Chapter 4:
(0,1) Random Number Generation

Refer to Text Book:

- "Simulation Modeling and ARENA", Manuel Rossetti, Ch. 2
- "Operations Research: Applications and Algorithms" By Wayne L. Winston ,Ch. 21
- "Operations Research: An Introduction" By Hamdi Taha, Ch. 16

LEARNING OBJECTIVES

- To be able to describe and use linear congruential pseudorandom number generation methods
- To be able to define and use key terms in pseudorandom number generation methods such as streams, seeds, and period.
- To be able to explain the key issues in pseudorandom number testing.

Review Last Lecture

- **Steps for Simulation Modeling**

  How to conduct a complete simulation modeling analysis?

- **Applications on Simulation Modeling**

  The Manufacturing model:

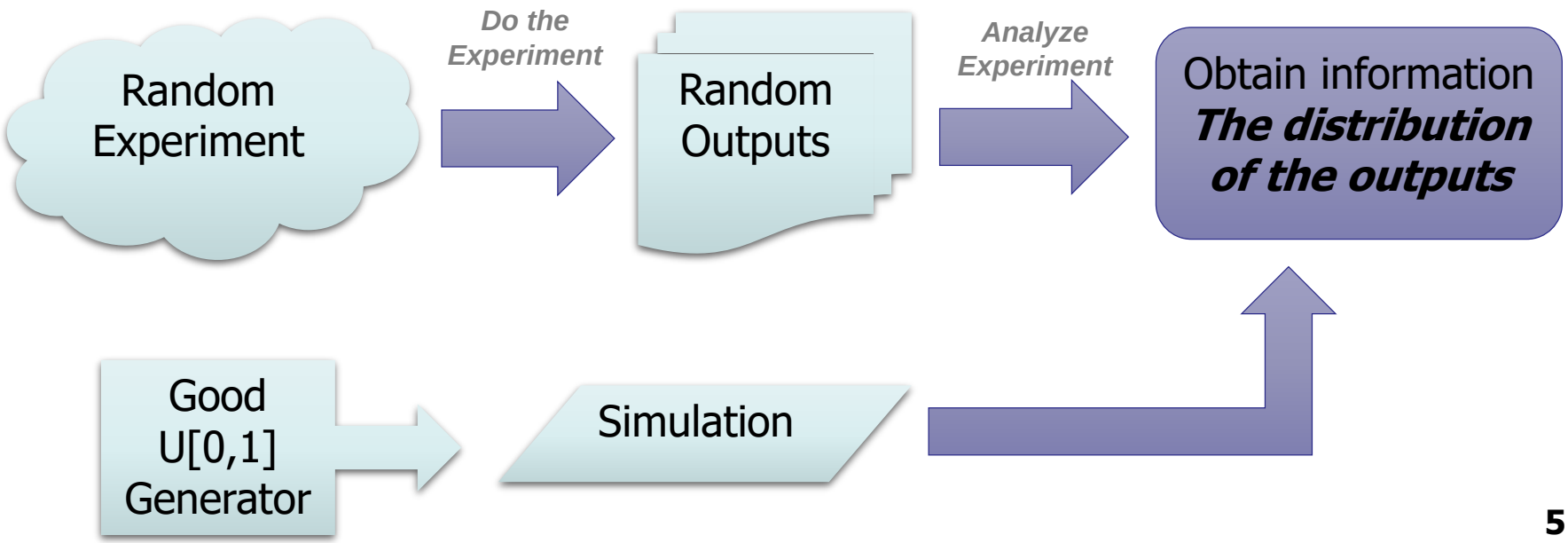
  - how to formulate the problem statement.
  - how to define goals of the simulation
  -how to determine the missing information
  - what are the data needed.

Today's Lecture Plan

- Idea of Random Number Generators
- Pseudo-Random Numbers
- Linear congruential generator (LCG)
  - Definitions
  - Conditions for LCG Full Cycle
  - Examples
- Random Streams

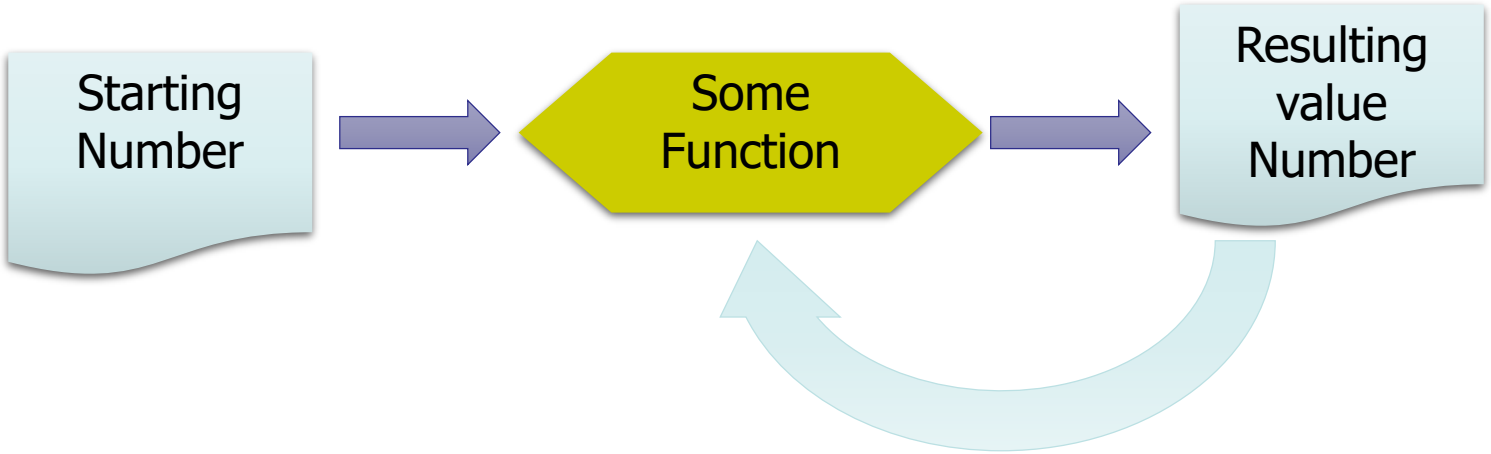
Random Number Generation

- Generating any random number from any distribution depends on U[0,1].



Pseudo-Random Numbers

- Random means Not Known for certain
- The random numbers used in a simulation are not really random!



- You can get all the numbers in advance

Pseudo-Random Numbers

- ***pseudo-random numbers***

Definition:

A sequence of ***pseudo-random*** numbers, U(i), is a <u>deterministic</u> sequence of numbers in [0,1] having the same relevant statistical properties as a sequence of truly random U(0,1) numbers.
(Ripley 1987)

Pseudo-Random Numbers

- ***linear congruential generator (LCG)***

- a recursive algorithm for producing a sequence of pseudorandom numbers.
- Each new pseudorandom number from the algorithm depends on the previous pseudorandom number
- There must be a starting value called the ***seed***

Linear congruential generator (LCG)

Definition 2.2 (LCG) *An LCG defines a sequence of integers,* $R_0, R_1, \ldots$ *between 0 and* $m-1$ *according to the following recursive relationship, where* $i = 0, 1, 2, \ldots$*:*

$$R_{i+1} = (aR_i + c) \bmod m \quad \text{for } i = 0, 1, 2, \ldots. \tag{2.1}$$

where R_0 *is called the seed of the sequence,* a *is called the constant multiplier,* c *is called the increment, and* m *is called the modulus.* (m, a, c, R_0) *are integers with* $a > 0$, $c \geq 0, m > a, m > c, m > R_0$, *and* $0 \leq R_i \leq m-1$.

To compute the corresponding pseudorandom uniform number, we use

$$U_i = \frac{R_i}{m} \tag{2.2}$$

Linear congruential generator (LCG)

- choice of the parameters of the LCG : seed, constant multiplier, increment, and modulus, that is, the, will determine the properties of the sequences

- properly chosen parameters, an LCG can be made to produce pseudorandom numbers look like real random.

Linear congruential generator (LCG)

EXAMPLE

Consider an LCG with parameters (m = 8, a = 5, c = 1, R0 = 5). Compute the first nine values for R_i and U_i from the defined sequence.

- how to compute using the mod operator. The mod operator is defined as

$$z = y \bmod m$$
$$= y - m\left\lfloor \frac{y}{m} \right\rfloor$$

where $\lfloor \cdot \rfloor$ is the floor operator,

Linear congruential generator (LCG)

EXAMPLE

$$z = y \bmod m$$
$$= y - m\left\lfloor \frac{y}{m} \right\rfloor$$

$$z = 17 \bmod 3$$
$$= 17 - 3\left\lfloor \frac{17}{3} \right\rfloor$$
$$= 17 - \lfloor 5.\overline{66} \rfloor$$
$$= 17 - 3 \times 5 = 2$$

In our example

m = 8

a = 5

c = 1

$R_0 = 5$

$$R_1 = (5R_0 + 1) \bmod 8 =$$
$$R_2 = (5R_1 + 1) \bmod 8 =$$
$$R_3 = (5R_2 + 1) \bmod 8 =$$
$$R_4 = (5R_3 + 1) \bmod 8 =$$

Linear congruential generator (LCG)

EXAMPLE

$$R_1 = (5R_0 + 1) \bmod 8 = 26 \bmod 8 = 2 \Rightarrow U_1 = 0.25$$

$$R_2 = (5R_1 + 1) \bmod 8 = 11 \bmod 8 = 3 \Rightarrow U_2 = 0.375$$

$$R_3 = (5R_2 + 1) \bmod 8 = 16 \bmod 8 = 0 \Rightarrow U_3 = 0.0$$

$$R_4 = (5R_3 + 1) \bmod 8 = 1 \bmod 8 = 1 \Rightarrow U_4 = 0.125$$

$$R_5 = 6 \Rightarrow U_5 = 0.75$$

$$R_6 = 7 \Rightarrow U_6 = 0.875$$

$$R_7 = 4 \Rightarrow U_7 = 0.5$$

$$R_8 = 5 \Rightarrow U_8 = 0.625$$

$$R_9 = 2 \Rightarrow U_9 = 0.25$$

Linear congruential generator (LCG)

Notes to conceder on LCG:

- the $\boldsymbol{U_i}$ are simple fractions involving m = 8.
- Certainly, this sequence does not appear very random. (pseudo-random) … Why?
- The $\boldsymbol{U_i}$ can only take one of the rational values:

$$0, \frac{1}{m}, \frac{2}{m}, \frac{3}{m}, \ldots, \frac{(m-1)}{m} \text{ since } 0 \leq R_i \leq m-1$$

- if m is small, there will be big gaps on the interval [0, 1)
- if m is large, then the $\boldsymbol{U_i}$ will be distributed on [0, 1).

Linear congruential generator (LCG)

Cycle of LCG:

- **Definition:** a sequence generates the same value as a previously generated value, then the sequence ***cycle***.
- **Definition:** The length of the cycle is called the ***period*** of the LCG.
- **Definition:** the LCG is said to achieve its full period if the cycle length is equals to **m**.
- LCG has a long cycle for good choices of parameters **a, m, c.**
- Most computers (32-bit) has value for

$$m = 2^{31} - 1 = 2{,}147{,}483{,}647$$

represents the largest integer number.

Linear congruential generator (LCG)

Theorem: (LCG Full Period Conditions)

An LCG has full period if and only if the following three conditions hold:

1. The only positive integer that (exactly) divides both m and c is 1 (i.e., $\boldsymbol{c}$ and $\boldsymbol{m}$ have no common factors other than 1).
2. If $\boldsymbol{q}$ is a prime number that divides $\boldsymbol{m}$ then $\boldsymbol{q}$ should divide $(\boldsymbol{a} - 1)$ (i.e., $(\boldsymbol{a} - 1)$ is a multiple of every prime number that divides m).
3. If 4 divides $\boldsymbol{m}$, then 4 should divide $(\boldsymbol{a} - 1)$

(i.e., $(\boldsymbol{a} - 1)$ is a multiple of 4 if m is a multiple of 4).

Linear congruential generator (LCG)

Example : (LCG Full Period Conditions)

To apply the theorem, you must check if each of the three conditions holds for the generator.

$$\mathbf{m = 8 \ , a = 5 \ , c = 1}$$

Cond-1. c and m have no common factors other than 1: factors of $m = 8$ are (1, 2, 4, 8), since c = 1 (with factor 1) condition 1 is true.

Cond-2. (a − 1) is a multiple of every prime number that divides m: The first few prime numbers are (1, 2, 3, 5, 7).

Linear congruential generator (LCG)

Example : (LCG Full Period Conditions)

To apply the theorem, you must check if each of the three conditions holds for the generator.

$$\mathbf{m = 8 \ , a = 5 \ , c = 1}$$

Cond. 2: (a − 1) is a multiple of every prime number that divides m.

The prime numbers, q, that divide $m = 8$ are ($q = 1, 2$). Since $a = 5$ and $(a - 1) = 4$, clearly $q = 1$ divides 4 and $q = 2$ divides 4. Thus, condition 2 is true.

Cond. 3: If 4 divides m, then 4 should divide (a − 1).

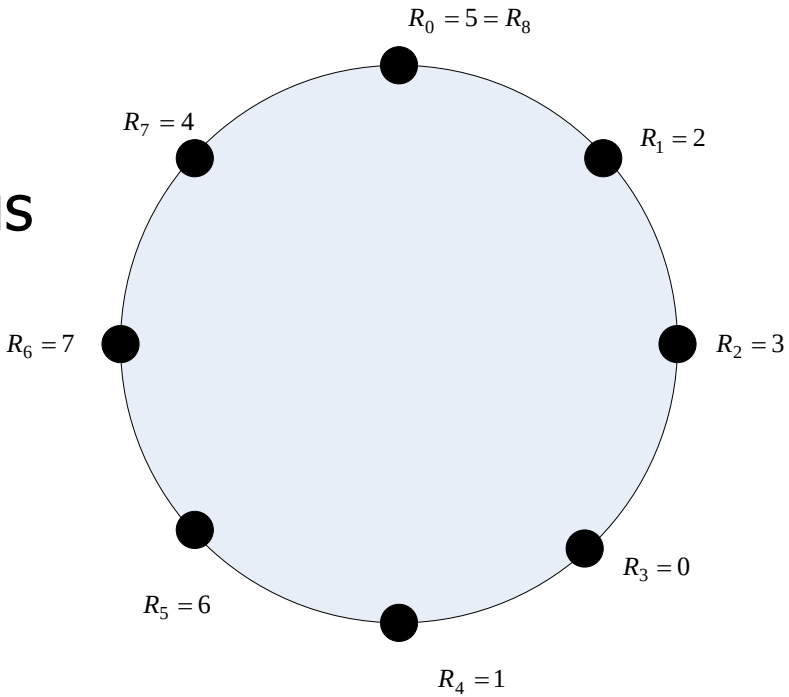
Since m = 8, clearly 4 divides m. Also, 4 divides (a − 1) = 4. Thus, condition 3 holds.

Linear congruential generator (LCG)

Random Stream

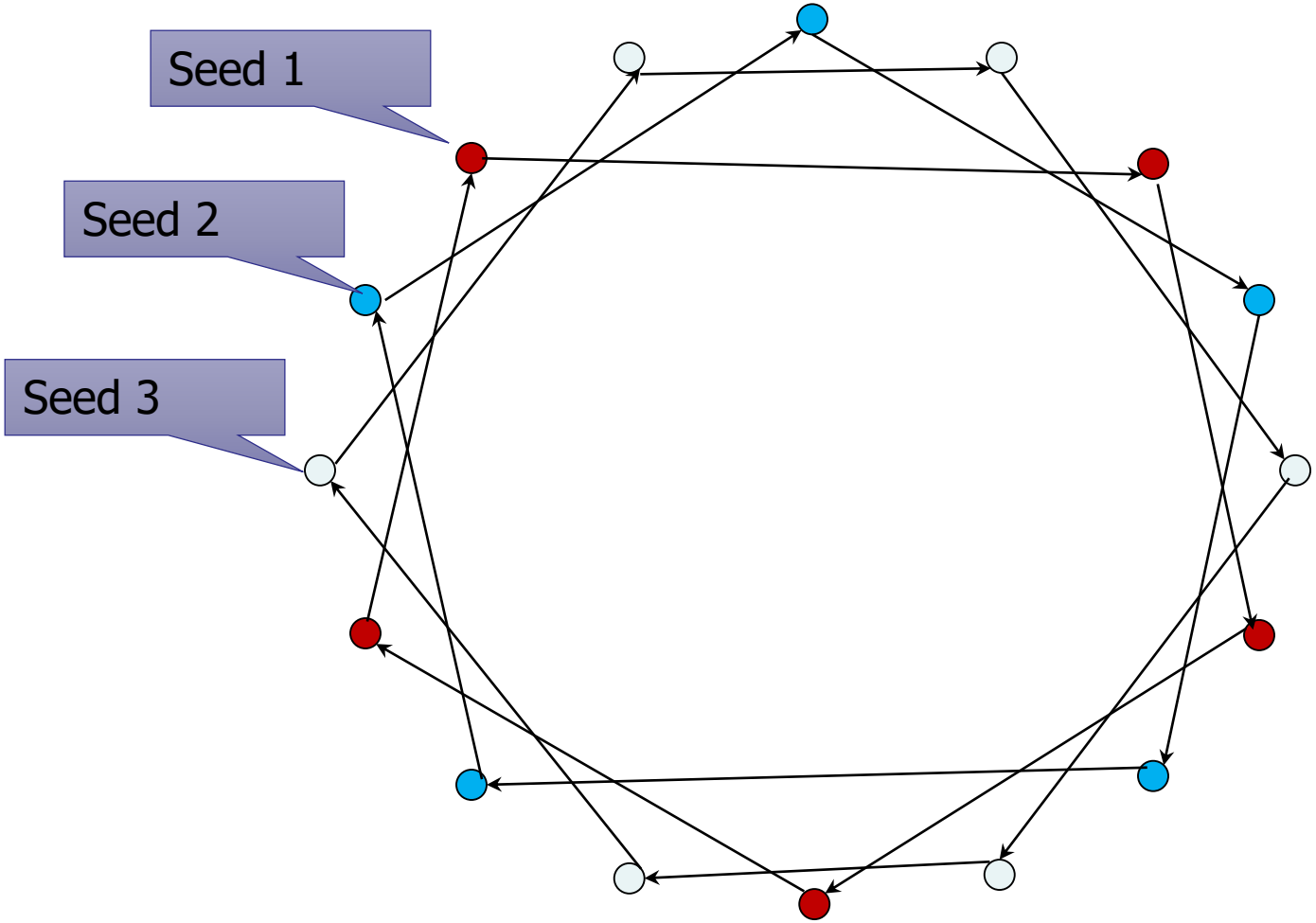
Definition (Random Number Stream): The subsequence of random numbers generated from a given seed is called a random number stream.

- A seed, e.g. $R_1 = 2$, defines a starting place in the cycle and thus a sequence.
- Small period easy to remember the random number streams
- With large m hard to remember the stream.



Linear congruential generator (LCG)

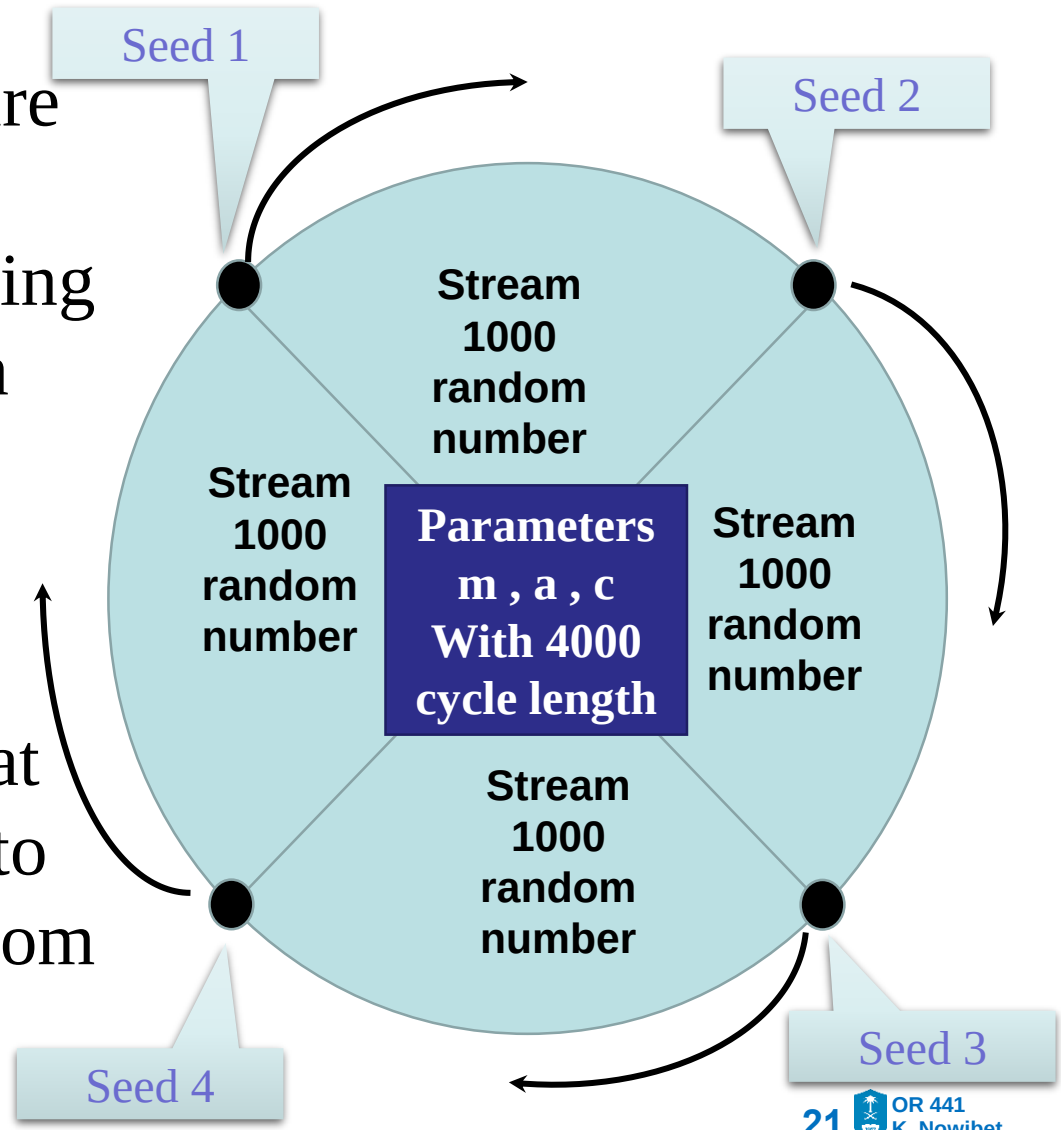
Random Stream



Linear congruential generator (LCG)

Random Stream

- choose to divide the entire sequence so that the number of non-overlapping random numbers in each stream is quite large
- computer simulation languages come with a default set of streams that divide the "circle" up into independent sets of random numbers



Linear congruential generator (LCG)

Random Stream

- The streams are only independent if you do not use up all the random numbers within the subsequence.
- To insure independence in the simulation, you can associate a specific stream with specific random processes in the model. For example:

1. Customers arrival process: stream 1.
2. Service time : stream 2.
3. Demand type: stream 3.

Linear congruential generator (LCG)

Exercise

Analyze the following LCG:

$$X(i) = (11\ X(i-1) + 5)(\mathrm{mod}(16)),\ X0 = 1$$

- What is the maximum possible period length for this generator?
- Does this generator achieve the maximum possible period length? Justify your answer.