In exercises 13 through 21, show that if y_1 is a solution of the differential equation, use the formula (5) to find an expression for a second linearly independent solution.

14) $xy'' + y' = 0$; $y_1 = \ln x$ and $x > 0$.

14) $y_2 = y_1 \cdot \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$ (Formula)

$\left(P(x) = \frac{y' \text{ معامل}}{y'' \text{ معامل}}\right)$

$P(x) = \frac{1}{x}$

$y_2 = \ln x \cdot \int \frac{e^{-\int \frac{1}{x}dx}}{(\ln x)^2}\,dx$

$= \ln(x) \cdot \int \frac{e^{-\ln x}}{(\ln x)^2}\,dx$

$= \ln x \cdot \int \frac{1}{x(\ln x)^2}\,dx$

$= \ln x \cdot -\frac{1}{\ln x}$

$= -1$

ملاحظة :

$\int \frac{1}{x}(\ln x)^{-2}\,dx$

$= \frac{(\ln x)^{-1}}{-1}$

$\int \frac{\frac{1}{x}}{(\ln x)^2}\,dx$

General solution is

$y = c_1 y_1 + c_2 y_2$

$= c_1 \cdot \ln x + c_2(-1)$

In exercises 23 through 26, use the reduction of order method to find the general solution of the differential equation.

23) $y'' + y = \sec x$; $0 < x < \frac{\pi}{2}$, where $y_1 = \sin x$ is a particular solution of

$$y'' + y = 0.$$

23) Assume $y = y_1 u$ is a solution of non-homog eq

$y = \sin x\, u$???

$y' = \sin x \cdot u' + \cos x \cdot u$

$y'' = \sin x \cdot u'' + \cos x \cdot u' + \cos x \cdot u' - \sin x \cdot u$

$= u'' \sin x + 2u' \cos x - \sin x \cdot u$

$u'' \sin x + 2u' \cos x - \cancel{u \sin x} + \cancel{u \sin x} = \sec(x)$

$u'' \sin x + 2u' \cos x = \sec(x)$

Assume $w = u' \longrightarrow w' = u''$

$w' \sin x + 2w \cos x = \sec(x)$ between w & x
(Linear) (1st order DE)

$P(x) = 2\cos x / \sin x$

$\mu = e^{\int P(x)dx} = e^{\int \frac{2\cos x}{\sin x}dx} = e^{2\ln(\sin x)} = \sin^2 x$

$\frac{d}{dx}(\mu \cdot w) = \mu \cdot \sec(x)$

$\mu \cdot w = \int \sin^2 x \cdot \sec x \, dx + c_1$

$\sin^2 x \cdot w = \int \sin^2 x \cdot \frac{1}{\cos x} dx + c_1$

$\sin^2 x \cdot w = \int \frac{1 - \cos^2 x}{\cos x} dx + c_1$

$\sin^2 x \cdot w = \int (\sec x - \cos x) dx + c_1$

$\sin^2 x \cdot w = \left[\ln|\sec x + \tan x| - \sin x\right] + c_1$

$u' = w = \frac{\ln|\sec x + \tan x|}{\sin^2 x} - \csc(x) + \frac{c_1}{\sin^2 x}$

$u = \int \frac{\ln|\sec x + \tan x|}{\sin^2 x} dx - \int \csc(x) dx + c_1 \int \csc^2(x) dx$

$u = -\cot x \ln|\sec x + \tan x| + / - / \quad - c_1 \cot x + c_2$

General solution is

$y = \sin x\left[-\cot x \ln|\sec x + \tan x| - c_1 \cot x + c_2\right]$

$= -\cos x \ln|\sec x + \tan x| - c_1 \cos x + c_2 \sin x$