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Solutions of fractional-stochastic Bao's system

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Abstract Systems of high-dimensional nonlinear ordinary differential equations play a significant role in Physics and applied sciences including big-data optimization, financial models, epidemic disease models. In this paper, we are concerned with numerical solutions of Bao's system that is a 4-dimensional hyperchaotic system introduced by Bo-Cheng and Zhong (2008). We solve the Bao's system with both the Crank-Nicolson and power series methods. Crank-Nicolson method is eventually evolved into a new system whose solution is presented in a quite neat algorithmic manner. By adding standard Brownian motion to each term in the model, we express the Bao's system as a system of stochastic differential equations. We solve the stochastic system with an Euler-type approximate solution method. By adding noise and expressing time derivatives with Caputo-type fractional derivative, we study on synchronization and parameter estimation of the models. To the best of our knowledge, Bao's system has not been numerically solved with the methods employed in this paper previously, and this paper considers fractional and stochastic Bao's system for the first time in the history of research. Techniques employed by us in this paper may serve as a framework for solutions of many other systems of ordinary differential equations including Lorenz types and epidemic models.

1. Introduction

High-dimensional and generalized systems of ordinary differential equations (stochastic, fractional and deterministic ones) appear and employed in modeling of many different phenom-

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ena in physics, engineering and applied sciences. Finding exact solutions of these nonlinear system of equations are impossible to obtain most of the time. Hence, approximate or numerical solutions of these systems are significantly important and useful tools in computational modern science. In the theory of high-dimensional systems of differential equations, studying solutions of problems such as chaos analysis, parameter estimation and identification, synchronization, optimal control problems associated with system of differential equations are quite active and significant research areas. Exact solutions of nonlinear system of ordinary differential equations having constant and variable coefficients is still challenging and significant area of research. For the numerical solutions of these types of systems of equations, a few methods like Taylor, Euler, Runge-Kutta have been developed in related scientific literature. There are many books and articles studying numerical solutions of systems of ODEs and PDEs, for instance, one can take a look at [10,12] among many other related good research works. The book [13] by Coddington and Levinson study theory of ODEs and systems of ODEs in a detailed and consistent manner.

Hyper-chaotic systems plays a significant role and have a broad range of applications, for instance, one can see the research works in [4–8,33–36] for other related studies in conjunction with present paper. In this paper, our major concerns are numerical solutions, parameter estimation, synchronization analysis of a 4-dimensional hyperchaotic system, namely, Bao's system, in determinstic, stochastic, fractional settings. Solution techniques employed in this paper may be applied to some nonlinear systems of differential equations appearing in energy control problems, financial mathematics including pension funds, insurance mathematics and cell signalling, system biology and epidemic models.

Lorenz-like systems (e.g., [5,7,8]) consist of nonlinear systems of ordinary differential equations having applications in the modeling of many different phenomena in meteorology, reactions in chemistry and biology, DC motors and so on. Analysis of chaotic behaviors of these systems of equations have been studied extensively. Hyperchaotic Bao system is a 4-dimensional Lorenz like system consisting of nonlinear ordinary differential equations introduced in [1]. The authors studied this model as a hyperchaotic attractor from the Lü's model. Some research works regarding solutions and analysis of Lü's model may be seen, for example, in [6]. Vaidyanathan and Rasappan studied [2] chaos synchronization of Bao's system with an active nonlinear control. The 4-dimensional deterministic ODE system known as Bao's system is described as:

$$\begin{aligned} \dot{x}_1(t) &= ax_2(t) - ax_1(t) + x_4(t), \\ \dot{x}_2(t) &= cx_2(t) - x_1(t)x_3(t), \\ \dot{x}_3(t) &= x_1(t)x_2(t) - bx_3(t), \\ \dot{x}_4(t) &= kx_1(t) + dx_2(t)x_3(t), \end{aligned} \tag{1}$$

where a, b, c, d, k are parameters and $x_i(t)$ terms are time dependent state variables. In the next section, we consider fractional-order Bao's system in which we investigate chaotic behavior of this system.

2. Fractional-order Bao's system

In recent two decades, research works regarding applications of fractional-order calculus (e.g. [25,26]) became a significant research area for scientists and engineers (e.g. see, [28–31,37–39]). It is because scientific events modeled by fractional-order derivatives and integrals take into account historical effects of events under consideration at each time step, and evaluates local changes in the systems more correctly than deterministic operators do. Fractional-order mathematics is a branch of differential equations and calculus generated by the fractional-order operators. Riemann-Liouville, Caputo, Atangana-Baleanu, Grünwald-Letnikov are some of the most popular fractional operators, see, [32]. In this section, we consider derivatives on the left-hand-side of system (1) as a Caputo-type time derivative which is described as follows:

Definition 2.1. Let α, c, t be real numbers with $\alpha > 0, t > c$. The operator

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \frac{f^n(y)}{(t-y)^{\alpha-n+1}} dy, \qquad n-1 < \alpha < n,$$

is called a fractional derivative of order α of the function f in the Caputo sense, where n is an integer and $\Gamma(\cdot)$ is the Gamma function. One can comprise boundary and initial supplementary conditions in a model via caputo-type derivative and Caputo-type derivative of a constant is 0 which make the Caputo-type derivative more advantageous over the other fractional derivative operators. [23] may be given as a reference guide for fundamental properties of fractional operators.

Letting $a = 2, b = c = k = 1, d = -1$ in the system (1), we express the fractional-order Bao's system as

$$\begin{aligned} D_t^\alpha x_1(t) &= 2x_2(t) - 2x_1(t) + x_4(t), \\ D_t^\alpha x_2(t) &= x_2(t) - x_1(t)x_3(t), \\ D_t^\alpha x_3(t) &= x_1(t)x_2(t) - x_3(t), \\ D_t^\alpha x_4(t) &= x_1(t) - x_2(t)x_3(t). \end{aligned} \tag{2}$$

The fixed points (steady-state points or equilibrium points) of the fractional-order system (2) are obtained by letting the right-hand-side of the system equal to 0. Hence, the fixed point is determined as $F_0^* = (x_1^*, x_2^*, x_3^*, x_4^*) = (1, -1, -1, 4)$. Stability criteria for the fractional-order system (2) is given either by the following Theorem or Routh-Hurwitz stability criteria [24].

Theorem 1. For n-dimensional fractional-order dynamical systems of order α, if all eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of the Jacobian matrix of an equilibrium point satisfy the condition:

$$|arg(\lambda_i)| > \frac{\alpha\pi}{2}, \ \alpha = \max\{\alpha_1, \ldots, \alpha_n\}, \ i = 1, \ldots, n,$$

then, the fractional dynamical system is locally asymptotically stable at that steady-state point.

The Jacobian matrix of fractional-order Bao's system is determined in F_0^* as

$$J(1, -1, -1, 4) = \begin{bmatrix} -2 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \tag{3}$$

From the characteristic equation $\det(J(1, -1, -1, 4) - \lambda I_{4\times4}) = 0$, we obtain the eigenvalues as $\lambda_1 = A + B,\ \lambda_2 = A - B,\ \lambda_3 = A + iC,\ \lambda_4 = A - iC$, where

$$A = -\frac{1}{2},\ B = \frac{\sqrt{9+8\sqrt{2}}}{2},\ C = \frac{\sqrt{-9+8\sqrt{2}}}{2}.$$

Now, it is clear that

$$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{9+8\sqrt{2}}}{2}$$

is a positive real number located on the positive real axis. This implies that

$$0 = |arg(\lambda_1)| < \frac{\alpha\pi}{2} \text{ for all } 0 < \alpha < 1.$$

Therefore, the fractional-order Bao's system (2) is locally asymptotically unstable in $F_0^* = (1, -1, -1, 4)$ and demonstrates chaotic behavior in this equilibrium point. It is possible to study chaos analysis of the system for different parameter values, hence, one can follow similar steps to analyses chaotic behavior in some other steady-state points.

3. Power series solutions of Bao's system

Power series solutions of systems of differential equations is an efficient and a classical solution method in the modern science and engineering applications of mathematics. In this section, we solve the deterministic Bao's model via a classical power series method. Next, we describe and apply the power series method.

Let us express state variables $x_1(t), x_2(t), x_3(t), x_4(t)$ of system (1) in terms of power series as follows:

$$\begin{aligned} x_1(t) &= \sum_{n=0}^{\infty} a_n t^n, \\ x_2(t) &= \sum_{n=0}^{\infty} b_n t^n, \\ x_3(t) &= \sum_{n=0}^{\infty} c_n t^n, \\ x_4(t) &= \sum_{n=0}^{\infty} d_n t^n, \end{aligned} \tag{4}$$

where a_n, b_n, c_n, d_n are the coefficients to be determined. Writing the notations in (4) into system (1), we obtain new system of equations:

$$\begin{aligned} \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n &= \sum_{n=0}^{\infty}(a(b_n - a_n) + d_n)t^n, \\ \sum_{n=0}^{\infty}(n+1)b_{n+1}t^n &= \sum_{n=0}^{\infty}\left(cb_n - \sum_{k=0}^{n} a_{n-k}c_k\right)t^n, \\ \sum_{n=0}^{\infty}(n+1)c_{n+1}t^n &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n-k}b_k - bc_n\right)t^n, \\ \sum_{n=0}^{\infty}(n+1)d_{n+1}t^n &= \sum_{n=0}^{\infty}\left(ka_n + d\sum_{k=0}^{n} b_{n-k}c_k\right)t^n, \end{aligned} \tag{5}$$

from which we easily obtain the recursive system of equations of coefficients as

$$\begin{aligned} a_{n+1} &= \tfrac{1}{n+1}(a(b_n - a_n) + d_n), \\ b_{n+1} &= \tfrac{1}{n+1}\left(cb_n - \sum_{k=0}^{n} a_{n-k}c_k\right), \\ c_{n+1} &= \tfrac{1}{n+1}\left(\sum_{k=0}^{n} a_{n-k}b_k - bc_n\right), \\ d_{n+1} &= \tfrac{1}{n+1}\left(ka_n + d\sum_{k=0}^{n} b_{n-k}c_k\right), \end{aligned} \tag{6}$$

for $n = 0, 1, 2, 3, \ldots$ We assume that the parameters a, b, c, d, and initial coefficients a_0, b_0, c_0, d_0 of a_n, b_n, c_n, d_n, respectively, are given or known. By having these constants, we obtain solutions $x_1(t), x_2(t), x_3(t), x_4(t)$. In the next section we solve the deterministic Bao's system via Crank-Nicolson method.

4. Solutions of Bao's system by Crank-Nicolson method

Crank-Nicolson method is an efficient and powerful numerical solution technique. This method has been applied to the computational solutions of many different types of differential equations like ordinary, partial, fractional and stochastic in the literature. Interested reader may take a look at any well-written numerical analysis book to be familiarized with the algorithmic structure of Crank-Nicolson method, for instance, to [9]. In this paper, we apply Crank-Nicolson method to Bao's 4-dimensional system. By applying Crank-Nicolson method to the Bao's system (1), we obtain that:

$$\begin{aligned} \tfrac{x_1^{n+1} - x_1^n}{\Delta t} &= ax_2^{n+\frac{1}{2}} - ax_1^{n+\frac{1}{2}} + x_4^{n+\frac{1}{2}}, \\ \tfrac{x_2^{n+1} - x_2^n}{\Delta t} &= cx_2^{n+\frac{1}{2}} - (x_1x_3)^{n+\frac{1}{2}}, \\ \tfrac{x_3^{n+1} - x_3^n}{\Delta t} &= (x_1x_2)^{n+\frac{1}{2}} - bx_3^{n+\frac{1}{2}}, \\ \tfrac{x_4^{n+1} - x_4^n}{\Delta t} &= kx_1^{n+\frac{1}{2}} + d(x_2x_3)^{n+\frac{1}{2}}. \end{aligned} \tag{7}$$

By using the following approximation to the right-hand of system (7), we obtain a new system as

$$\begin{aligned} \tfrac{x_1^{n+1} - x_1^n}{\Delta t} &= \tfrac{1}{2}\left[a\left(x_2^n + x_2^{n+1}\right) - a\left(x_1^n + x_1^{n+1}\right) + x_4^n + x_4^{n+1}\right], \\ \tfrac{x_2^{n+1} - x_2^n}{\Delta t} &= \tfrac{1}{2}\left[c\left(x_2^n + x_2^{n+1}\right) - x_1^n x_3^n - x_1^{n+1}x_3^{n+1}\right], \\ \tfrac{x_3^{n+1} - x_3^n}{\Delta t} &= \tfrac{1}{2}\left[x_1^n x_2^n + x_1^{n+1}x_2^{n+1} - b\left(x_3^n + x_3^{n+1}\right)\right], \\ \tfrac{x_4^{n+1} - x_4^n}{\Delta t} &= \tfrac{1}{2}\left[k\left(x_1^n + x_1^{n+1}\right) + d\left(\left(x_2^n x_3^n + x_2^{n+1}x_3^{n+1}\right)\right]\right.. \end{aligned} \tag{8}$$

Letting $X_1 := x_1^{n+1}$, $X_2 := x_2^{n+1}$, $X_3 := x_3^{n+1}$, $X_4 := x_4^{n+1}$, $X_1^{(1)} := x_1^n$, $X_2^{(1)} := x_2^n$, $X_3^{(1)} := x_3^{(n)}$, $X_4^{(1)} := x_4^{(n)}$, we write system (8) as follows:

$$\begin{aligned} S_{X_1}(X_1, X_2, X_3, X_4) := {} & X_1 - X_1^{(1)} \\ & - \tfrac{\Delta t}{2}\left[a\left(X_2^{(1)} + X_2\right) - a\left(X_1^{(1)} + X_1\right) + X_4^{(1)} + X_4\right] = 0, \\ S_{X_2}(X_1, X_2, X_3, X_4) := {} & X_2 - X_2^{(1)} \\ & - \tfrac{\Delta t}{2}\left[c\left(X_2^{(1)} + X_2\right) - X_1^{(1)}X_3^{(1)} - X_1X_3\right] = 0, \\ S_{X_3}(X_1, X_2, X_3, X_4) := {} & X_3 - X_3^{(1)} - \tfrac{\Delta t}{2}\left[X_1^{(1)}X_2^{(1)} + X_1X_2 - b\left(X_3^{(1)} + X_3\right)\right] = 0, \\ S_{X_4}(X_1, X_2, X_3, X_4) := {} & X_4 - X_4^{(1)} - \tfrac{\Delta t}{2}\left[k\left(X_1^{(1)} + X_1\right) + d\left(X_2^{(1)}X_3^{(1)} + X_2X_3\right)\right] = 0. \end{aligned} \tag{9}$$

Assume that $X_1^-, X_2^-, X_3^-, X_4^-$ are approximations to X_1, X_2, X_3, X_4, in order. Solving the system (9) with respect to the unknowns X_1, X_2, X_3, X_4, we get a new system:

$$\begin{aligned} X_1(t) &= \frac{X_1^{(1)} + \frac{\Delta t}{2}\left[a\left(X_2^{(1)} + X_2^-\right) - aX_1^{(1)} + X_4^{(1)} + X_4^-\right]}{1 + a\frac{\Delta t}{2}}, \\ X_2(t) &= \frac{X_2^{(1)} + \frac{\Delta t}{2}\left[cX_2^{(1)} - X_1^{(1)}X_3^{(1)} - X_1^-X_3^-\right]}{1 - c\frac{\Delta t}{2}}, \end{aligned} \tag{10}$$

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$$X3(t) = \frac{X_3^{(1)} + \frac{\Delta t}{2}\left[X_1^{(1)}X_2^{(1)} + X_1^- X_2^- + bX_3^{(1)}\right]}{1 - b\frac{\Delta t}{2}},$$
$$X_4(t) = X_4^{(1)} + \frac{\Delta t}{2}\left[k\left(X_1^{(1)} + X_1^-\right) + d\left(X_2^{(1)}X_3^{(1)} + X_2^- X_3^-\right)\right].$$

Let us notice that one must update the state variables as

$$X_1^- \leftarrow X_1,\ X_2^- \leftarrow X_2,\ X_3^- \leftarrow X_3,\ X_4^- \leftarrow X_4$$

before each recursion.

Next we express the system $S := S(S_{X_1}, S_{X_2}, S_{X_3}, S_{X_4})$ as $S(w) := 0$ with $w = (X_1, X_2, X_3, X_4)$. Hence, the Jacobian matrix of Newton's method,

$$J = \begin{bmatrix} \frac{\partial}{\partial X_1}S_{X_1} & \frac{\partial}{\partial X_2}S_{X_1} & \frac{\partial}{\partial X_3}S_{X_1} & \frac{\partial}{\partial X_4}S_{X_1} \\ \frac{\partial}{\partial X_1}S_{X_2} & \frac{\partial}{\partial X_2}S_{X_2} & \frac{\partial}{\partial X_3}S_{X_2} & \frac{\partial}{\partial X_4}S_{X_2} \\ \frac{\partial}{\partial X_1}S_{X_3} & \frac{\partial}{\partial X_2}S_{X_3} & \frac{\partial}{\partial X_3}S_{X_3} & \frac{\partial}{\partial X_4}S_{X_3} \\ \frac{\partial}{\partial X_1}S_{X_4} & \frac{\partial}{\partial X_2}S_{X_4} & \frac{\partial}{\partial X_3}S_{X_4} & \frac{\partial}{\partial X_4}S_{X_4} \end{bmatrix},$$

is clearly represented by

$$J = \begin{bmatrix} 1 + \frac{\Delta t}{2}a & -\frac{\Delta t}{2}a & 0 & -\frac{\Delta t}{2} \\ \frac{\Delta t}{2}X_3 & 1 + \frac{\Delta t}{2}c & \frac{\Delta t}{2}X_1 & 0 \\ -\frac{\Delta t}{2}X_2 & -\frac{\Delta t}{2}X_1 & 1 + \frac{\Delta t}{2}b & 0 \\ -\frac{\Delta t}{2}k & -\frac{\Delta t}{2}dX_3 & -\frac{\Delta t}{2}dX_2 & 1 \end{bmatrix}.$$

The Newtonian system of equations looks as follows:

$$J(w^-)\delta w = -S(w^-),$$

which must be solved in each recursion, is hence expressed as

$$J\left(X_1^-, X_2^-, X_3^-, X_4^-\right)\begin{bmatrix} \delta X_1 \\ \delta X_2 \\ \delta X_3 \\ \delta X_4 \end{bmatrix} = -S\left(X_1^-, X_2^-, X_3^-, X_4^-\right).$$

We can state Newton's method of solution as an algorithm:

Given that $S(w) = 0$ and an initial guess point $w^- = \left(X_1^-, X_2^-, X_3^-, X_4^-\right)$, iterate the algorithm until it converges:

i. Solve $J(w^-)\delta w = -S(w^-)$ with respect to δw,
ii. Update w as $w := w^- + \epsilon\delta w$,
iii. $w^- \leftarrow w$,

where ϵ is an arbitrarily small positive number. In the Figs. 1–6, we illustrate behavior of the system discretized by Crank-Nicolson method for different discretization values. Fig. 7 demonstrates time series analysis of the system on different coordinate planes.

5. Numerical solutions of stochastic Bao's system

Stochastic differential equations and their systems, e.g., [16,17], are quite useful and significant types of differential equations mostly used in the modeling of systems involving noise, uncertainty, and randomness in coefficients, or in the supplementary conditions of the equation. A popular and introductory book written about stochastic differential equations is [14] by Oksendal. Higham's paper [11] presents some numerical methods for the solution of systems of stochastic differential equations. In this section, firstly, let us write the deterministic Bao's system (1) as

$$dX(t) = \alpha(X(t))dt,$$

where

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ X_4(t) \end{bmatrix}, \tag{11}$$

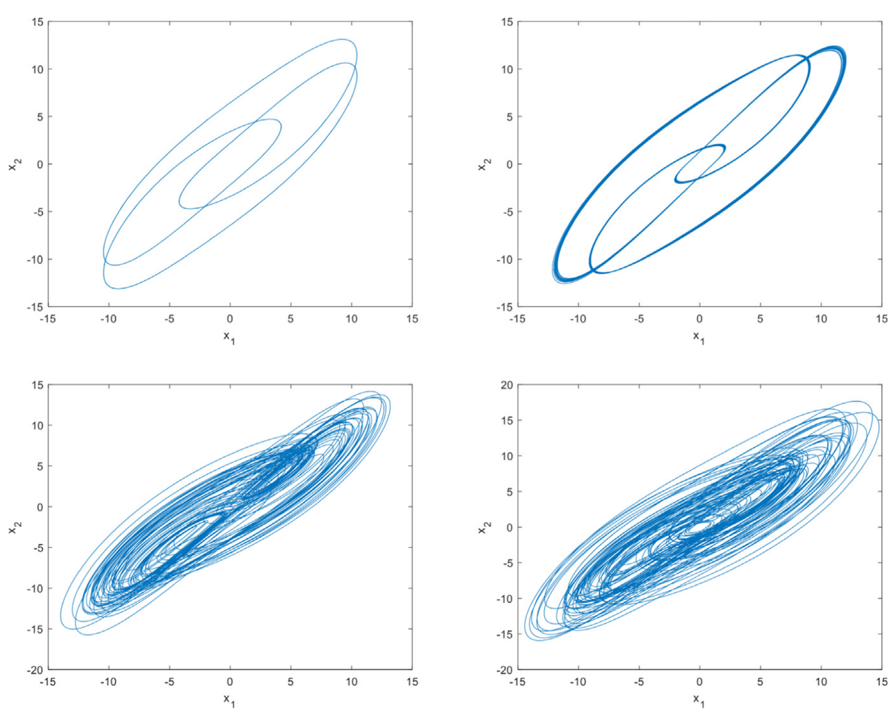

Fig. 1 Phase diagram in x1x2 plane.

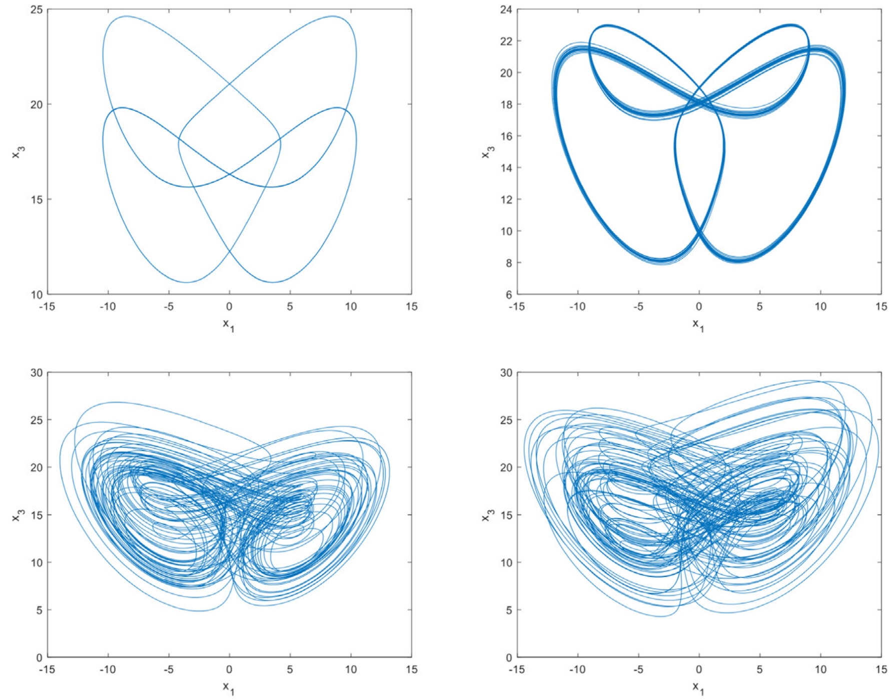

Fig. 2 Phase diagram in x1x3 plane.

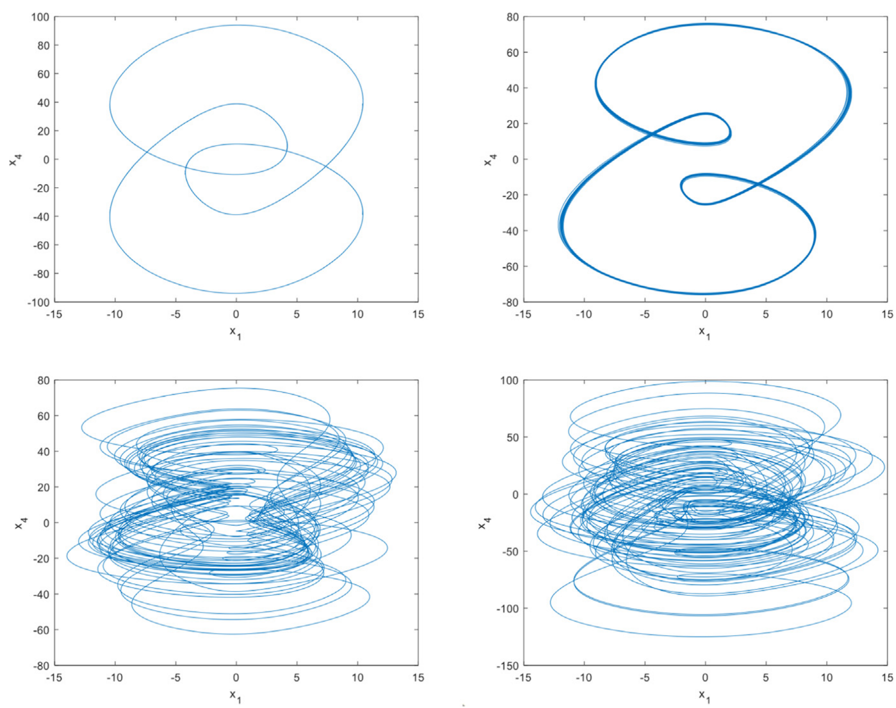

Fig. 3 Phase diagram in x1x4 plane.

and $\alpha(X(t))$ is the right-hand side of the system (1), i.e.,

$$\alpha(X(t),t)=\begin{bmatrix} aX_2(t)-aX_1(t)+X_4(t) \\ cX_2(t)-X_1(t)X_3(t) \\ X_1(t)X_2(t)-bX_3(t) \\ kX_1(t)+dX_2(t)X_3(t) \end{bmatrix}. \tag{12}$$

We perturb the Bao's system (1) from deterministic to a stochastic system by adding white noise which is written as a derivative of a standard Brownian motion under the assumption that each term in Bao's system includes some random terms or state variables are random variables. The terms in Bao's system may involve some random terms bacause, for example, Bao's system may be subject to some uncertain effects. In this case we add $\epsilon_i X_i(t)dW_i(t)$ to the right hand side

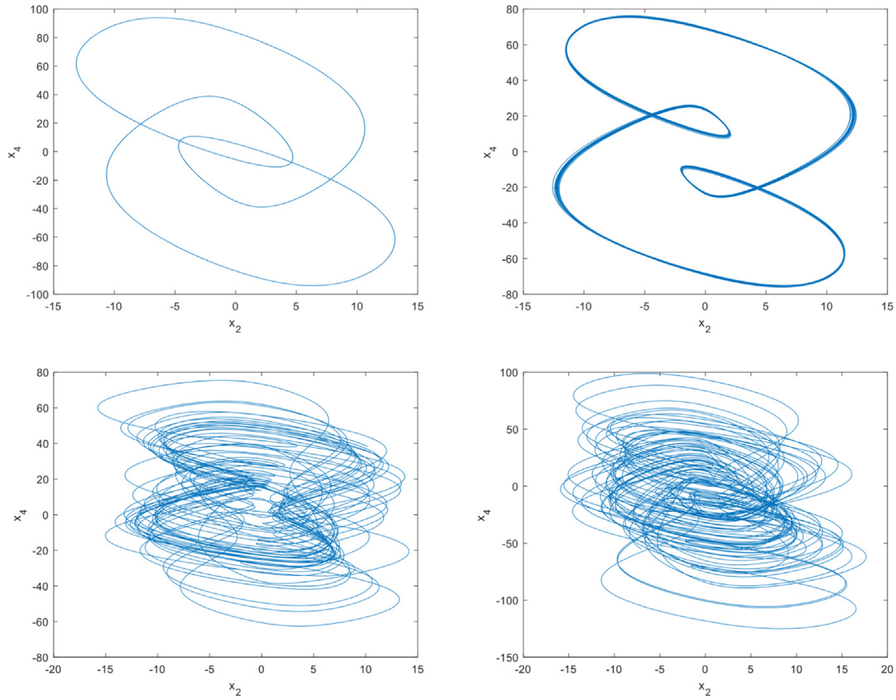

Fig. 4 Phase diagram in x2x4 plane.

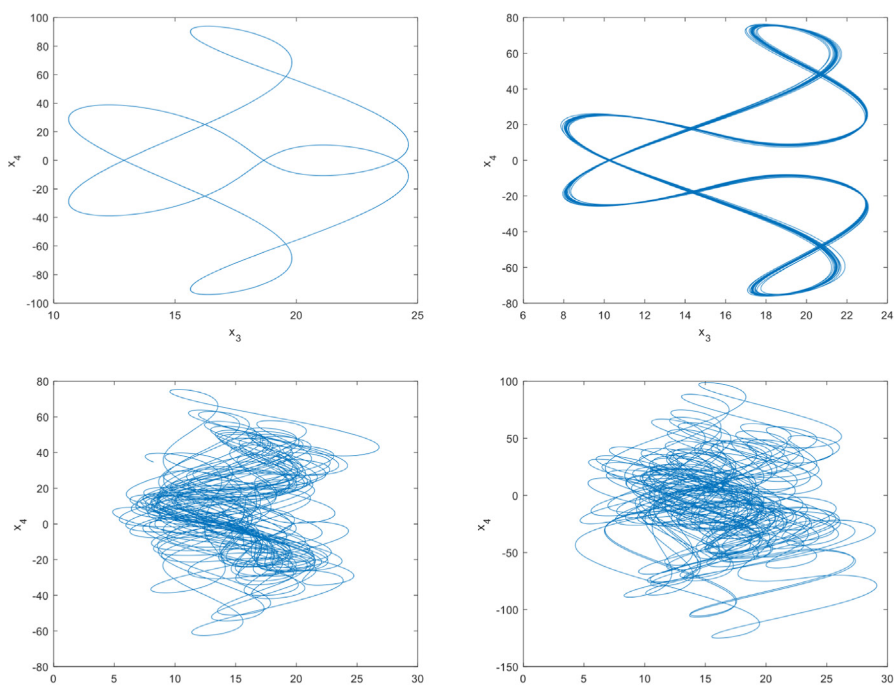

Fig. 5 Phase diagram in x3x4 plane.

of each term in $\alpha(X(t), t)$. Therefore, we state stochastic Bao's system as

$$dX(t) = \alpha(X(t), t)dt + \beta(X(t), t)dW(t) \text{ with } X(t_0) = X_0, t \in [t_0, t_f], \tag{13}$$

where $X(t)$ is defined in (11), $\alpha(X(t), t)$ is a function known as drift coefficient (α is a deterministic and continuous vector valued function) defined in (12), β is a random or uncertain continuous vector function, and

$$\beta(X(t)) = \begin{bmatrix} \epsilon_1 X_1(t) \\ \epsilon_2 X_2(t) \\ \epsilon_3 X_3(t) \\ \epsilon_4 X_4(t) \end{bmatrix}, \tag{14}$$

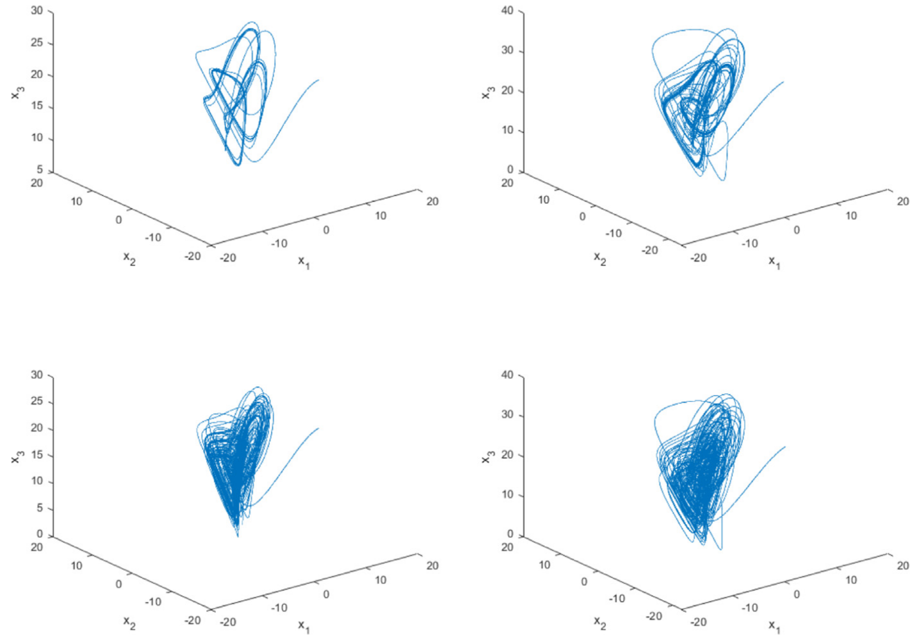

Fig. 6 Phase diagram in x1x2x3 space.

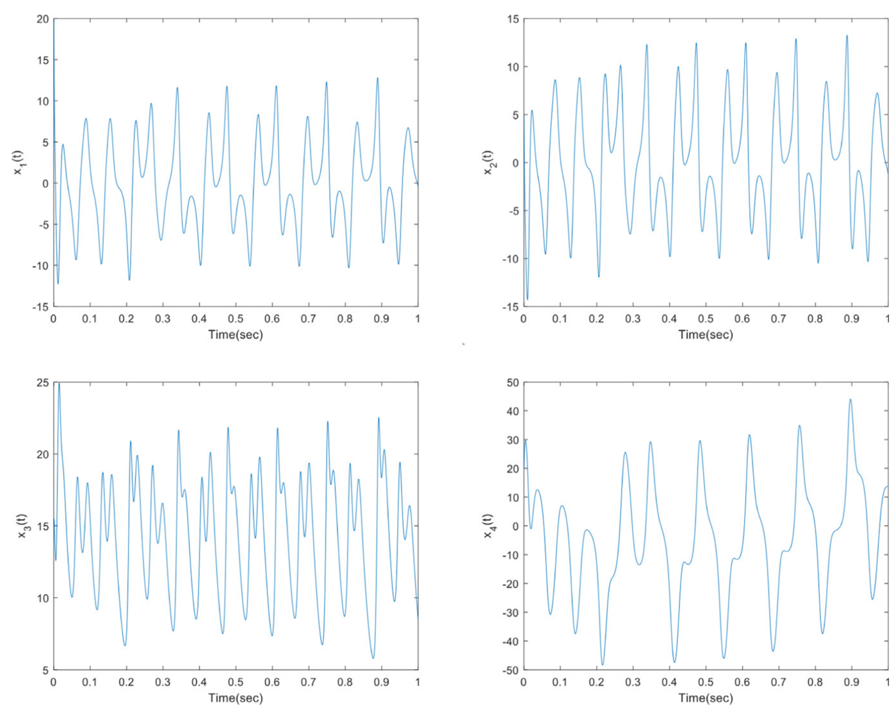

Fig. 7 Time-series representation of the model on different planes.

where the terms, ϵ_i, are arbitrarily small positive real numbers. Before we present stochastic Bao's system, let us give the definition of standard Brownian motion. A Wiener process or standard Brownian motion, eg. [18], is a stochastic process $(S_t)_{t\geqslant 0}$ defined on a probability space $(\mathscr{X}, \mathscr{F}, \mathscr{P})$ with assets:

i. $S_0 = 0$,
ii. the function $t \mapsto S_t$ is a continuous function with probability 1,
iii. the increments $S_{t+n} - S_t$ have a the normal $\mathscr{N}(0, n)$ distribution.

Independent increments mean $S_t - S_l$ and $S_k - S_m$ are independent random variables for $0 \leqslant l \leqslant t \leqslant m \leqslant k$. Detailed information regarding standard Brownian motion might be obtained from [14]. We write stochastic Bao's system explicitly as follows:

$$\begin{aligned} dX_1(t) = & \quad (aX_2(t) - aX_1(t) + X_4(t))dt + \epsilon_1 X_1(t) dW_1(t), \\ dX_2(t) = & \quad (cX_2(t) - X_1(t)X_3(t))dt + \epsilon_2 X_2(t) dW_2(t), \\ dX_3(t) = & \quad (X_1(t)X_2(t) - bX_3(t))dt + \epsilon_3 X_3(t) dW_3(t), \\ dX_4(t) = & \quad (kX_1(t) + dX_2(t)X_3(t))dt + \epsilon_4 X_4(t) dW_4(t), \end{aligned} \tag{15}$$

where t is a positive number and $W_i(t)$ is a standard Brownian motion.

We can write (13) in the integral form as

$$X(t) = X_0 + \int_{t_o}^{t} \alpha(X(s), s)ds + \int_{t_0}^{t} \beta(X(s), s)dW(s). \tag{16}$$

Notice that the integral $\int_{t_o}^{t} \alpha(X(s), s)ds$ is a Riemann integral (deterministic) and $\int_{t_0}^{t} \beta(X(s), s)dW(s)$ is a stochastic integral of Ito type. One can refer to [14] as a detailed study for solution and analysis of stochastic integrals, amongst many other references. Regarding the existence and uniqueness of the solutions of the system (13), we present the following theorem.

Theorem 1. *Assume* $\alpha, \beta : [t_0, t_f] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ *are continuous functions and satisfy the uniform Lipschitz condition:*

$$\begin{aligned} \|\alpha(X(t), t) - \alpha(Y(t), t))\|_{L_2} \leqslant & \quad c_1 \|X(t) - Y(t)\|_{L_2}, \\ \|\beta(X(t), t) - \beta(Y(t), t))\|_{L_2} \leqslant & \quad c_2 \|X(t) - Y(t)\|_{L_2}, \end{aligned}$$

for all $t \in [t_0, t_f]$, *and* c_1, c_2 *are constants and assume that expectation* $E(X_0)$ *is not infinite where* $\|\cdot\|_{L_2}$ *is a vector norm. Then, there exists a unique solution of the system of Eqs. (13). Furthermore, the Linear Growth condition*

$$\|\alpha(X(t), t)\|_{L_2} + \|\beta(X(t), t)\|_{L_2} \leqslant M\Big(1 + \|x(t)\|_{L_2} \tag{17}$$

guarantees existence of solution on the interval $[t_0, t_f]$.

It is obvious that the drift coefficient, $\beta(X(t), t)$ of (13) is a uniform Lipschitz globally continuous function and also satisfies the linear growth condition, 17. Therefore, the system of the equation given in Eq. (13) has a solution. The research work in the reference [12] might be considered as an interesting study on existence-and-uniqueness of solutions of stochastic differential equations and their systems.

Now, we obtain a numerical solution of (13) with an Euler type method that is defined for the system (16) as follows: Consider the time discretization of $[t_0, t_f]$ for $t_0 = 0 < t_1 < \ldots < t_{n+1} = t_f$, being $n+1$ discrete time steps:

$$\begin{aligned} Z_{n+1}^m = & Z_n^m + \alpha(Z_n^m, t_n)(t_{n+1} - t_n) + \beta(Z_n^m, t_n) \\ & \times (W^m(t_{n+1}) - W^m(t_n)), \end{aligned} \tag{18}$$

for $n = 0, 1, 2, \ldots n$. Here, (Z_n) is a sequence converging to X as n approaches to infinity. Next we give a discretization of each equation in (15). Let us consider the first equation,

$$dX_1(t) = (aX_2(t) - aX_1(t) + X_4(t))dt + \epsilon_1 X_1(t) dW_1(t),$$

of system (15). Applying Euler's method to this equation, we get

$$\begin{aligned} X_1(t_{i+1}) = & X_1(t_i) + (aX_2(t_i) - aX_1(t_i) + X_4(t_i))(t_{i+1} - t_i) \\ & + \epsilon_1 X_1(t_i) \sqrt{t_{i+1} - t_i} \rho_1(i), \end{aligned}$$

where $\rho_1(i)$ is a normal standard independent random variables, i.e., $\rho_1(i) \sim \mathcal{N}(0, 1)$. Next, let us consider the second equation,

$$dX_2(t) = (cX_2(t) - X_1(t)X_3(t))dt + \epsilon_2 X_2(t) dW_2(t),$$

of system (15). In a similar way, we discretize this equation as follows:

$$\begin{aligned} X_2(t_{i+1}) = & X_2(t_i) + (cX_2(t_i) - X_1(t_i)X_3(t_i))(t_{i+1} - t_i) + \epsilon_2 X_2(t_i) \\ & \times \sqrt{t_{i+1} - t_i} \rho_2(i), \end{aligned}$$

where $\rho_2(i) \sim \mathcal{N}(0; 1)$. Thirdly, our discretization of the equation,

$$dX_3(t) = (X_1(t)X_2(t) - bX_3(t))dt + \epsilon_3 X_3(t) dW_3(t),$$

is given by

$$\begin{aligned} X_3(t_{i+1}) = & X_3(t_i) + (X_1(t_i)X_2(t_i) - bX_3(t_i))(t_{i+1} - t_i) + \epsilon_3 X_3(t_i) \\ & \times \sqrt{t_{i+1} - t_i} \rho_3(i), \end{aligned}$$

where $\rho_3(i) \in \mathcal{N}(0, 1)$. Finally, the discretization of the equation

$$dX_4(t) = (kX_1(t) + dX_2(t)X_3(t))dt + \epsilon_4 X_4(t) dW_4(t)$$

is presented as

$$\begin{aligned} X_4(t_{i+1}) = & X_4(t_i) + (kX_1(t_i) + dX_2(t_i)X_3(t_i))(t_{i+1} - t_i) \\ & + \epsilon_4 X_4(t_i) \sqrt{t_{i+1} - t_i} \rho_4(i), \end{aligned}$$

where $\rho_3(i) \sim \mathcal{N}(0, 1)$.

6. Parameter estimation in stochastic Bao's system

In this section, we are concerned with estimation or identification of parameters, $p := (a, b, c, d, \epsilon_i), (i = 1, 2, 3, 4)$, in the stochastic Bao's system in (15). Let us represent the system (15) as

$$\dot{X}(t; p) = F(t, X(t; p); p), \text{ for } t_0(p) \leqslant t. \tag{19}$$

Suppose that we have a data set, $\widetilde{X}(t_i)$ which is an approximate value of $X(t_i; p^*)$ obtained from some measurements or observations of the random variables X_i, $i = 1, 2, 3, 4$. We aim to estimate the optimal values, p^*, of the set of parameters by minimizing a cost functional such as

$$C(p) := \sum_i \left[\widetilde{X}(t_i) - X(t_i; p)\right]^2, \tag{20}$$

with the conditions that each parameter $p \geqslant 0$ inside of the domain and $p = 0$ on the boundaries of the domain. One may solve this optimization problem with some well-achieved optimization techniques including sequential quadratic programming [20], Levenberg-Marquardt [21], and a gradient-Hessian based method [22]. In case one needs to use an gradient-Hessian based technique, the gradient and Hessian of this cost functional are respectively given by

$$\frac{\partial C(p)}{\partial p_j} = -2 \sum_i \left[\widetilde{X}(t_i) - X(t_i; p)\right] \frac{\partial X(t_i; p)}{\partial p_j},$$

$$\frac{\partial^2 C(p)}{\partial p_j \partial p_n} = 2 \sum_i \left[\left(\frac{\partial X(t_i; p)}{\partial p_j} \right) \left(\frac{\partial X(t_i; p)}{\partial p_n} \right) - \left[\widetilde{X}(t_i) - X(t_i; p)\right] \frac{\partial^2 X(t_i; p)}{\partial p_j \partial p_n} \right].$$

One needs to employ a sensitivity equations approach [15] to obtain the optimum values of the parameters.

7. Synchronization of fractional-stochastic system

Synchronization of differential equations (e.g. [3,4,27]) is an important and useful technique to understand the chaotic behavior of a system. Adaptive, Master-slave, sliding mode, feedback are some of the well-known synchronization types. [2] studied master-slave synchronization of deterministic Bao'system. In this section, we employ a sliding-mode synchronization technique to the fractional-order system (2). Now, define

$$D_t^\alpha X(t) = CX(t) + g(X(t)) + E(X(t))\dot{W}(t), \tag{21}$$

where $X(t) = [X_1(t), X_2(t), X_3(t), X_4(t)]^T$, $C \in \mathbf{R}^{4\times4}$, $g : \mathbf{R}^4 \mapsto \mathbf{R}^4$ is a nonlinear function, the smooth function $E(X(t))$ represents the intensity of noise and it is assume that $|E(X(t))| \leqslant K$ for some $K > 0, \dot{W}(t) = \left[\dot{W}_1(t), \dot{W}_2(t), \dot{W}_3(t), \dot{W}_4(t)\right]$ is a noise vector with mutually independent noises (i.e. $\dot{W}_i(t)$ and $\dot{W}_j(t)$ are statistically independent for any $i,j = 1,2,3,4$, and $D_t^\alpha X(t)$ is Caputo type fractional time derivative of $X(t)$. In a synchronization analysis, one must determine driving and response systems. Now, suppose that the fractional-order Bao's system (2) is driving system and response system with controller $u(t) = [u_1(t), u_2(t), u_3(t), u_4(t)]^T$ is given by

$$D_t^\alpha Y(t) = C_2 Y(t) + g_2(Y(t)) + u(t), \tag{22}$$

where $Y(t) = [Y_1(t), Y_2(t), Y_3(t), Y_4(t)]^T, C_2 \in \mathbf{R}^{4\times4}, g_2 : \mathbf{R}^4 \mapsto \mathbf{R}^4$. Now, define the error function $f(t) = [f_1(t), f_2(t), f_3(t), f_4(t)]^T = Y(t) - X(t)$. Under these settings, the main purpose of the sliding-mode control is to minimize the error function $\|Y(t) - X(t)\| = \|f(t)\| \mapsto 0$ as $\|t\| \mapsto \infty$.

Then, by the systems (21) and (22), we have

$$\begin{aligned} D_t^\alpha f(t) = &\quad C_2 Y(t) + g_2(Y(t)) + u(t) - CX(t) - g(X(t)) - E(X(t))\dot{W}(t) \\ = &\quad C_2 f(t) + H(X(t), Y(t)) - E(X(t))\dot{W}(t) + u(t), \end{aligned}$$

where

$$H(X(t), Y(t)) = g_2(Y(t)) - g(X(t)) + (C_2 - C)X(t).$$

Under these settings, synchronization of fractional-stochastic Bao's system with Lü system ([6]) may be investigated. In fact, this algorithm may be applied to some other fractional-stochastic differential equations, eg. [19], as well.

8. Conclusions and outlook

Many problems arising in physics, cyber security, finance, actuarial sciences, pension fund systems, game theory, neuroscience, information sciences, marketing and internal marketing are modeled by means of nonlinear systems of ordinary differential equations. In the theory of differential equations, there is not a standard or general method which can be applied to the solutions of systems of equations efficiently. The techniques applied to approximate solutions of Bao's system in this paper may serve as a general framework to the solutions of the systems appearing in many different disciplines. In this paper, we were interested in chaos analysis, parameter estimation and numerical solutions of fractional-stochastic Bao's systems. We solved the Bao's system with both Crank-Nicolson and Power series methods in the deterministic case. By adding standard Brownian motion to each term in the Bao's system, we express the Bao's system as a system of stochastic differential equations. We solve the resulting stochastic system with an Euler type numerical method. To the best of our knowledge, Bao's system has not been numerically solved with the methods employed in this paper previously and this paper considers the fractional-order and stochastic Bao's systems first time in the literature. Techniques applied to the solutions of deterministic and stochastic Bao's systems might be applied to some other nonlinear models appearing in engineering and science as well. In a future extension of this work, we will investigate the applicability of techniques employed in this paper to the systems appearing in quantum theory, physics and all disciplines of the modern engineering.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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