financial Derivatives

<u>ch2</u> : BOPM

Notes.

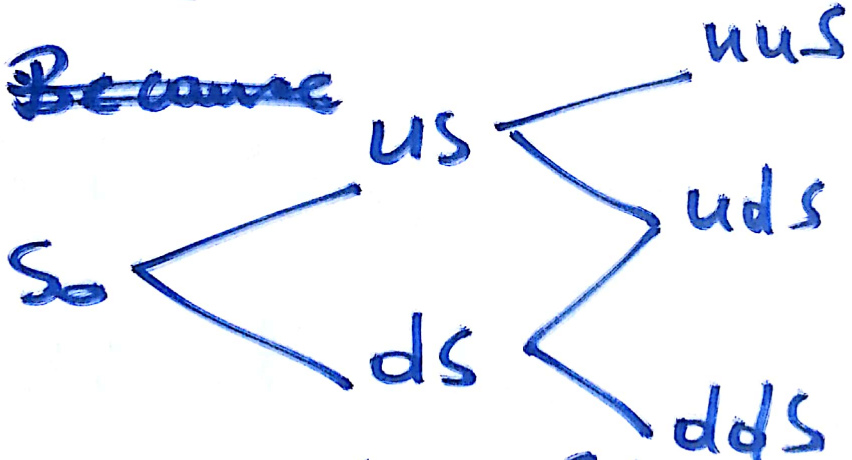
Lecture 6.

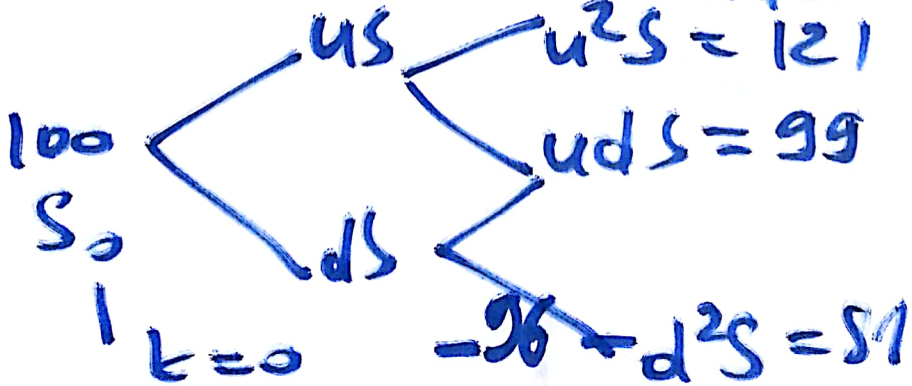
<u>Nov, 22nd, 2017</u>

The 2 periods BOPM

Extending the BOPM to 2 periods where $u = 1.1$, $D = 0.9$. give the Stock price outcome and the possible values of a Long call at expiration.

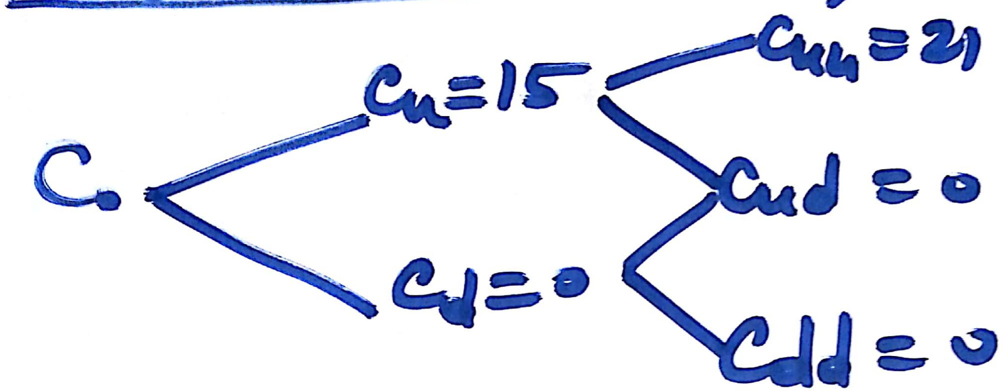
~~Because~~





$t = 0$

Long One Call: $(K=100)$



$$C_{uu}=\mathrm{Max}\left[0,u^2S-d\right]=$$

$$=\mathrm{Max}\left[0,121-100\right]=21$$

$$C_{ud}=\mathrm{Max}\left[0,udS-100\right]=0$$

$$C_{dd}=\mathrm{Max}\left[0,d^2S-100\right]$$

$$=\mathrm{Max}\left[0,81-100\right]=0$$

Because we know that we can create a risk free hedge portfolio at any node of the Binominal Tree, we can calculate C using Bakward induction from the known values of Cuu, Cud and Cdd in our binominal formula.

Here, we are using RNV.

For example, considering the 2 upper branches in Fig. we have

$$C_u = \frac{1}{R}\left[q\,C_{uu} + (1-q)\,C_{ud}\right]$$

$$= \frac{1}{1.05}\left[0.75 \times 21 + 0.25 \times 0\right]$$

$$= \underline{15}$$

From the 2 lower Branches we Obtain:

$$C_d = \frac{1}{R}\left[q\,C_{ud} + (1-q)\,C_{dd}\right]$$

$$C_d = \frac{1}{1.05}\left[0.75 \times 0 + 0.25 \cdot 0\right]$$

$$= 0$$

So, $C_d = 0$ and $C_u = 0$

We can now Solve for C the premium for this 2 periods option problem:

$$C = \frac{1}{R}\left[q C_u + (1-q) C_d\right]$$

$$= \frac{1}{1.05}\left[0.75 \times 15 + 0.25 \times 0\right]$$

$$\boxed{C_0 = 10.71\$}$$, So, we have

$$\begin{tikzcd}[column sep=small]
 & & C_{uu}=21 \\
 & C_u=15 \arrow[ur, dash] \arrow[dr, dash] & \\
C_0=10.71 \arrow[ur, dash] \arrow[dr, dash] & & C_{ud}=0 \\
 & C_d=0 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & C_{dd}=0
\end{tikzcd}$$

Rq:

The call premium for the option with 2 periods to maturity has a higher value than our identical option with one-period to maturity. where we found $C_0 = 7.14$ \$.

Many periods:

If we use the Eq of C_u and C_d and we substitute them in the final Eq of the call (C_0), we obtain

$$C_0 = \frac{1}{R^2}\left[1q^2 C_{uu} + 2q(1-q)C_{ud} + 1(1-q)^2 C_{dd}\right]$$

The Option Price is equal to the Expected value using RNP of the option payoffs at expiry, discounted at the RFR

For $n=3$

$$C_0 = \frac{1}{R^3}\left[q^3 C_{uuu} + 3q^2(1-q)C_{uud} + 3q(1-q)^2 C_{udd} + \right.$$

$(1-q)^3 C_{dddd}\Big]$.

* ~~For n~~ <u>we use</u>:

$n=$ 1.

$n=$ 1 2 1 —— $n=2$

$n=3$ 1 3 3 1 —— $n=3$

1 4 6 4 1 —— $n=4$

1 5 10 10 5 1 —— $n=5$

1 6 15 20 15 6 1 —— $n=6$

<u>For $n=4$</u>:

$$C_0 = \frac{1}{R^4}\Big[q^4 C_{uuuu} + 4q(1-q)^3 C_{uuud}$$
$$+ 6q^2(1-q)^2 C_{uudd} +$$
$$4q(1-q)^3 C_{uddd} +$$

$$+ (1-q)^4 C_{dddd} \Big]$$

With:

$$C_{uuuu} = Max\left[0, u^4 S - K\right]$$

$$C_{uuud} = Max\left[0, u^3 d S - K\right]$$

$$C_{uudd} = Max\left[0, u^2 d^2 S - K\right]$$

$$C_{uddd} = Max\left[0, u d^3 S - K\right]$$

Where U and D come From?

At $t=0$, K and S are known

$$\boxed{u = e^{\sigma\sqrt{\Delta t}}}$$

$$D = e^{-\sigma\sqrt{\Delta t}}$$

σ is the observed standard deviation of the stock return.

T is time to expiration in years (or fraction of years).

n is the number of steps of BOPM.

$$\Delta T = T/n.$$

Example:

$T = 3$ months $= \frac{1}{4}$ year.

$n = 30$

$$\Delta t = \frac{T}{n} = \frac{0,25}{30} = 0.00833$$

of 1 year.

if $\sigma = 0,015$ (annual)

So, $u = e^{0.015\sqrt{0.00833}}$

$$d = e^{-0.015\sqrt{0.00833}}$$

$$u = e^{0,015 \times 0,0912}$$

$$= e^{0.001369} = \boxed{1.001 = u}$$

$$\boxed{d = 0,001}$$

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— Exercices —

Q1: $S_0 = 50$, $K = 45$, RFR $= 5\%$.
$u = 1.2$, $D = 0.8$, $n = 1$

calculate the Call Premium:

a- using the Binomial Model Formula.

b- By Constructing the Binomial Tree (Step by Step).

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a.

$S_0 = 50$ → $uS = 60$, $dS = 40$

$= ?$ → $C_u = \text{Max}[0, uS - K]$, $C_d = \text{Max}[0, dS - K]$

$C_0 = ?$

$$C_0 \begin{cases} C_u = \text{Max}[0, 60-45] = 15 \\ C_d = \text{Max}[0, 40-45] = 0 \end{cases}$$

$$C_0 = \frac{1}{1.05}\left[q\,C_u + (1-q)\,C_d\right]$$

$$q = \frac{R-D}{U-d} = \frac{1.05-0.8}{1.2-0.8} = \underline{0.625}$$

$$1-q = 1-0.625 = 0.375$$

$$C_0 = \frac{1}{1.05}\left[0.625 \times 15 + 0.375 \times 0\right]$$

$$= \frac{0.625 \times 15}{1.05} = 8.92\,\$$$

So,

$$C_0 = 892\$ \begin{cases} 15\$ \\ 0\$. \end{cases}$$

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b - <u>Using the BOPM Steps :</u>

We construct of risk neutral portfolio :

$$\begin{cases} \text{Short call} \\ \text{Long on } H \text{ stock} . \end{cases}$$

$$\underset{t=0}{HS_0 - C_0} \begin{cases} HuS - Cu \\ HdS - Cd \end{cases} \quad t = 1$$

Hedge Ratio :

$H = ?$,

$HuS - C_u = HdS - C_d$

$$\Rightarrow \boxed{H = \frac{C_u - C_d}{S(u-d)}}$$

$$H = \frac{15.0}{50(1.2-0.8)} = \frac{15}{50 \times 0.4}$$

$$= \frac{15}{20} = 7.5$$

$\boxed{h = 7.5}$

we have to buy 7.5 Stocks in our portfolio, to have a risk neutral Portfolio.

The discounted ~~to~~ values of the final portfolio payoffs: are equal to the cost of the hedged Portfolio.

$$\frac{HuS - Cu}{1 + RFR} = \frac{HdS - Cd}{1 + RFR}$$

$$= HS_0 - C_0$$

From this equation, we find C_0:

$$C_0 = \frac{1}{1+RFR}\left[\cancel{Cuu}\ q \cdot Cuu + (1-q)\, Cdd\right]$$

$$C_0 = \frac{1}{1.05}\left[0.625 \times 15 + 0\right]$$

$$= 8.92\ \$.$$

Q2: calculate the put premium ~~for~~ having the same ~~st~~ K, S_0, T,

we use the call-put parity:

$$C - P = S - Ke^{-rT}$$

$$P = C - S + Ke^{-rT}$$

$T = 1,\ K = 45,\ S = 50,\ C_0 = 8.92$

$$P = 8.92 - 50 + 45e^{-0.05 \times 1}$$

$$= 8.92 - 50 + 45 \times 0.9512$$

$$\boxed{P_0 = 1.72\$}$$

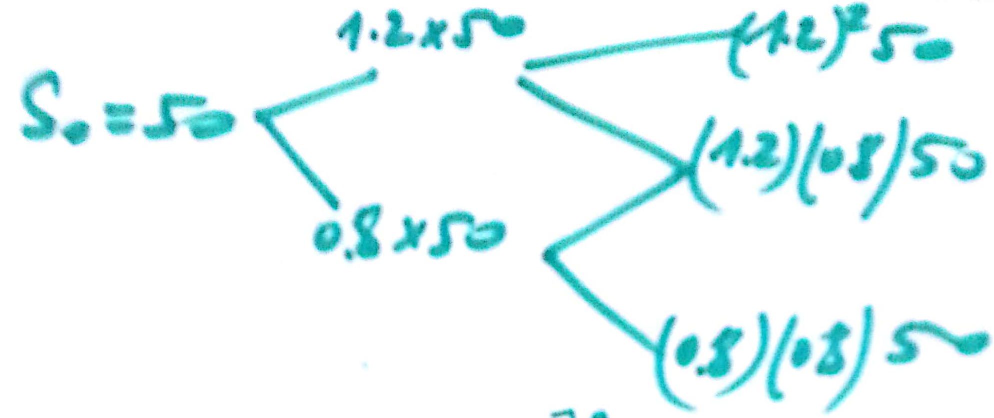
Q_3: Calculate C_0, for $\underline{n = 2}$

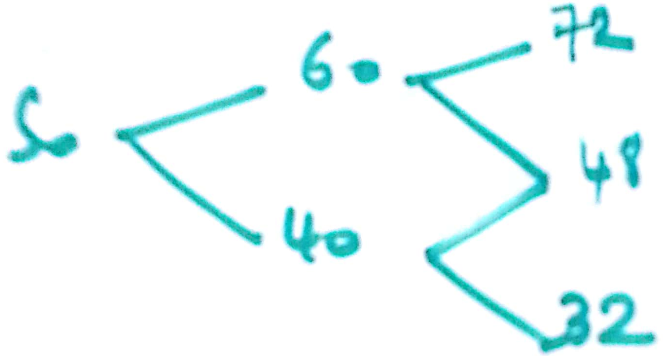
$S_0 = 50$, $K = 45$, $RFR = 5\%$.

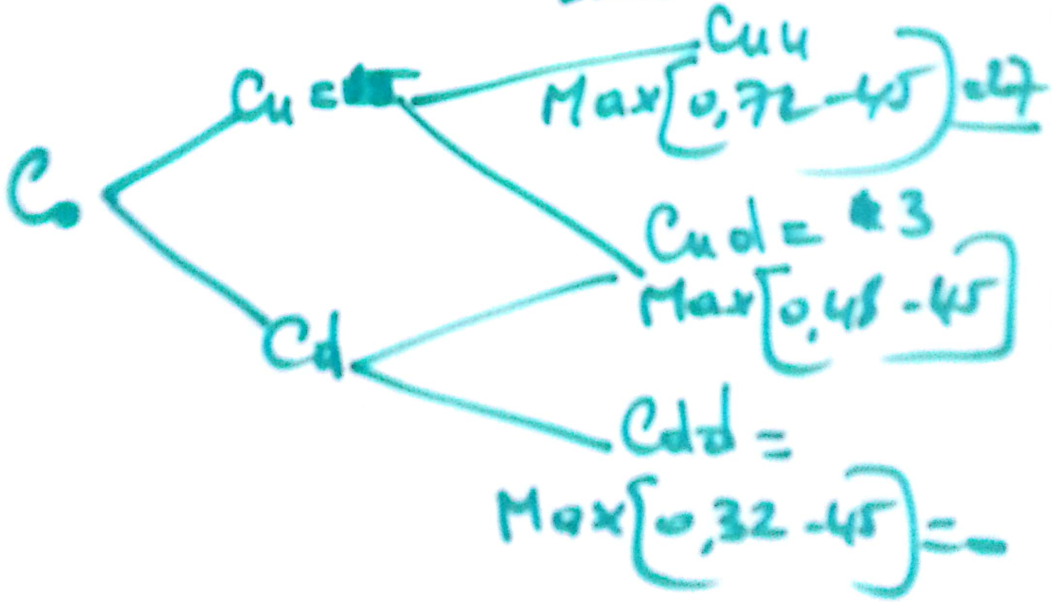
$u = 1.2$, $d = 0.8$.

using the Binomial Tree

$$\begin{tikzcd}[row sep=small]
 & & u^2S \\
 & uS \arrow[ur] \arrow[dr] & \\
S_0 = 50 \arrow[ur] \arrow[dr] & & udS \\
 & dS \arrow[ur] \arrow[dr] & \\
 & & d^2S
\end{tikzcd}$$


S0=50
1.2x50
0.8x50
(1.2)²50
(1.2)(0.8)50
(0.8)(0.8)50


S0
60
40
72
48
32


C0
Cu
Cd
Cuu
Max[0,72-45]=27
Cud=3
Max[0,48-45]
Cdd=
Max[0,32-45]=0

$$C_o = \frac{1}{R^2}\left[q^2 C_{uu} + 2q(1-q) C_{ud} + (1-q)^2 C_{dd}\right]$$

$q = 0.625$, $1-q = 0,325$

$$C_o = \frac{1}{R^2}\left[(0.625)^2 \times 27 + 2(0.625)(0.325) \times 3 + (0.325)^2 \times 0\right] =$$

$$C_o = \frac{1}{(1.05)^2}\left[10.54 + 0.703\right]$$

$$= \frac{11.24}{1.1025} = \underline{\underline{10.19}}$$