APPLICATIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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1. Orthogonal Trajectories

Suppose that we have a family of two curves given by

$$f(x, y, c) = 0, \quad and \quad g(x, y, k) = 0.$$

These two curves are said to be **orthogonal** at a point if and only is their tangent lines T_1 and T_2 are perpendicular at the point of intersection.

Definition 1. *orthogonal trajectories are two families of curves that always intersect perpendicularly.*

In other words, an orthogonal trajectory is any one curve that intersects every curve of another family at right angles.

Now we will show a method to how to find the family of orthogonal trajectories

2. Method

Given a family of curve $f(x, y, c) = 0$

- **STEP 1** Find the differential equation for the given family of curves, by differentiating $f(x, y, c) = 0$.
- **STEP 2** Find the differential equation of the Orthogonal Trajectories

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} \quad or \quad \frac{dx}{dy} = -f(x, y)$$

- **STEP 3** Separating variables and integrating the above differential equation, we get the algebraic equation of the family of orthogonal trajectories.

Example 1. *Find the orthogonal trajectories of all parabolas with vertices at the origin and foci on the x-axis*

$$y^2 = 4ax, \tag{1}$$

Solution *We can rewrite the equation (1) as follows*

$$\frac{y^2}{x} = 4a, \tag{2}$$

- *Differentiate equation* (2)*:*

$$\frac{x \times 2y\frac{dy}{dx} - y^2 \times (1)}{x^2} = 0$$

$$2xy\frac{dy}{dx} = y^2$$

$$\frac{dy}{dx} = \frac{y}{2x}$$

- *Differential equation of the orthogonal trajectories*

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} \Longrightarrow \frac{dy}{dx} = -\frac{2x}{y}$$

- *Solving the differential equation by method of separation of variables*

$$\frac{dy}{dx} = -\frac{2x}{y}$$

$$-ydy = 2xdx$$

$$2xdx + ydy = 0$$

$$\int 2xdx + \int ydy = \int 0$$

$$x^2 + \frac{1}{2}y^2 = C, \quad C \text{ is an integration constant.}$$

$$2x^2 + y^2 = C_1, \quad \text{where } C_1 = 2C.$$

Hence the orthogonal trajectories are

$$2x^2 + y^2 = b^2, \quad \textit{family of ellipse.}$$

Example 2. *Find the orthogonal trajectories of*

$$y(x^2 + c) + 2 = 0, \tag{3}$$

Solution

- *Differentiate equation* (3)*:*

$$\frac{dy}{dx}(x^2 + c) + y(2x) = 0$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + c}, \tag{4}$$

To find c: From equation (3) *we have* $c = \frac{-2-x^2y}{y}$. *Substitute in* (4) *we get:*

$$\frac{dy}{dx} = \frac{-2xy}{x^2 + \frac{-2-x^2y}{y}},$$

$$\frac{dy}{dx} = \frac{-2xy^2}{x^2y + (-2 - x^2y)},$$
$$\frac{dy}{dx} = \frac{-2xy^2}{-2},$$
$$\frac{dy}{dx} = xy^2.$$

- *Differential equation of the orthogonal trajectories*

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} \Longrightarrow \frac{dy}{dx} = -\frac{1}{xy^2}$$

- *Solving the differential equation by method of separation of variables*

$$\frac{dy}{dx} = -\frac{1}{xy^2}$$
$$y^2 dy = -\frac{1}{x}dx$$
$$\int y^2 dy = \int -\frac{1}{x}dx$$
$$\frac{1}{3}y^3 = -\ln|x| + C, \quad C \textit{ is an integration constant.}$$
$$y^3 = -3\ln|x| + 3C,$$

Hence the orthogonal trajectories are

$$y^3 = -3\ln|x| + C_1, \quad \textit{where } C_1 = 3C.$$

Example 3. *Find the orthogonal trajectories of*

$$x^2 - y^2 = cx, \tag{5}$$

Solution *Find c:*

$$c = \frac{x^2 - y^2}{x}$$

- *Differentiate equation (5):*

$$2x - 2y\frac{dy}{dx} = c$$
$$2x - 2y\frac{dy}{dx} = \frac{x^2 - y^2}{x},$$
$$x - y\frac{dy}{dx} = \frac{x^2 - y^2}{2x},$$

$$-y\frac{dy}{dx} = \frac{x^2 - y^2}{2x} - x,$$
$$-y\frac{dy}{dx} = \frac{x^2 - y^2 - 2x^2}{2x},$$
$$-y\frac{dy}{dx} = \frac{-y^2 - x^2}{2x},$$
$$\frac{dy}{dx} = \frac{y^2 + x^2}{2xy},$$

- *Differential equation of the orthogonal trajectories*

$$\frac{dx}{dy} = -f(x,y) \Longrightarrow \frac{dx}{dy} = -\frac{x^2 + y^2}{2xy}$$
$$\frac{dx}{dy} = -\frac{1}{2}\Big(\frac{x}{y} + \frac{y}{x}\Big)$$

We know that $\frac{dx}{dy} = -\frac{1}{2}\Big(\frac{x}{y} + \frac{y}{x}\Big)$ *is homogeneous equation because* $f(x,y) = -\frac{1}{2}\Big(\frac{x}{y} + \frac{y}{x}\Big)$ *is homogeneous function.*

$$f(tx,ty) = -\frac{1}{2}\Big(\frac{tx}{ty} + \frac{ty}{tx}\Big) = t^0 f(x,y)$$

- *Solving the differential equation by method of homogeneous equation.*

Let

$$v = \frac{x}{y} \to x = vy$$
$$\longrightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

So,

$$v + y\frac{dv}{dy} = -\frac{1}{2}\Big(v + \frac{1}{v}\Big)$$
$$-2v - 2y\frac{dv}{dy} = v + \frac{1}{v}$$
$$-2y\frac{dv}{dy} = v + \frac{1}{v} + 2v$$
$$-2y\frac{dv}{dy} = 3v + \frac{1}{v}$$
$$-2y\frac{dv}{dy} = \frac{3v^2 + 1}{v}$$
$$-2yv\,dv = (3v^2 + 1)dy$$

$$\frac{-2v}{3v^2+1}dv = \frac{1}{y}dy$$
$$-2\int \frac{v}{3v^2+1}dv = \int \frac{1}{y}dy$$
$$\frac{-2}{6}\int \frac{6v}{3v^2+1}dv = \int \frac{1}{y}dy$$
$$\frac{-1}{3}\ln|3v^2+1| = \ln|y| + c,$$
$$\ln|3v^2+1| = -3\ln|y| - 3c,$$
$$\ln|3v^2+1| = \ln|y|^{-3} - 3c,$$
$$|3v^2+1| = e^{-3c}|y|^{-3},$$
$$3v^2+1 = \pm e^{-3c}y^{-3},$$
$$3v^2+1 = Ky^{-3}, \textit{ where } K = \pm e^{-3c},$$
$$3\frac{x^2}{y^2} + 1 = K\frac{1}{y^3}$$
$$3x^2y + y^3 = K$$

Hence the orthogonal trajectories are

$$y(3x^2+y^2) = K.$$

3. EXERCISES

Find the orthogonal trajectories of the following equations:

(1) $y(x^2+1) = cx$,
(2) $y^2(2x^2+y^2) = c^2$,
(3) $y = c\sin x$,
(4) $y^2 = x^2(1-cx)$.