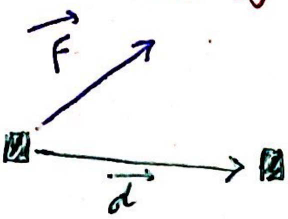
★ The work done by a force $\vec{F}$:



$$\vec{W} = \vec{F} \cdot \vec{d}$$

★ Direction Angles :

The meaning: Let $\vec{v} = \langle a, b, c \rangle$ be a vector. The direct Angles are the angles

α with $\vec{i}$ $\left(\cos\alpha = \frac{a}{|v|} \right)$

β with $\vec{j}$ $\left(\cos\alpha = \frac{b}{|v|} \right)$

γ with $\vec{k}$ $\left(\cos\gamma = \frac{c}{|v|} \right)$

★ Vector Product ≡ Cross Product :

Let $v_1 = \langle a_1, a_2, a_3 \rangle$ $\quad v_2 = \langle b_1, b_2, b_3 \rangle$ then

$$v_1 \times v_2 = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Also,

$$v_1 \times v_2 = |v_1|\,|v_2|\,(\sin\theta)\,\vec{u}$$

where θ is the angle Between v_1 and v_2 & $\vec{u}$ is the unite vector orthognal over $\vec{v}_1, \vec{v}_2$.

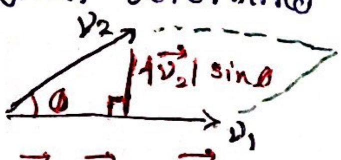
So,

(1) $v_1 \times v_2$ is orthogonal over v_1 and v_2 at the same time (i.e. $(\vec{v_1} \times \vec{v_2}) \cdot v_1 = (\vec{v_1} \times \vec{v_2}) \cdot v_2 = 0$)

(2) If $\vec{v_1}$ and $\vec{v_2}$ are Parallel $\Rightarrow \vec{a} \times \vec{b} = 0$

(3) $|\vec{v_1} \times \vec{v_2}|$ = The area of Paralledgram determind by $\vec{v_1}$ and $\vec{v_2}$

$= \|\vec{v_1}\|\,\|\vec{v_2}\| \sin\theta$



(4) $\vec{i} \times \vec{j} = \vec{k}$ $\quad \vec{j} \times \vec{i} = -\vec{k}$ $\quad \vec{i} \times \vec{k} = j$ $\quad \vec{k} \times \vec{i} = -\vec{j}$

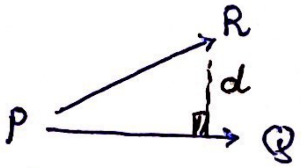
$\vec{j} \times \vec{k} = \vec{i}$ $\quad \vec{k} \times \vec{j} = -\vec{i}$

★ Scalar triple Product :

$v_1 = \langle a_1, a_2, a_3 \rangle$
$v_2 = \langle b_1, b_2, b_3 \rangle \Rightarrow (v_1 \times v_2) \cdot v_3 = v_1 \cdot (v_1 \times v_3)$
$v_3 = \langle c_1, c_2, c_3 \rangle$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

★ distance of a point :



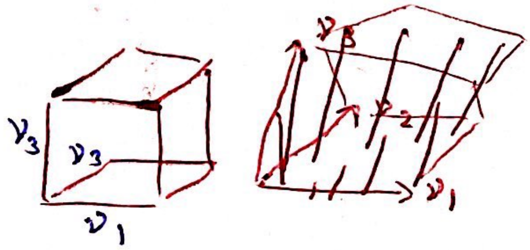
$$\boxed{d = \| \overrightarrow{PR} \| \sin\theta}$$

$$= \frac{\| \overrightarrow{PR} \| \sin\theta}{\| PQ \|} \cdot \| PQ \|$$

$$\boxed{d = \frac{\| \overrightarrow{PQ} \times \overrightarrow{PR} \|}{\| PQ \|}}$$

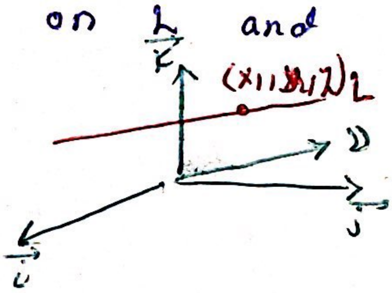
★ volume of a box :

$$v = \underbrace{\| v_1 \times v_2 \|}_{\text{Area of Base}} \cdot \underbrace{\| v_3 \| \cos\theta}_{\text{Height}}$$



★ Equation of Line :

Determind by a point (x_1, y_1, z_2) lies on L and a parallel vector $v = \langle v_1, v_2, v_3 \rangle$



$$\frac{x - x_1}{v_1} = \frac{y - y_1}{v_2} = \frac{z - z_1}{v_3} \quad \text{(symmetric Form)}$$

or

$$\left.\begin{aligned} x &= x_1 + v_1 t \\ y &= y_1 + v_2 t \\ z &= z_1 + v_3 t \end{aligned}\right\} \quad \text{(Parametric Form)} \quad ; t \in \mathbb{R}.$$

★ Orthogonal and Parallel Lines :

Let $\vec{\nu_1}$ and $\vec{\nu_2}$ Lie one L_1 and L_2 respectively.

Then $L_1 \parallel L_2$ if $\vec{\nu_1} \parallel \vec{\nu_2}$ (i.e. $\vec{\nu_1} = K \cdot \vec{\nu_2}$)

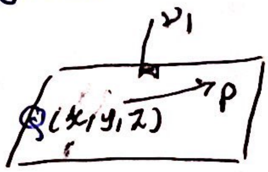
$L_1 \perp L_2$ if $\vec{\nu_1} \perp \vec{\nu_2}$ (i.e. $\vec{\nu_1} \cdot \vec{\nu_2} = 0$)

★ Angle between two Lines L_1 and L_2

= the angle between the vectors that Parallel L_1 and L_2 respectively. (we can deduce the Parallel vectors from the symetric equation, directly).

★ Equation of Plane :

Determined by a point $P(x_1, y_1, z_1)$ Lies on the Plane and a vector $\nu_1 = (a, b, c)$ that is (normal) orthogonal over the Plane



So, $$\boxed{\vec{\nu_1} \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0}$$

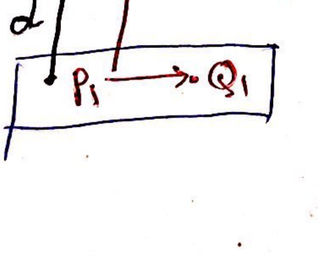
So, The equation of plane on the form:

$$ax + by + cz + d = 0 \quad \text{where } a, b, c, d \in \mathbb{R}$$

(i) Distance from point (x_0, y_0, z_0) to the Plane $ax + by + cz + d = 0$:

$$\text{height} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(ii) Shortest distance d between P_1 and P_2



$$d = \frac{(\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}) \cdot \overrightarrow{P_1P_2}}{\| \overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2} \|}$$

★ Two Planes P_1 and P_2 are (Paralled) or (orthogonal) if their normal vectors are (Parallel) or (orthogonal)

Ex Let $P(-3, 0, 5)$, $Q(2, -1, -3)$, $R(4, 1, -1)$ be the point in a plane,

(i) Find a vector perpendicular to the plane determined by P, Q and R

(ii) Find the area of parallelogram made by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$,

(iii) Find the area of triangle PQR

(iv) Find a unit vector perpendicular to the plane determined by P, Q and R.

Solution

$$\overrightarrow{PQ} = \langle 2+3, -1+0, -3-5 \rangle = \langle 5, -1, -8 \rangle$$

$$\overrightarrow{PR} = \langle 4+3, 1+0, -1-5 \rangle = \langle 7, 1, -6 \rangle$$

(i)
$$\vec{A} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 5 & -1 & -8 \\ 7 & 1 & -6 \end{vmatrix} = \begin{vmatrix} -1 & -8 \\ 1 & -6 \end{vmatrix} i - \begin{vmatrix} 5 & -8 \\ 7 & -6 \end{vmatrix} j + \begin{vmatrix} 5 & -1 \\ 7 & 1 \end{vmatrix} k$$

$$= 14i - 26j + 12k$$

(ii) Area of $\|$gram is $\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{(14)^2 + (-26)^2 + (12)^2} = \sqrt{196 + 676 + 144}$$
$$= \sqrt{1016}$$
$$= 31.87 \text{ unit}^2$$

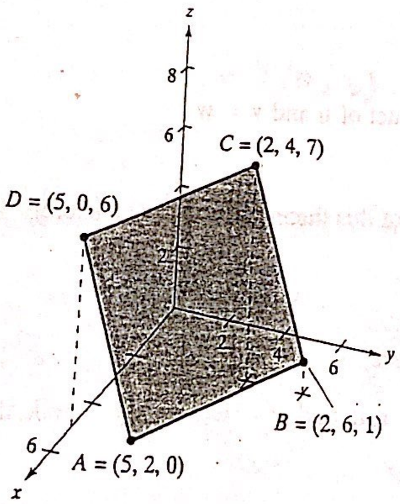
(iii) Area of triangle PQR $= \frac{1}{2} \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2} \cdot (31.87) = 15.94 \text{ unit}^2$

(iv)
$$u = \frac{\vec{A}}{\|\vec{A}\|} = \frac{1}{31.87} \langle 14, -26, 12 \rangle$$

EXAMPLE 3 Geometric Application of the Cross Product

Show that the quadrilateral with vertices at the following points is a parallelogram, and find its area.

$$A = (5, 2, 0) \qquad B = (2, 6, 1)$$
$$C = (2, 4, 7) \qquad D = (5, 0, 6)$$



The area of the parallelogram is approximately 32.19.
Figure 10.38

Solution From Figure 10.38 you can see that the sides of the quadrilateral correspond to the following four vectors.

$$\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k} \qquad \overrightarrow{CD} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} = -\overrightarrow{AB}$$
$$\overrightarrow{AD} = 0\mathbf{i} - 2\mathbf{j} + 6\mathbf{k} \qquad \overrightarrow{CB} = 0\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} = -\overrightarrow{AD}$$

Thus, $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and $\overrightarrow{AD}$ is parallel to $\overrightarrow{CB}$, and you can conclude that the quadrilateral is a parallelogram with $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as adjacent sides. Moreover, because

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix}$$
$$= 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$$

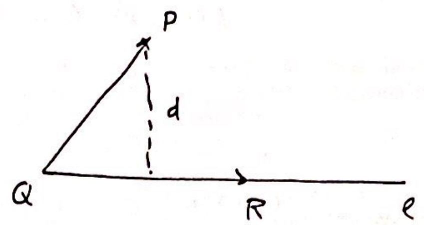
the area of the parallelogram is

$$\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{1036} \approx 32.19.$$

Is the parallelogram a rectangle? You can tell whether it is by finding the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

Ex. Find the distance from $P(3,1,-2)$ to the line through $Q(2,5,1)$ and $R(-1,4,2)$.

Solution



$$d = \frac{\|\overrightarrow{QP} \times \overrightarrow{QR}\|}{\|\overrightarrow{QR}\|}$$

$$\overrightarrow{QP} = \langle 3-2, 1-5, -2-1 \rangle = \langle 1, -4, -3 \rangle$$

$$\overrightarrow{QR} = \langle -1-2, 4-5, 2-1 \rangle = \langle -3, -1, 1 \rangle.$$

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ 1 & -4 & -3 \\ -3 & -1 & 1 \end{vmatrix} = -7i + 8j - 13k$$

$$\|\overrightarrow{QR} \times \overrightarrow{QP}\| = \sqrt{49+64+169} = \sqrt{282} = 16.79$$

$$\|\overrightarrow{QR}\| = \sqrt{9+1+1} = \sqrt{11} = 3.3$$

$$d = \frac{\|\overrightarrow{QP} \times \overrightarrow{QR}\|}{\|\overrightarrow{QR}\|} = \frac{16.79}{3.3} = 5.09 \text{ unit}$$

Ex Find the volume of box having adjacent sides AB, AC and AD where $A(2,1,-1)$, $B(3,0,2)$, $C(4,-2,1)$, $D(5,-3,0)$

Solution $a = \overrightarrow{AB} = \langle 1, -1, 3 \rangle$

$b = \overrightarrow{AC} = \langle 2, -3, 2 \rangle$

$c = \overrightarrow{AD} = \langle 3, -4, 1 \rangle$

$$V = |(a \times b) \cdot c| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & -3 & 2 \\ 3 & -4 & 1 \end{vmatrix} = 5 - 4 + 3$$

$$= 4 \text{ unit}^3$$

Ex. Find equation of the line passing Point $P(4,3,2)$ and parallel to vector $a = i + 2j + 3k$.

Solution. Parametric Form $x = 4 + t$

$y = 3 + 2t$

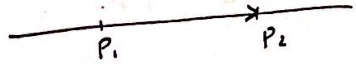
$z = 2 + 3t$

Symmetric Form $\frac{x-4}{1} = \frac{y-3}{2} = \frac{z-2}{3}$

Ex. Find equation of line passing through $P_1(-3, 1, -1)$ and $P_2(7, 11, -8)$

Solution $\overrightarrow{P_1P_2} = \langle 7+3, 11-1, -8+1 \rangle$

$= \langle 10, 10, -7 \rangle$



Eq. of line passing through point $P_1(-3, 1, -1)$ or $P_2(7, 11, -8)$ and parallel to vector $\overrightarrow{P_1P_2} = \langle 10, 10, -7 \rangle$ is

Parametric form

$$x = -3 + 10t \qquad \text{or} \qquad x = 7 + 10t$$
$$y = 1 + 10t \qquad\qquad y = 11 + 10t$$
$$z = -1 - 7t \qquad\qquad z = -8 - 7t$$

Symmetric form $\dfrac{x+3}{10} = \dfrac{y-1}{10} = \dfrac{z+1}{-7}$

or $\dfrac{x-7}{10} = \dfrac{y-11}{10} = \dfrac{z+8}{10}$

Ex. If l has parametric equations $x = 5 - 3t$
$y = -2 + t$
$z = 1 + 9t$,

find parametric equations for the line through $P(-6, 4, -3)$ that is parallel to l.

Solution. Vector parallel to line l: $a = \langle -3, 1, 9 \rangle$

Parametric equations passing through point $P(-6, 4, -3)$ and parallel to vector $a = \langle -3, 1, 9 \rangle$ are

$$x = -6 - 3t$$
$$y = 4 + t$$
$$z = -3 + 9t$$

Example. (a) Find parametric equations for the line, ℓ, passing through the points $A(-2, 2, 1)$, $B(1, 3, 2)$.

(b) At what point does 'ℓ' intersect xy-plane?

Solution

a. Direction of line 'ℓ' is same as the vector

$$\overrightarrow{AB} = \langle 1+3,\ 3-2,\ 2-1 \rangle = \langle 4, 1, 1 \rangle$$

Equation of line passing through point $A(-2, 2, 1)$ and parallel to vector $\overrightarrow{AB} = \langle 4, 1, 1 \rangle$ is

$$x = -2 + 4t, \quad y = 2 + t, \quad z = 1 + t, \quad t \in \mathbb{R}.$$

b. At the point where the line intersects the xy-plane is when $z = 0$

From equation of line $1 + t = 0 \Rightarrow t = -1$.

Substituting this value into the parametric equation, we obtained the required point

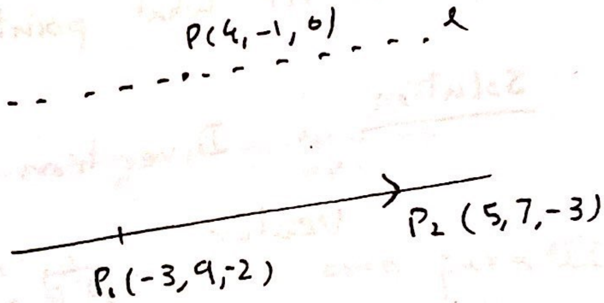
$$x = -2 - 4 = -6$$

$$y = 2 - 1 = 1$$

$$z = 1 - 1 = 0$$

Point is $(-6, 1, 0)$.

Ex. Find parametric equations for the line through the point $P(4,-1,0)$ that is parallel to the line through the points $P_1(-3,9,-2)$ and $P_2(5,7,-3)$



Solution

$$\overrightarrow{P_1P_2} = \langle 5+3\ ,\ 7-9\ ,\ -3+2 \rangle$$

$$= \langle 8\ ,\ -2,\ -1 \rangle$$

Eq. of line passing through $P(4,-1,0)$ and parallel to vector $\overrightarrow{P_1P_2} = \langle 8,-2,-1 \rangle$ is

$$x = 4 + 8t$$

$$y = -1 - 2t$$

$$z = 0 - t$$

Ex. Determine whether the lines

$l_1 : \quad x = 4 - 2t, \quad y = 1 + 4t, \quad z = 3 + 10t$

$l_2 : \quad x = v, \quad y = 6 - 2v, \quad z = \frac{1}{2} - 5v$ are parallel.

olution.

$a = \langle -2, 4, 10 \rangle$

$b = \langle 1, -2, -5 \rangle$

Since $\frac{-2}{1} = \frac{4}{-2} = \frac{10}{-5}$ $\Rightarrow$ lines are parallel

or $\quad a = -2b \quad \Rightarrow$ lines are parallel.

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Ex. Determine whether the lines

$l_1: \; x = -6 - t, \quad y = 10 + 3t, \quad z = 3 + 2t$

$l_2: \; x = 3 + 2v, \quad y = -5 - 4v, \quad z = -1 + 7v$

are orthogonal.

Solution. vectors are

$a = \langle -1, 3, 2 \rangle$

$b = \langle 2, -4, 7 \rangle$

$a \cdot b = -2 - 12 + 14 = 0$

$\Rightarrow$ l_1 and l_2 are orthogonal.

Ex.

Determine whether the lines

$l_1: \; x = 1 - 6t, \quad y = 3 + 2t, \quad z = 1 - 2t$

$l_2: \; x = 2 + 2v, \quad y = 6 + v, \quad z = 2 + v$

intersects, and if so, find the point of intersection.

Solution.

Let the point of intersection be $P_0(x_0, y_0, z_0)$ is common to both lines, we must have

$$\begin{aligned} 1 - 6t_0 &= 2 + 2v_0 \\ 3 + 2t_0 &= 6 + v_0 \\ 1 - 2t_0 &= 2 + v_0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 6t_0 + 2v_0 &= -1 \quad \rightarrow 1 \\ 2t_0 - v_0 &= 3 \quad \rightarrow 2 \\ 2t_0 + v_0 &= -1 \quad \rightarrow 3 \end{aligned}$$

We solve any two the equations simultaneously and use the remaining equation as check. taking Eq. 2. and Eq. 3.

$E_2 + E_3 \Rightarrow 4t_0 = 2$ or $t_0 = \frac{1}{2}$

$E_2 - E_3 \Rightarrow -2v_0 = 4$ or $v_0 = -2$

Substituting in Eq. 1 $\quad 6(\frac{1}{2}) + 2(-2) = 3 - 4 = -1$

$\Rightarrow$ l_1 and l_2 intersects.

To find point of intersection, we substitute $v_0 = -2$ in l_2

$$\Rightarrow \begin{aligned} x_0 &= -2 \\ y_0 &= 4 \\ z_0 &= 0 \end{aligned} \quad \Rightarrow \text{Point of intersection is } P_0(-2, 4, 0)$$

///

Ex. Find angle between l_1 and l_2

$l_1 : \quad x = 5 + 3t, \quad y = 4 - t, \quad z = 3 + 2t$

$l_2 : \quad x = -t, \quad y = 1 - 2t, \quad z = 3 + t$

Solution. Angle between lines is angle between their parallel vectors.

$a = \langle 3, -1, 2 \rangle \qquad \|a\| = \sqrt{9+1+4} = \sqrt{14}$

$b = \langle -1, -2, 1 \rangle \qquad \|b\| = \sqrt{1+4+1} = \sqrt{6}$

$$\theta = \cos^{-1} \frac{a \cdot b}{\|a\|\|b\|} = \cos^{-1} \frac{-3+2+2}{\sqrt{14}\sqrt{6}} = \cos^{-1} \frac{1}{\sqrt{14}\sqrt{6}} \cong 84^{\circ}$$

Ex. Find an equation of the plane through the point $P(-11, 4, -2)$ with normal vector $a = 6i - 5j - k$

Soln. $a_1 (x - x_1) + a_2 (y - y_1) + a_3 (z - z_1) = 0$

$6(x + 11) - 5(y - 4) - (z + 2) = 0$

$6x + 66 - 5y + 20 - z - 2 = 0$

$6x - 5y - z + 84 = 0$ ///

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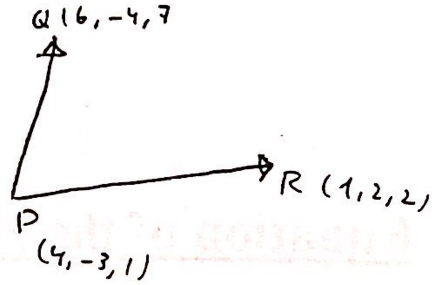
Ex. Find an equation of the plane determined by the Points $P(4,-3,1)$, $Q(6,-4,7)$; and $R(1,2,2)$



Soln.

$\overrightarrow{PQ} = \langle 2, -1, 6 \rangle$

$\overrightarrow{PR} = \langle -3, 5, 1 \rangle$

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is normal to the plane determined by P, Q and R

$$n = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 2 & -1 & 6 \\ -3 & 5 & 1 \end{vmatrix} = -31i - 20j + 7k$$

Eq. of plane with $P(4,-3,1)$ and normal vector $\hat{n}$

$$-31(x-4) - 20(y+3) + 7(z-1) = 0$$

$$-31x - 20y + 7z + 57 = 0$$

Ex

Show that planes $4x-2y+6z=3$ and $-6x+3y-9z=4$ are parallel and find the distance between the planes.

Solution. Two planes are parallel, when vectors normal to the planes are parallel.

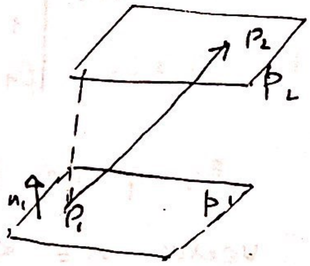
vector normal to plane P_1 is $n_1 = \langle 4,-2,6\rangle$

vector normal to plane P_2 is $n_2 = \langle -6,3,-9\rangle$

n_1 and n_2 are parallel because

$$\frac{4}{-6} = -\frac{2}{3} = \frac{6}{-9} \qquad \text{or} \qquad n_2 = -\frac{3}{2}n_1$$

distance between planes p_1 and p_2 will be $\text{Comp}_{n_1}\overrightarrow{P_1P_2}$



To find P_1, Let $x=1, y=1 \Rightarrow z=\frac{1}{6}$

To find P_2 Let $x=1, y=1 \Rightarrow z=-\frac{7}{9}$

2nd method

$P_2(0,0,-\frac{4}{9})$

$$d = \frac{|6\times(-\frac{4}{9})-3|}{\sqrt{16+4+36}} = \frac{17}{3\sqrt{56}} = \frac{17}{6\sqrt{14}}$$

$P_1(1,1,\frac{1}{6})$ on p_1

$P_2(1,1,-\frac{7}{9})$ on p_2

$\overrightarrow{P_1P_2} = \langle 0,0,\frac{17}{18}\rangle$

$\hat{n} = \langle 4,-2,6\rangle$

$$h = \text{Comp}_n\overrightarrow{P_1P_2} = \frac{|\overrightarrow{P_1P_2}\cdot n|}{|n|} = \frac{17/3}{\sqrt{16+4+36}} = \frac{17}{6\sqrt{14}}$$

Ex. Find the distance from $P(1,-1,2)$ to the plane $3x-7y+z-5=0$

Solution

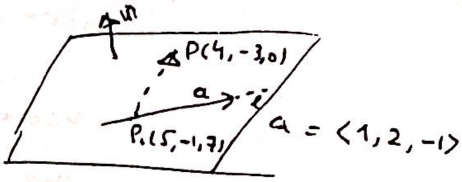
$$h = \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|3(1)-7(-1)+1(2)-5|}{\sqrt{9+49+1}} = \frac{7}{\sqrt{59}}$$

Ex Find the equation of the plane that contains the point $P(4,-3,0)$ and the line $x = t+5$, $y = 2t-1$, $z = -t+7$

Solution.

vector $a = \langle 1,2,-4\rangle$ is parallel to line l



and $P_1\,(5,-1,7)$ is point on the line.

$$\overrightarrow{P_1P} = \langle 5-4,\ -1+3,\ 7-0\rangle$$
$$= \langle 1,\ 2,\ 7\rangle$$

A vector normal to the plane is $n = a \times \overrightarrow{P_1P}$

$$n = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 1 & 2 & 7 \end{vmatrix} = 16i - 8j + 0k$$

Eq. of the plane containing point $P(4,-3,0)$ with normal vector $n = \langle 16,-8,0\rangle$ is

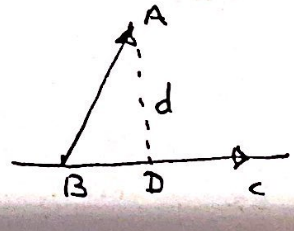
$$16(x-4) - 8(y-3) + 0(z-0) = 0$$

$$16x - 8y - 88 = 0$$

Ex. Use dot product to find the distance from $A(2,-6,1)$ to the line through $B(3,4,-2)$ and $C(7,-1,5)$

Solution

$\overrightarrow{BA} = \langle -1,-10,3\rangle$, $\overrightarrow{BC} = \langle 4,-5,7\rangle$



$$|BD| = \mathrm{Comp}_{\overrightarrow{BC}}\overrightarrow{BA} = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\|\overrightarrow{BC}\|} = \frac{67}{\sqrt{90}}$$

$$\|\overrightarrow{BA}\| = \sqrt{110}$$

$$AD = \sqrt{\|\overrightarrow{BA}\|^2 - \|\overrightarrow{BD}\|^2} = \sqrt{110 - \frac{(67)^2}{90}} = \sqrt{\frac{5411}{90}} \approx 7.75$$

Note. The graph of every linear equation $ax+by+cz+d=0$ is a plane with normal vector $\langle a,b,c\rangle$

Ex. Find parametric equations for the line of intersection of the planes.

$$x + 2y - 9z = 7 \quad \longrightarrow P_1$$
$$2x - 3y + 17z = 0 \quad \longrightarrow P_2$$

Solution. Vector n_1 normal to P_1 is $\langle 1, 2, -9 \rangle$

vector n_2 normal to P_2 is $\langle 2, -3, 17 \rangle$

Vector parallel to line of intersection of planes is $n_1 \times n_2$

$$a = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & -9 \\ 2 & -3 & 17 \end{vmatrix} = 7i - 35j - 7k$$

To find a point on the line of intersection of P_1 and P_2, Let $z = 0$

$$x + 2y = 7 \quad \rightarrow E_1$$
$$2x - 3y = 0 \quad \rightarrow E_2$$

$-2E_1 + E_2$ $\quad$ $-2x - 4y = -14$
$\quad 2x - 3y = 0 \quad \Rightarrow \quad -7y = -14 \Rightarrow y = 2$

$x = 7 - 2y$

$x = 7 - 4 = 3$

Point is $(3, 2, 0)$.

Equation of line through point $(3, 2, 0)$ parallel to vector

$a = \langle 7, -35, -7 \rangle$ is $\quad x = 3 + 7t$

$y = 2 - 35t$

$z = -7t$

METHOD 2. by Linear Algebra

$$\begin{bmatrix} 1 & 2 & -9 & 7 \\ 2 & -3 & 17 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -9 & 7 \\ 0 & -7 & 35 & -14 \end{bmatrix} -2R_1 + R_2$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -5 & 2 \end{bmatrix} \begin{matrix} -2R_2' + R_1 \\ -\frac{1}{7}R_2 \end{matrix}$$

$x + z = 3 \qquad x = 3 - z$

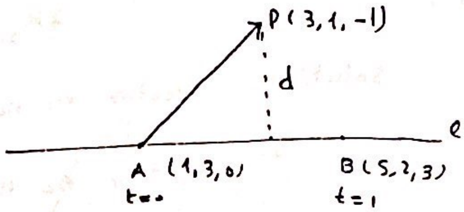
$y - 5z = 2 \qquad y = 2 - 5z$

Let $z = t$

$x = 3 - t$
$y = 2 - 5t$
$z = t$ are parametric equations of the line of intersection.

Ex. Find the distance from the point $P(3,1,-1)$ to the line: $x = 1+4t$, $y = 3-t$, $z = 3t$.

Solution.



$$d = \frac{\| \vec{AB} \times \vec{AP} \|}{\| \vec{AB} \|}$$

$\vec{AB} = \langle 4, -1, 3 \rangle$

$\vec{AP} = \langle 2, -2, -1 \rangle$

$$\vec{AB} \times \vec{AP} = \begin{vmatrix} i & j & k \\ 4 & -1 & 3 \\ 2 & -2 & -1 \end{vmatrix} = 7i + 10j - 6k$$

$\| \vec{AB} \times \vec{AP} \| = \sqrt{49 + 100 + 36} = \sqrt{185}$

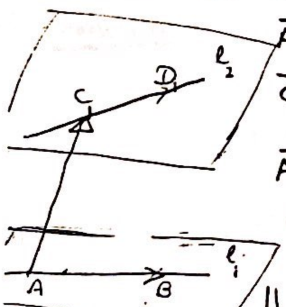
$\| \vec{AB} \| = \sqrt{16+1+9} = \sqrt{26}$

$$d = \sqrt{\frac{185}{26}} = 2.67$$

Ex. Find the shortest distance between the lines l_1 through the points $A(1,-2,3)$, $B(2,0,5)$ and line l_2 through the points $C(4,1,-1)$, $D(-2,3,4)$. l_1 and l_2 are skew lines

Soln. Two lines are skew if they are not parallel and donot intersect.

$$h = \frac{|(\vec{AB} \times \vec{CD}) \cdot \vec{AC}|}{\| \vec{AB} \times \vec{CD} \|}$$



$\vec{AB} = \langle 1, 2, 2 \rangle$

$\vec{CD} = \langle -6, 2, 5 \rangle$

$\vec{AC} = \langle 3, 3, -4 \rangle$

$$\vec{AB} \times \vec{CD} = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ -6 & 2 & 5 \end{vmatrix} = 6i - 17j + 14k$$

$\| \vec{AB} \times \vec{CD} \| = \sqrt{36 + 289 + 196} = \sqrt{521} = 22.8$

$(\vec{AB} \times \vec{CD}) \cdot \vec{AC} = \langle 6, -17, 14 \rangle \cdot \langle 3, 3, -4 \rangle = 18 - 51 - 56 = -89$

$$h = \frac{|-89|}{22.8} = \frac{89}{22.8} = 3.9$$