{2}$, or about the pole.

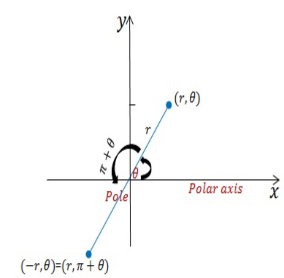
### (1) Symmetry in Polar Coordinates

**Theorem 27**

1. ***Symmetry about the polar axis.***
   *The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if replacing $(r, \theta)$ with $(r, -\theta)$ or with $(-r, \pi - \theta)$ does not change the equation.*
2. ***Symmetry about the vertical line $\theta = \frac{\pi}{2}$.***
   *The graph of $r = f(\theta)$ is symmetric with respect to the vertical line if replacing $(r, \theta)$ with $(r, \pi - \theta)$ or with $(-r, -\theta)$ does not change the equation.*
3. ***Symmetry about the pole $\theta = 0$.***
   *The graph of $r = f(\theta)$ is symmetric with respect to the pole if replacing $(r, \theta)$ with $(-r, \theta)$ or with $(r, \theta + \pi)$ does not change the equation.*




**Example 113**
1. *The graph of $r = 4\cos\theta$ is symmetric about the polar axis since*
$$\cos(-\theta) = \cos\theta\ .$$
2. *The graph of $r = 4\sin\theta$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$ since*
$$\sin(\pi - \theta) = \sin\theta \quad and \quad -r\sin(-\theta) = r\sin\theta\ .$$
3. *The graph of $r^2 = a^2\sin 2\theta$ is symmetric about the pole since*
$$(-r)^2 = a^2\sin 2\theta,$$
$$\Rightarrow r^2 = a^2\sin 2\theta\ .$$

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Also,

$$\begin{aligned} r^2 &= a^2 \sin[2(\pi+\theta)], \\ &= a^2 \sin(2\pi + 2\theta), \\ &= a^2 \sin 2\theta \ . \end{aligned}$$

### (2) Sketch of Polar Curves

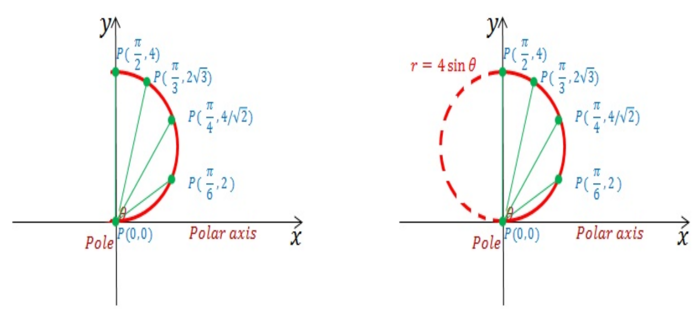
Here, we take two examples to explain how to plot polar curves.

**Example 114** *Sketch the graph of $r = 4\sin\theta$.*

Solution:

Note that, $r = 4\sin\theta$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$ since $\sin(\pi - \theta) = \sin\theta$. Therefore, we restrict our attention to the interval $[0, \pi/2]$. The following table displays some solution of $r = 4\sin\theta$:

| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $r$ | $0$ | $2$ | $4/\sqrt{2}$ | $2\sqrt{3}$ | $4$ |

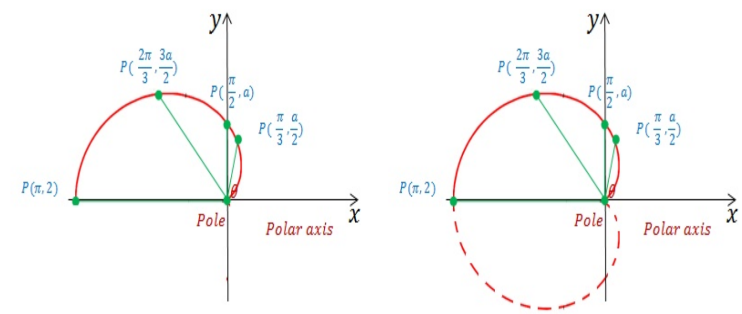



**Example 115** *Sketch the graph of $r = a(1 - \cos\theta)$ where $a > 0$.*

Solution:

The equation is symmetric about the polar axis since $\cos(-\theta) = \cos\theta$. Therefore, we restrict our attention to the interval $[0, \pi]$. The following table display some solution of the equation $r = a(1 - \cos\theta)$:

| $\theta$ | $0$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $r$ | $0$ | $a/2$ | $a$ | $3a/2$ | $2a$ |




### (3) Some Special Polar Graphs

- **Lines in polar coordinates**

1. General equation of a straight line is $ax + bx = c$. Its polar equation is
$$r = \frac{c}{a\cos\theta + b\sin\theta} \ .$$

2. Equation of a vertical line is $x = k$. Its polar equation is
$$r = k\sec\theta \ .$$
To see this, put $r = k\sec\theta \Rightarrow r = \frac{k}{\cos\theta}$. This implies $r\cos\theta = k \Rightarrow x = k$.

3. Equation of a horizontal line is $y = k$. Its polar equation is
$$r = k\csc\theta \ .$$
Put $r = k\csc\theta \Rightarrow r = \frac{k}{\sin\theta}$. This implies $r\sin\theta = k \Rightarrow y = k$.

4. Equation of a line that passes the origin point and makes an angle $\theta_0$ is $\theta = \theta_0$ .

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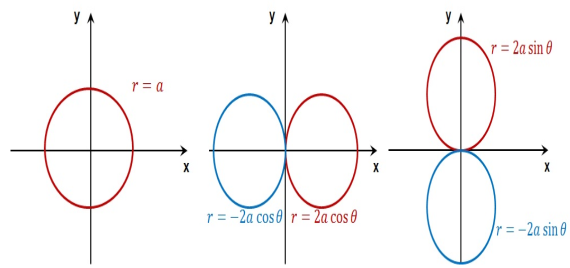
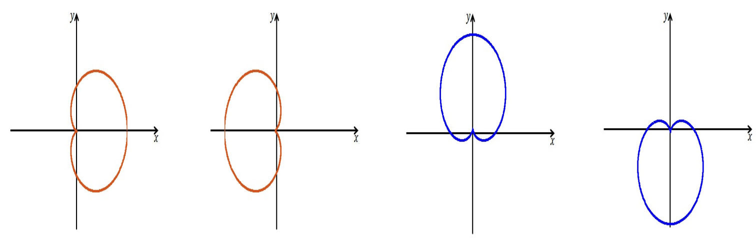
- **Circles in polar coordinates**
  1. A circle its center at $O$ and radius $a$: $r = a$ .
  2. A circle its center at $(a, 0)$ and radius $|a|$: $r = 2a\cos\theta$ .
  3. A circle its center at $(0, a)$ and radius $|a|$: $r = 2a\sin\theta$ .




- **Cardioid**

$r = a(1 \pm \cos\theta)$ OR $r = a(1 \pm \sin\theta)$

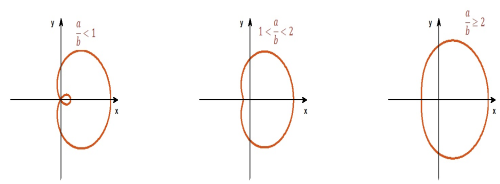
$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$



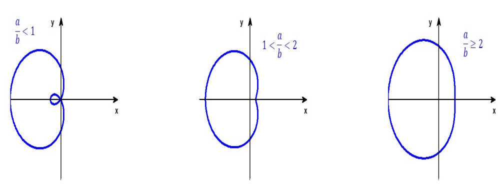
- **Limacons**

$r = a \pm b\cos\theta$ OR $r = a \pm b\sin\theta$

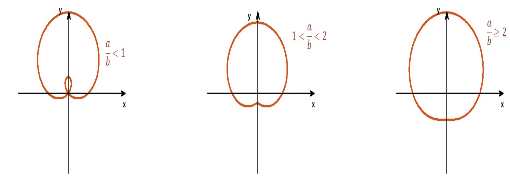
(1) $r = a \pm b\cos\theta$
  1. $r = a + b\cos\theta$



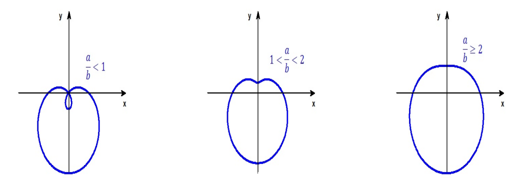

  2. $r = a - b\cos\theta$




(2) $r = a \pm b\sin\theta$
  1. $r = a + b\sin\theta$

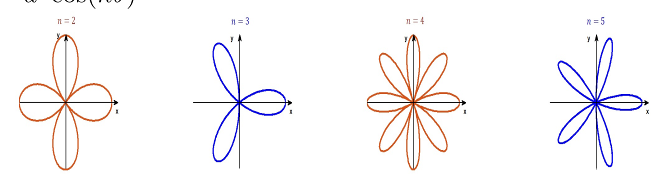



  2. $r = a - b\sin\theta$

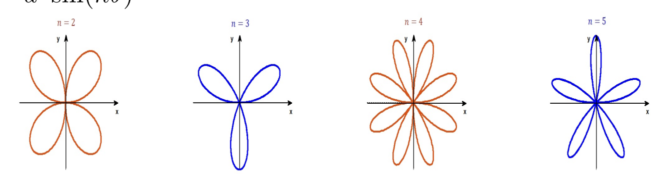



- **Roses**

$r = a\ \cos(n\theta)$ OR $r = a\ \sin(n\theta)$ where $n \in \mathbb{N}$.
  1. $r = a\ \cos(n\theta)$



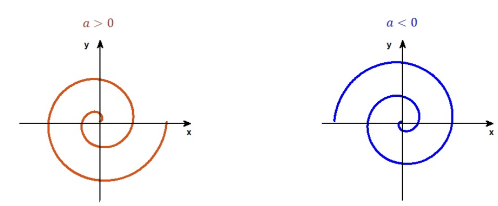

  2. $r = a\ \sin(n\theta)$




**Note:** If $n$ is odd, there are $n$ petals. If $n$ is even, there are $2n$ petals.

- **Spiral of Archimedes**

$r = a\ \theta$



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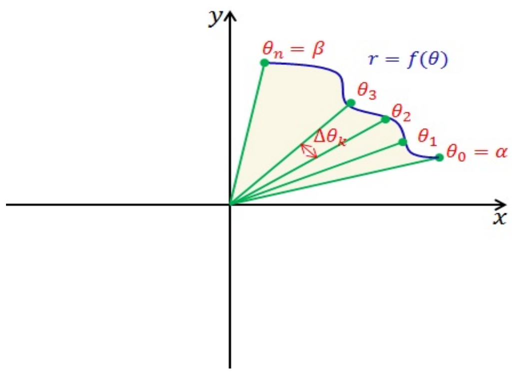
### (D) Area in Polar Coordinates

Let $r = f(\theta)$ be a continuous function on the interval $[\alpha, \beta]$ such that $0 \le \alpha \le \beta \le 2\pi$. Let $f(\theta) > 0$ over that interval and $R$ be a polar region bounded by the polar equations $r = f(\theta)$, $\theta = \alpha$ and $\theta = \beta$ as shown in Figure .




To find the area of $R$, we assume $P = \{\theta_1, \theta_2, ..., \theta_n\}$ is a regular partition of the interval $[\alpha, \beta]$. Consider the interval $[\theta_{k-1}, \theta_k]$ where $\triangle\theta_k = \theta_k - \theta_{k-1}$. By choosing $\omega_k \in [\theta_{k-1}, \theta_k]$, we have a circular sector where its angle and radius are $\triangle\theta_k$ and $f(\omega_k)$, respectively. The area between $\theta_{k-1}$ and $\theta_k$ can be approximated by the circular sector ( see Figure ). The area of the circular sector is $\frac{[f(\omega_k)]^2 \triangle\theta_k}{2}$, thus the area of $R$ is

$$A = \sum_{k=1}^{n} \frac{1}{2}[f(\omega_k)]^2 \triangle \theta_k \ .$$

From Riemann sum where the approximation is more accurate as $n \to \infty$, we have

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \big(f(\theta)\big)^2 \, d\theta$$

Similarly, assume $f$ and $g$ are continuous on the interval $[\alpha, \beta]$ such that $f(\theta) > g(\theta)$. The area bounded by the curves of $f$ and $g$ on the interval $[\alpha, \beta]$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[\big(f(\theta)\big)^2 - \big(g(\theta)\big)^2\right] d\theta$$

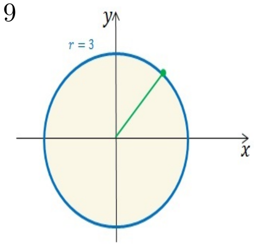
**Example 116** *Find the area of the region bounded by the graph of the polar equation*

*1.* $r = 3$ *2.* $r = 2\cos\theta$

Solution:

1. From the figure, the area is

$$A = \frac{1}{2}\int_0^{2\pi} 3^2 \, d\theta = \frac{9}{2}\int_0^{2\pi} d\theta = \frac{9}{2}\Big[\theta\Big]_0^{2\pi} = 9$$

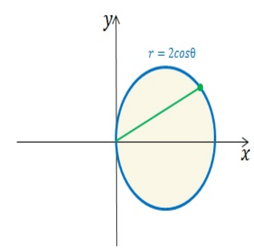



Note that, one can evaluate the area in the first quadrant and multiply the result by 4 to find the area of the whole region i.e.,

$$A = 4\Big(\frac{1}{2}\int_0^{\frac{\pi}{2}} 3^2 \, d\theta\Big) = 18\int_0^{\frac{\pi}{2}} d\theta = 18\Big[\theta\Big]_0^{\frac{\pi}{2}} = 9\pi \ .$$

2. We find the area of the half circle and multiply the result by 2 as follows:

$$\begin{aligned} A = 2\Big(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2\cos\theta)^2 \, d\theta\Big) &= \int_0^{\frac{\pi}{2}} 4\cos^2\theta \, d\theta \\ &= 2\int_0^{\frac{\pi}{2}} 1 + \cos 2\theta \, d\theta \\ &= 2\Big[\theta + \frac{\sin 2\theta}{2}\Big]_0^{\frac{\pi}{2}} \\ &= 2\Big[\frac{\pi}{2} - 0\Big] \\ &= \pi \ . \end{aligned}$$



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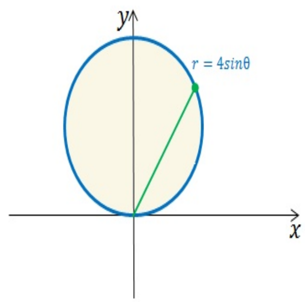
**Example 117** *Find the area of the region bounded by the graph of the polar equation*

*1.* $r = 4\sin\theta$ *2.* $r = 6 - 6\sin\theta$

Solution:

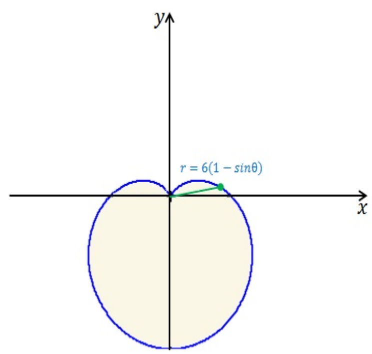
1. The area of the region is

$$A = \frac{1}{2}\int_0^{\pi}(4\sin\theta)^2\,d\theta = \frac{16}{4}\int_0^{\pi} 1-\cos 2\theta\,d\theta$$
$$= 4\Big[\theta - \frac{\sin 2\theta}{2}\Big]_0^{\pi}$$
$$= 4\Big[\pi - 0\Big]$$
$$= 4\pi\ .$$




The area of the region is

$$A = \frac{1}{2}\int_0^{2\pi} 36(1-\sin\theta)^2\,d\theta$$
$$= 18\int_0^{2\pi} 1 - 2\sin\theta + \sin^2\theta\,d\theta$$
$$= 18\Big[\theta + 2\cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big]_0^{2\pi}$$
$$= 18\Big[(2\pi + 2 + \pi) - 2\Big]$$
$$= 54\pi\ .$$



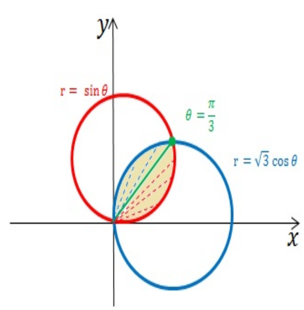

**Example 118** *Find the area of the region that is inside the graphs of both the equations* $r = \sin\theta$ , $r = \sqrt{3}\cos\theta$.

Solution:

First, we find the intersection point of the two curves.

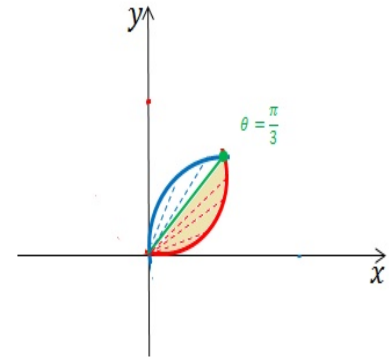
$$\sin\theta = \sqrt{3}\cos\theta \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}\ .$$

From the figure, the region is divided into two small regions: below and above the line $\frac{\pi}{3}$.



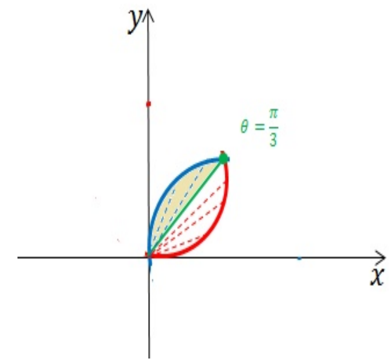

Area of the region below the line $\frac{\pi}{3}$:

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2\theta\,d\theta$$
$$= \frac{1}{4}\int_0^{\frac{\pi}{3}} 1 - \cos 2\theta\,d\theta$$
$$= \frac{1}{4}\Big[\theta - \frac{\sin 2\theta}{2}\Big]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{4}\Big[\frac{\pi}{3} - \frac{\sin\frac{2\pi}{3}}{2}\Big]$$
$$= \frac{1}{4}\Big[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\Big]$$




Area of the region above the line $\frac{\pi}{3}$:

$$A_2 = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{3}\cos\theta)^2\,d\theta$$
$$= \frac{3}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1 + \cos 2\theta\,d\theta$$
$$= \frac{3}{4}\Big[\theta + \frac{\sin 2\theta}{2}\Big]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$
$$= \frac{3}{4}\Big[(\frac{\pi}{2} - 0) - (\frac{\pi}{3} + \frac{\sqrt{3}}{4})\Big]$$
$$= \frac{3}{4}\Big[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\Big]$$




Total area $= A_1 + A_2 = \frac{5\pi}{24} - \frac{\sqrt{3}}{4}$.

**Example 119** *Find the area of the region that is outside the graph* $r = 3$ *and inside the graph* $r = 2 + 2\cos\theta$.

Solution:

## (E) Arc Length and Surface of Revolution in Polar Coordinates

### (1) Arc Length in Polar Coordinates

Let the polar function $r = f(\theta)$, $\alpha \leq \theta \leq \beta$ be smooth. We know that

$$x = f(\theta)\cos\theta \quad \text{and} \quad y = f(\theta)\sin\theta, \quad \alpha \leq \theta \leq \beta \ .$$

Thus,

$$L = \int_\alpha^\beta \sqrt{r^2 + (\frac{dr}{d\theta})^2} \ d\theta$$

**Example 120** *Find the length of the curves:*

1. $r = 2$
2. $r = 2\sin\theta$
3. $r = e^{-\theta}$ *where* $0 \leq \theta \leq 2\pi$
4. $r = 2 - 2\cos\theta$

Solution:

1. $r^2 + (\frac{dr}{d\theta})^2 = 4$

$$L = \int_0^{2\pi} \sqrt{4}\ d\theta = 2\Big[\theta\Big]_0^{2\pi} = 4\pi \ .$$

2. $r^2 + (\frac{dr}{d\theta})^2 = 4\sin^2\theta + 4\cos^2\theta = 4(\sin^2\theta + \cos^2\theta) = 4$

$$L = \int_0^{\pi} \sqrt{4}\ d\theta = 2\Big[\theta\Big]_0^{\pi} = 2\pi \ .$$

### (2) Surface of Revolution in Polar Coordinates

Let the polar curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$ be smooth. Then,

$$x = f(\theta)\cos\theta \quad \text{and} \quad y = f(\theta)\sin\theta, \quad \alpha \leq \theta \leq \beta \ .$$

From Section **??** in the previous chapter, we have the following:

1. the surface area of revolution about the polar axis (x-axis) is

$$S.A = 2\pi \int_\alpha^\beta r\sin\theta\sqrt{r^2 + (\frac{dr}{d\theta})^2} \ d\theta$$

2. the surface area of revolution about the line $\theta = \frac{\pi}{2}$ (y-axis) is

$$S.A = 2\pi \int_\alpha^\beta r\cos\theta\sqrt{r^2 + (\frac{dr}{d\theta})^2} \ d\theta$$

**Example 121** *For the curve* $C$*:* $r = 2\sin\theta$*, find the area of the surface generated by revolving the curve* $C$ *about*

1. *the line polar axis.*
2. *the line* $\theta = \frac{\pi}{2}$*.*

Solution:

1. We use the formula $S = 2\pi\int_\alpha^\beta \mid y \mid \sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta$ .

$$r^2 + (\frac{dr}{d\theta})^2 = 4\sin^2\theta + 4\cos^2\theta = 4(\sin^2\theta + \cos^2\theta) = 4 \ .$$

Thus,

$$\begin{aligned} S.A &= 2\pi \int_0^\pi 2\sin^2\theta\sqrt{4}\ d\theta \\ &= 8\pi \int_0^\pi (1 - \cos 2\theta)\ \sqrt{4}\ \ d\theta \\ &= 4\pi\big[\theta - \frac{\sin 2\theta}{2}\big]_0^\pi \\ &= 4\pi\big[\pi - 0\big] = 4\pi^2 \ . \end{aligned}$$

2. We use the formula $S = 2\pi\int_\alpha^\beta \mid x \mid \sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta$ . Note that, $\mid r\cos\theta \mid = r\cos\theta \ \forall x \in [0, \frac{\pi}{2}]$. Therefore,

$$S.A = 2\pi \int_0^{\frac{\pi}{2}} 2\sin\theta\cos\theta\sqrt{4}\,d\theta = -\frac{8\pi}{2}\big[\cos^2\theta\big]_0^{\frac{\pi}{2}} = -4\pi\big[0{-}1\big] = 4\pi \ .$$

$\sec x$ $\cos x$ $\sin x$ $\tan x$ $\cot x$ $csc\ x$ $sech\ x$ $\tanh\ x$