Chapter 3: Random Variables and Probability Distributions:

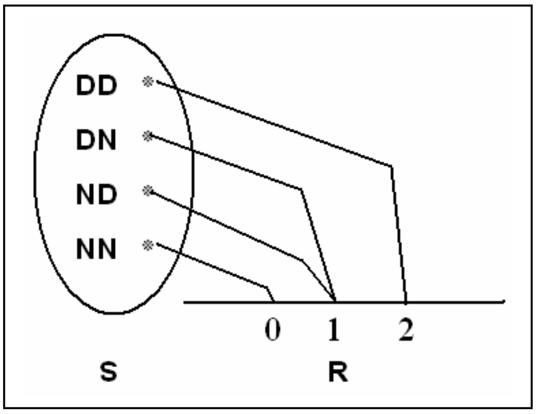
3.1 Concept of a Random Variable:

- In a statistical experiment, it is often very important to allocate numerical values to the outcomes.

Example:

- Experiment: testing two components.
  (D=defective, N=non-defective)
- Sample space: S={DD,DN,ND,NN}
- Let X = number of defective components when two components are tested.
- Assigned numerical values to the outcomes are:

Sample point (Outcome)	Assigned Numerical Value (x)
DD	2
DN	1
ND	1
NN	0



- Notice that, the set of all possible values of the random variable X is {0, 1, 2}.

Definition 3.1:

A random variable X is a function that associates each element in the sample space with a real number (i.e., $X : S \to \mathbf{R}$.)

Notation: " X " denotes the random variable .
" x " denotes a value of the random variable X.

Types of Random Variables:

- A random variable X is called a **discrete** random variable if its set of possible values is countable, i.e.,
  $x \in \{x_1, x_2, \ldots, x_n\}$ or $x \in \{x_1, x_2, \ldots\}$
- A random variable X is called a **continuous** random variable if it can take values on a continuous scale, i.e.,
  $x \in \{x: a < x < b;\ a, b \in R\}$

- In most practical problems:
    - A discrete random variable represents count data, such as the number of defectives in a sample of k items.
    - A continuous random variable represents measured data, such as height.

3.2 Discrete Probability Distributions

- A discrete random variable X assumes each of its values with a certain probability.

Example:

- Experiment: tossing a non-balance coin 2 times independently.
- H= head , T=tail
- Sample space: S={HH, HT, TH, TT}
- Suppose P(H)=½P(T) ⇔ P(H)=1/3 and P(T)=2/3
- Let X= number of heads

Sample point (Outcome)	Probability	Value of X (x)
HH	P(HH)=P(H) P(H)=1/3×1/3 = 1/9	2
HT	P(HT)=P(H) P(T)=1/3×2/3 = 2/9	1
TH	P(TH)=P(T) P(H)=2/3×1/3 = 2/9	1
TT	P(TT)=P(T) P(T)=2/3×2/3 = 4/9	0

- The possible values of X are: 0, 1, and 2.
- X is a discrete random variable.
- Define the following events:

Event (X=x)	Probability = P(X=x)
(X=0)={TT}	P(X=0) = P(TT)=4/9
(X=1)={HT,TH}	P(X=1) =P(HT)+P(TH)=2/9+2/9=4/9
(X=2)={HH}	P(X=2) = P(HH)= 1/9

- The possible values of X with their probabilities are:

x	0	1	2	Total
P(X=x)=f(x)	4/9	4/9	1/9	1.00

The function f(x)=P(X=x) is called the probability function (probability distribution) of the discrete random variable X.

Definition 3.4:

The function f(x) is a probability function of a discrete random variable X if, for each possible values x, we have:

1. f(x) ≥ 0
2. $\sum_{all\ x} f(x) = 1$
3. f(x)= P(X=x)

Note:

- P(X∈A) = $\sum_{all\ x \in A} f(x) = \sum_{all\ x \in A} P(X = x)$

Example:

For the previous example, we have:

x	0	1	2	Total
f(x)= P(X=x)	4/9	4/9	1/9	$\sum_{x=0}^{2} f(x) = 1$

P(X<1) = P(X=0)=4/9

P(X≤1) = P(X=0) + P(X=1) = 4/9+4/9 = 8/9

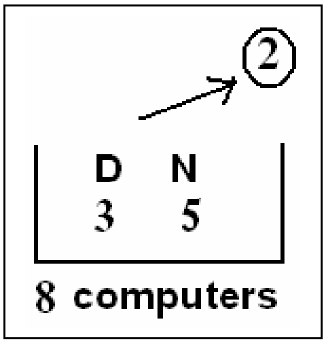
P(X≥0.5) = P(X=1) + P(X=2) = 4/9+1/9 = 5/9

P(X>8) = P(ϕ) = 0

P(X<10) = P(X=0) + P(X=1) + P(X=2) = P(*S*) = 1

Example 3.3:

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective and 5 are non-defective.
If a school makes a random purchase of 2 of these computers, find the probability distribution of the number of defectives.



Solution:

We need to find the probability distribution of the random variable: X = the number of defective computers purchased.
Experiment: selecting 2 computers at random out of 8

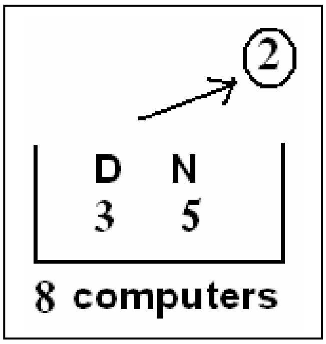
$$n(S) = \binom{8}{2} \text{ equally likely outcomes}$$

The possible values of X are: x=0, 1, 2.
Consider the events:

(X=0)={0D and 2N} ⇒ n(X=0)= $\binom{3}{0} \times \binom{5}{2}$

(X=1)={1D and 1N} ⇒ n(X=1)= $\binom{3}{1} \times \binom{5}{1}$

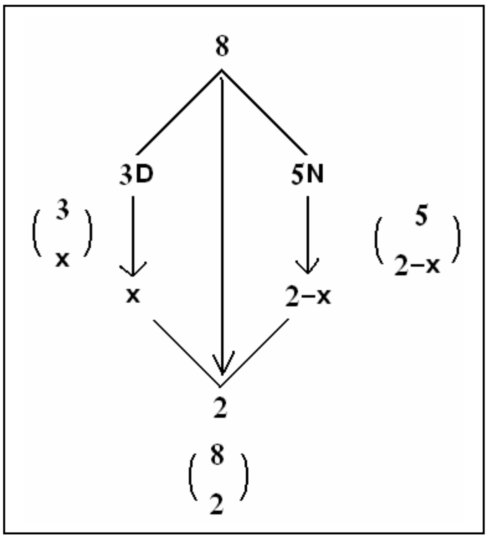
(X=2)={2D and 0N} ⇒ n(X=2)= $\binom{3}{2} \times \binom{5}{0}$



f(0)=P(X=0)= $\frac{n(X=0)}{n(S)} = \frac{\binom{3}{0}\times\binom{5}{2}}{\binom{8}{2}} = \frac{10}{28}$

f(1)=P(X=1)= $\frac{n(X=1)}{n(S)} = \frac{\binom{3}{1}\times\binom{5}{1}}{\binom{8}{2}} = \frac{15}{28}$

f(2)=P(X=2)= $\frac{n(X=2)}{n(S)} = \frac{\binom{3}{2}\times\binom{5}{0}}{\binom{8}{2}} = \frac{3}{28}$



In general, for x=0,1, 2, we have:

f(x)=P(X=x)= $\frac{n(X=x)}{n(S)} = \frac{\binom{3}{x}\times\binom{5}{2-x}}{\binom{8}{2}}$

The probability distribution of X can be given in the following table:

x	0	1	2	Total
f(x)= P(X=x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	1.00

The probability distribution of X can be written as a formula as follows:

$$f(x) = P(X=x) = \begin{cases} \frac{\binom{3}{x}\times\binom{5}{2-x}}{\binom{8}{2}}; & x = 0, 1, 2 \\ 0; & otherwise \end{cases}$$

Hypergeometric Distribution

Definition 3.5:

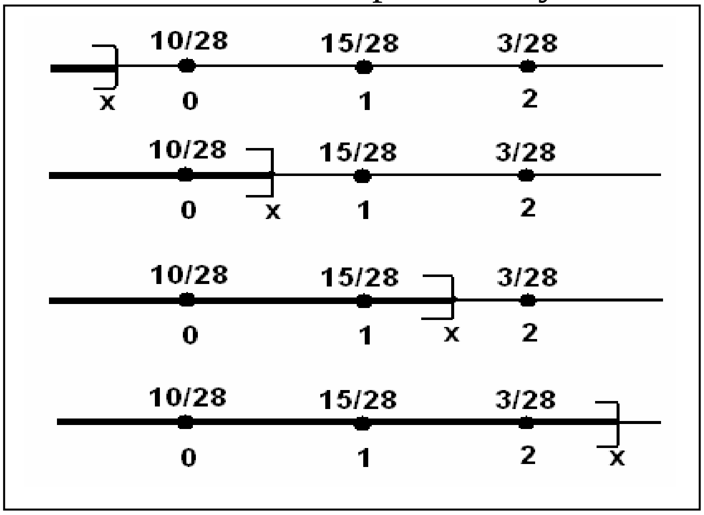
The cumulative distribution function (CDF), F(x), of a discrete random variable X with the probability function f(x) is given by:

$$F(x) = P(X \le x) = \sum_{t \le x} f(t) = \sum_{t \le x} P(X = t) \ ; \ \text{for } -\infty < x < \infty$$

Example:

Find the CDF of the random variable X with the probability function:

x	0	1	2
f(x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$



Solution:

$F(x)=P(X\le x)$ for $-\infty<x<\infty$

For $x<0$: $F(x)= 0$

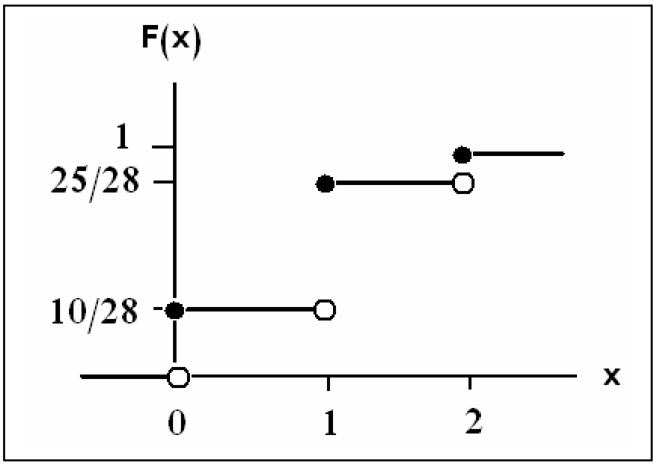
For $0\le x<1$: $F(x)=P(X=0)= \frac{10}{28}$

For $1\le x<2$: $F(x)=P(X=0)+P(X=1)= \frac{10}{28}+\frac{15}{28}=\frac{25}{28}$

For $x\ge 2$: $F(x)=P(X=0)+P(X=1)+P(X=2)= \frac{10}{28}+\frac{15}{28}+\frac{3}{28}=1$

The CDF of the random variable X is:

$$F(x) = P(X \le x) = \begin{cases} 0 \ ; & x<0 \\ \frac{10}{28} \ ; & 0 \le x < 1 \\ \frac{25}{28} \ ; & 1 \le x < 2 \\ 1 \ ; & x \ge 2 \end{cases}$$



Note:

$F(-0.5) = P(X\le -0.5)=0$

$F(1.5)=P(X\le 1.5)=F(1) =\frac{25}{28}$

$F(3.8) =P(X\le 3.8)F(2)= 1$

Another way to find F(x):

x	< 0	**0**	**1**	**2**	> 2
f(x)		$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	
F(x)	0	$\frac{10}{28}$	$\frac{25}{28}$	$\frac{28}{28}$	1

$$F(x) = \begin{cases} 0 \ ; & x < 0 \\ \frac{10}{28} \ ; & 0 \le x < 1 \\ \frac{25}{28} ; & 1 \le x < 2 \\ 1 \ ; & x \ge 2 \end{cases}$$

Result:

P(a < X ≤ b) = P(X ≤ b) – P(X ≤ a) = F(b) – F(a)

P(a ≤ X ≤ b) = P(a < X ≤ b) + P(X=a) = F(b) – F(a) + f(a)

P(a < X < b) = P(a < X ≤ b) – P(X=b) = F(b) – F(a) – f(b)

Result:

Suppose that the probability function of X is:

x	x_1	x_2	x_3	…	x_n
f(x)	$f(x_1)$	$f(x_2)$	$f(x_3)$	…	$f(x_n)$

Where $x_1 < x_2 < \ldots < x_n$. Then:

$F(x_i) = f(x_1) + f(x_2) + \ldots + f(x_i)$; i=1, 2, …, n

$F(x_i) = F(x_{i-1}) + f(x_i)$; i=2, …, n

$f(x_i) = F(x_i) - F(x_{i-1})$

Example:

In the previous example,

$P(0.5 < X \le 1.5) = F(1.5) - F(0.5) = \frac{25}{28} - \frac{10}{28} = \frac{15}{28}$

$P(1 < X \le 2) = F(2) - F(1) = 1 - \frac{25}{28} = \frac{3}{28}$

3.3. Continuous Probability Distributions

For any continuous random variable, X, there exists a non-negative function f(x), called the probability density function (p.d.f) through which we can find probabilities of events expressed in term of X.

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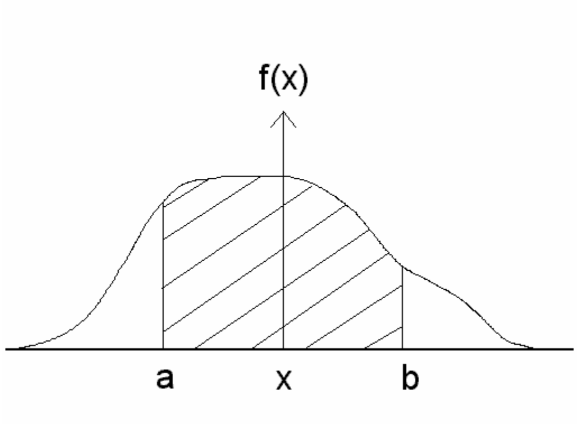
<td>
$f: R \to [0, \infty)$</td>
<td>$P(a < X < b) = \int_a^b f(x)\,dx$
= area under the curve of f(x) and over the interval (a,b)

$P(X \in A) = \int_A f(x)\,dx$
= area under the curve of f(x) and over the region A</td>
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Definition 3.6:

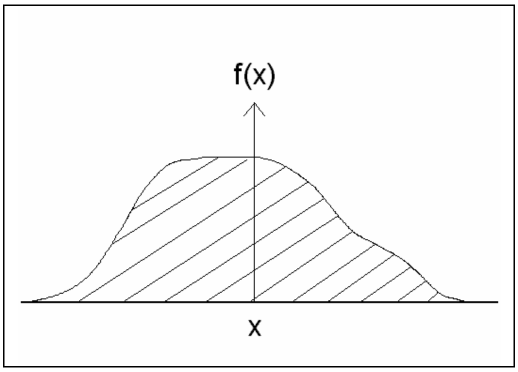
The function f(x) is a probability density function (pdf) for a continuous random variable X, defined on the set of real numbers, if:

1. $f(x) \geq 0 \quad \forall\, x \in R$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P(a \leq X \leq b) = \int_a^b f(x)\,dx \quad \forall\, a, b \in R;\ a \leq b$

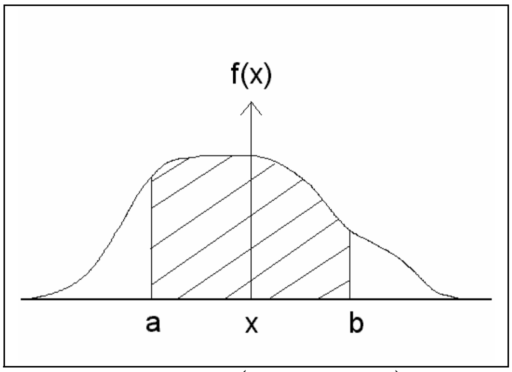
Note:

For a continuous random variable X, we have:

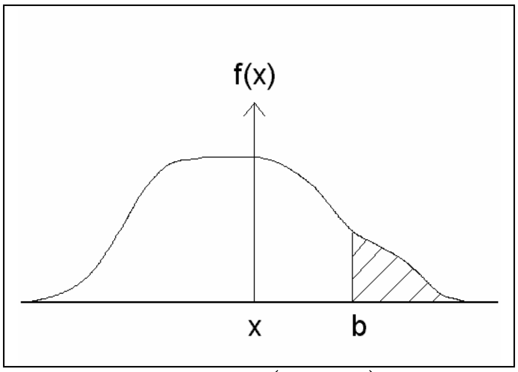
1. $f(x) \neq P(X=x)$ (in general)
2. $P(X=a) = 0$ for any $a \in R$
3. $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$
4. $P(X \in A) = \int_A f(x)\,dx$



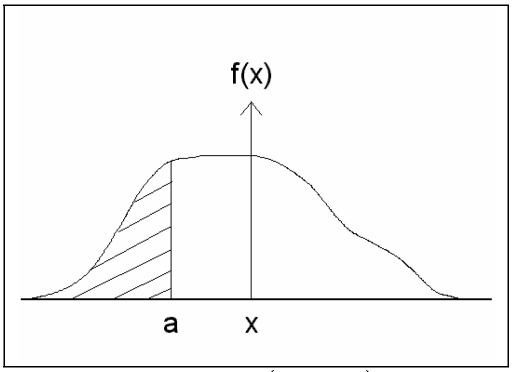
$$Total\ area = \int_{-\infty}^{\infty} f(x)\,dx = 1$$



$$area = P(a \le X \le b) = \int_{a}^{b} f(x)dx$$



$$area = P(X \ge b) = \int_{b}^{\infty} f(x)\,dx$$

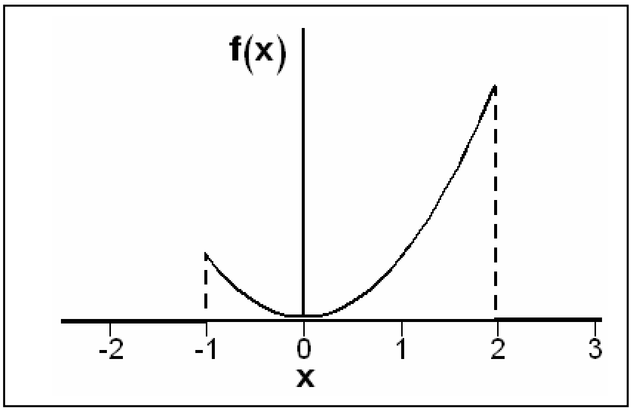


$$area = P(X \le a) = \int_{-\infty}^{a} f(x)dx$$

Example 3.6:

Suppose that the error in the reaction temperature, in ^{o}C, for a controlled laboratory experiment is a continuous random variable X having the following probability density function:

$$f(x) = \begin{cases} \frac{1}{3}x^2 \ ; -1 < x < 2 \\ 0 \ ; \ elsewhere \end{cases}$$



1. Verify that:
   (a) $f(x) \ge 0$
   (b) $\int_{-\infty}^{\infty} f(x)\,dx = 1$
2. Find $P(0 < X \le 1)$

Solution:

X = the error in the reaction temperature in ^{o}C.

X is continuous r. v.

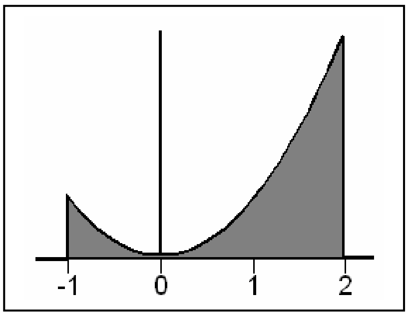
$$f(x)=\begin{cases}\frac{1}{3}x^2\ ;\ -1<x<2\\ 0\ ;\ elsewhere\end{cases}$$

1. (a) f(x) ≥ 0 because f(x) is a quadratic function.

(b) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{2}\frac{1}{3}x^2\,dx + \int_{2}^{\infty} 0\,dx$

$= \int_{-1}^{2}\frac{1}{3}x^2\,dx = \left[\frac{1}{9}x^3 \,\middle|\, \begin{matrix}x=2\\x=-1\end{matrix}\right]$

$= \frac{1}{9}(8-(-1)) = 1$

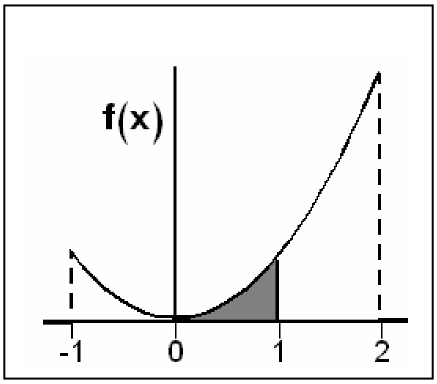


2. P(0<X≤1) = $\int_{0}^{1} f(x)\,dx = \int_{0}^{1}\frac{1}{3}x^2\,dx$

$= \left[\frac{1}{9}x^3 \,\middle|\, \begin{matrix}x=1\\x=0\end{matrix}\right]$

$= \frac{1}{9}(1-(0))$

$= \frac{1}{9}$



Definition 3.7:

The cumulative distribution function (CDF), F(x), of a continuous random variable X with probability density function f(x) is given by:

$$F(x) = P(X\le x)= \int_{-\infty}^{x} f(t)\,dt\ ;\ \text{for } -\infty<x<\infty$$

Result:

P(a < X ≤ b) = P(X ≤ b) – P(X ≤ a) = F(b) – F(a)

Example:

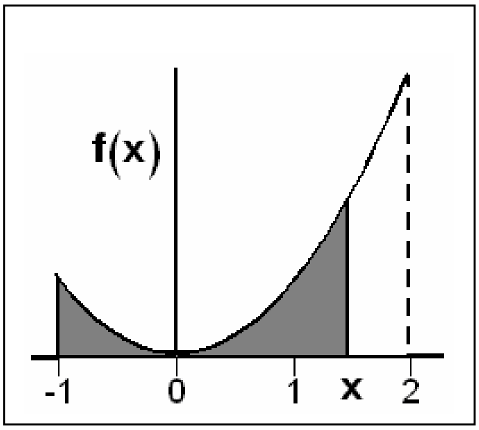
in Example 3.6,

1. Find the CDF
2. Using the CDF, find P(0<X≤1).

Solution:

$$f(x)=\begin{cases}\frac{1}{3}x^2\ ;\ -1<x<2\\ 0\ ;\ elsewhere\end{cases}$$

(1) Finding F(x):



$F(x) = P(X\le x) = \int_{-\infty}^{x} f(t)\,dt$; for $-\infty<x<\infty$

For $x<-1$:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

For $-1\le x<2$:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{x} \frac{1}{3}t^2\,dt$$

$$= \int_{-1}^{x} \frac{1}{3}t^2\,dt$$

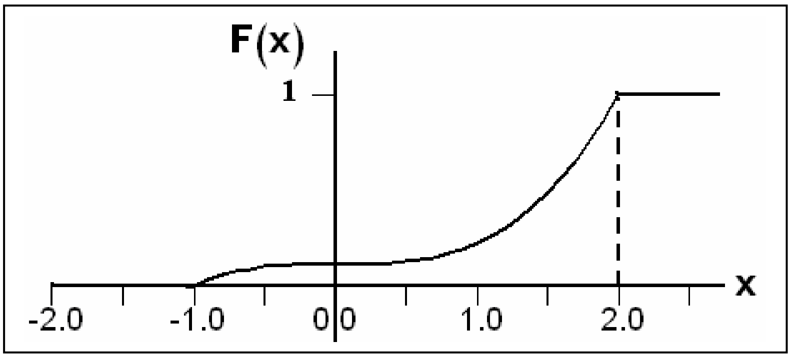
$$= \left[\frac{1}{9}t^3 \;\Big|_{t=-1}^{t=x}\right] = \frac{1}{9}(x^3-(-1)) = \frac{1}{9}(x^3+1)$$

For $x\ge 2$:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{2} \frac{1}{3}t^2\,dt + \int_{2}^{x} 0\,dt = \int_{-1}^{2} \frac{1}{3}t^2\,dt = 1.$$

Therefore, the CDF is:

$$F(x) = P(X\le x) = \begin{cases} 0 & ; x<-1 \\ \frac{1}{9}(x^3+1) & ; -1\le x<2 \\ 1 & ; x\ge 2 \end{cases}$$



2. Using the CDF,

$$P(0<X\le 1) = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$