PRODUCTS OF BERGMAN SPACE TOEPLITZ OPERATORS AND BROWN-HALMOS TYPE THEOREMS

By

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To my marvelous mother

Sabah Abdul-Jabbar

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Abstract

The Bergman space is the Hilbert subspace of the Lebesgue space of square summable functions on the unit disk consisting of the analytic ones. A Toeplitz operator on the Bergman space is a multiplication by a fixed function (called the symbol) followed by a projection onto the Bergman space. A theorem is called of Brown-Halmos type if it answers the following question: When the product of two Toeplitz operators is again a Toeplitz operator? The original Brown-Halmos theorem was discovered by themselves in the Hardy space setting by 1963. Very recently, in their excellent paper [A theorem of Brown-Halmos type for Bergman space Toeplitz operators. Journal of Functional Analysis, 187 (2001), 200–210.], P. Ahern and Ž. Čučković, have established a Brown-Halmos type theorem for Bergman space Toeplitz operators with bounded harmonic symbols. In a subsequent paper, namely [Some Examples related to the Brown-Halmos Theorem for the Bergman Space. Acta. Sci. Math. (Szeged) 70 (1-2), (2004), 373-378.], they have discussed the case of radial symbols. P. Ahern in [On the range of the Berezin transform. Journal of Functional Analysis, 215 (1), (2004), 206- -216.], has somewhat generalized the results of the first paper. The aim of this thesis is to present their results in a significantly detailed framework. Our contribution consists of the following:

- Collecting various background elements and surrounding the underlying concepts.
- Detailing the proofs and commenting on the results.
- Extracting perspectives and suggesting affordable questions for further research.
- Establishing generalizations of some of their results and pointing out new corollaries.

The main outcomes of this work consist of: a deep initiation to the theory of operators on function spaces and a contribution in establishing more Brown-Halmos type theorems.

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Introduction

Let D be the unit disk in the complex plane $\mathbb{C}$ and let ∂D be its boundary. Denote by dA the normalized Lebesgue measure on D. $L^2(D, dA)$ is the well-known Lebesgue space of square summable functions in D. The Bergman space L_a^2 is the Hilbert subspace of $L^2(D, dA)$ consisting of analytic functions. Denote by T_f the Toeplitz operator with symbol f, (a priori, such symbol is supposed to be an L^1-function). It is defined, on L_a^2, to be a multiplication by f followed by a projection onto L_a^2. A related operator is the Hankel operator H_f, which is defined on L_a^2 to be a multiplication by f followed by a projection onto $\left(L_a^2\right)^\perp$, the orthogonal complement of L_a^2, instead.

The study of Toeplitz and Hankel operators on the Bergman space has begun by the eighties. The main concerns of such studies are related to the characterization of symbols giving rise to bounded, compact, finite rank and certain Schatten class operators. In addition to the problem of products of such operators and so many other questions.

In particular, the following question represents our main task in this thesis: how to characterize the product of two Toeplitz operators in order to be again a Toeplitz operator? In the classical case, namely in the framework of the Hardy space, various aspects of such problem become nowadays well understood. The main contribution in this direction is the work of Brown and Halmos in their fascinating paper [2]. As a matter of fact, this paper is acknowledged by experts to be the pillar of this area. Among the famous Brown-Halmos results, the one answering the above question. Precisely speaking, it asserts that a necessary and sufficient condition for the product T_fT_g of two Hardy space Toeplitz operators T_f and T_g to be again a Toeplitz operator is that either g be holomorphic or f be co-holomorphic; in either case $T_fT_g = T_{fg}$. This solves the problem completely in the Hardy space case.

However, unlike the Hardy space case, the Bergman space situation is still less understood despite the heavy efforts done mainly by P. Ahern and Ž. Čučkovič. Indeed, only very recently, P. Ahern and Ž. Čučkovič in [3] have started the study of the Brown-Halmos conjecture, (i.e. the Bergman space analog of the latter assertion), for Bergman space Toeplitz operators. In that nice piece of work they considered the problem for bounded harmonic symbols with bounded harmonic conjugates. Namely, they assumed that $f = f_1 + \overline{f_2}$, $g = g_1 + \overline{g_2}$ and $h = h_1 + \overline{h_2}$ with f_i, g_i and h_i are bounded and holomorphic on D. Then they gave several necessary conditions for $T_fT_g = T_h$ to hold in a non-trivial way, (by non-trivial way it is meant here that it is not the case that both f and g are holomorphic, or that both conjugate holomorphic or that f or g is constant). Actually, if one of these four situations takes place, one sees immediately that $T_fT_g = T_{fg}$. They mentioned also that they don't know any example of such harmonic f, g and h such that $T_fT_g = T_h$ holds in a non-trivial way. As a matter of fact it turns out, as we will see later, that their feeling was pretty exact, namely there is no such f, g and h such that $T_fT_g = T_h$ in the above meaning of non-trivial way! They have given, in particular, some necessary conditions for $T_fT_g = T_h$ to hold in a non-trivial way, namely

1. f_1 and g_2 lie in the Zygmund class Λ_*.
2. There is no subset $E \subset \partial D$ of positive measure such that both f_2' and g_1' have continuous extensions to each point of E.
3. fg is not harmonic.
4. The function $\varphi = fg - h$ extends continuously to $\overline{D}$ and vanishes on ∂D.
5. T_φ is in the Schatten class S_r for all $r > \frac{1}{2}$.
6. If $h = 0$, then f_1, f_2, g_1 and g_2 are cyclic vectors for the backward shift on the Hardy space H^2.

The elegant and exciting techniques used in that paper are very useful and completely new. This makes that significantly rich paper, in our opinion of course, the key to the solution of the Brown-Halmos conjecture in the subsequent papers by P. Ahern and Ž. Čučković themselves [1, 2, 4]. Although it turns out that the set of such triples (f, g, h) satisfying $T_fT_g = T_h$ in the above meaning of non-trivial way is an empty set, (and thus the above necessary conditions, or any other arbitrary necessary condition, must be trivially satisfied), the techniques introduced in that paper give insight on the adequate approach towards the solution of the relevant problem, and we did enjoy it.

Having concluded that this first attempt to define the non-trivial way in the relevant context seems to be inadequate, P. Ahern and Ž. Čučković have corrected it in their subsequent excellent paper [2]. It turns out therefore that the right notion of non-trivial way should be simply either g be holomorphic or f be co-holomorphic. In this paper they showed that a Brown-Halmos type theorem holds under some additional hypothesis on the symbols f and g but fails in general. More precisely they showed the following: if f and g are bounded harmonic and h is bounded C^2 with bounded invariant Laplacian, then $T_fT_g = T_h$ holds if and only if either $\overline{f}$ or g is holomorphic, and then $fg = h$. An important special case is the one when f, g and h are bounded harmonic; then $T_fT_g = T_h$ holds if and only if both f and g are holomorphic, or both f and g are conjugate holomorphic or f or g is constant. This shows that indeed there are no such triple (f, g, h) satisfying $T_fT_g = T_h$ in the original sense of non-trivial way.

It should be fairly convenient to mention that the celebrated paper [2] by P. Ahern and Ž. Čučkovič opens the gate to at least two additional research directions. The first one is the problem of characterizing symbols giving rise to finite rank Toeplitz operators. Actually, they proved an auxiliary lemma asserting that there is no non-zero rank one operators with bounded symbols. In the perspectives, by the end of this thesis, we are going to formulate the question properly. The second research direction suggested implicitly by that work is the problem of characterization of the range of the Berezin transform. They addressed this problem in its simplest form, namely: does there exist a function $u \in L^1(D, dA)$ such that $z\overline{z} = Bu(z)$? It turns out that the answer is positive with $u(\zeta) = 1 - \log \frac{1}{|\zeta|^2}$. P. Ahern in [1] pursued the investigation of such characterization of non-constant holomorphic functions f, g with $f\overline{g} = Bu$. He proved that a few of such couples exist, namely f and g must be polynomials, of lower orders, in disk automorphisms. Such characterization proves useful in establishing a more general Brown-Halmos type theorem.

As we have already mentioned, a Brown-Halmos type theorem can be obtained for bounded harmonic symbols but fails in general. Accordingly, P. Ahern and Ž. Čučkovič in [4] have given several examples on some classes of symbols where a Brown-Halmos type theorem can be obtained and other cases where it fails. The particularly important class considered in that paper is the class of radial symbols. A powerful technique is well explored in that work, in addition to the Berezin transform, namely the Mellin transform. Among integral transforms, the Mellin transforms proves to be extremely useful whenever it is about radial functions. Note that the use of such technique in Toeplitz operators was initiated by Ž. Čučkovič and N. V. Rao in [23].

The study of the commutativity problem related to Toeplitz operators with bounded harmonic symbols was initiated by Čučkovič in his dissertation [22], where the construction of the fundamental elements of the theory was established. A series of related work, namely [12] and [21] were subsequently published. It is remarkable that, unlike the Brown-Halmos conjecture, the techniques used in this direction are not relied on the Berezin transform. It is one of our contributions to make use of the Berezin transform to re-establish some of the results obtained in [13, 21], and to see whether one can go further in this direction using this pioneering tool.

Having given an idea on the main recent developments regarding these two directions, we now return to the description of our contribution in this topic. Our thesis mainly hinges on the work of Ž. Čučkovič and his collaborators S. Axler, P. Ahern and N. P. Rao, namely [1, 2, 3, 4, 12, 13, 21], and it consists of two rather distinct parts. The first part is the main one. It hinges on the results of Ž. Čučkovič and P. Ahern [1], [2], and [4], which are our source of inspiration throughout this thesis. Actually, unless mentioned, most of the assertions are due to them by default. We present their results in a significantly detailed way through our own comprehension of the subject with no claim of originality. The main concern of this part is to establish a Brown-Halmos type theorem for Bergman space Toeplitz operators with bounded harmonic symbols using a powerful tool, namely the

Berezin transform. Taking into account the challenging difficulties that may be encountered when dealing with the weighted Bergman space Toeplitz operators, we address however such a case. Fortunately, we have established most of the assertions yielding a satisfactory answer to the weighted Brown-Halmos conjecture analog, except some remaining steps that seem to be somewhat resistant. Precisely speaking, for the weighted Bergman space situation, the following assertions are new: Remark 1.1.36, Theorem 2.3.3 and Section 3.4. Part (3) of Remark 4.1.2 on the diagonal Toeplitz operators with radial symbols and Remark 4.2.7 which characterizes radial Toeplitz operators can also be considered as new contributions. Among our contributions we have suggested proofs to many facts that are probably familiar to experts but we could not localize any corresponding reference, namely: Lemma 2.2.2, Lemma 2.4.2, part (1) of Proposition 3.1.3 and Lemma 3.1.5. Also we have proved some well known results by our own methods just as Proposition 2.1.3, Corollary 3.1.17 and Theorem 5.1.2. Another interesting assertion, which is considered as one of our main contributions, namely Lemma 4.2.10. Actually it is the converse of Proposition 0.2. of [68] for Toeplitz operators and it proves to be of great help subsequently. We have also extracted some questions and perspectives for further research in this exciting topic that interplays operator theory with complex function theory.

Our thesis is organized as follows: in Chapter 1 we give some preparatory material needed for the rest of the thesis. First we survey the results obtained in both Hardy space and Bergman space settings from a wide point of view. We collect some facts from various areas serving our own purpose. Most importantly are the main results obtained so far, regarding Toeplitz and Hankel operators on both Bergman and Hardy spaces, that we supply in details and in a well organized framework. This gives a good initiation to the theory of operators on analytic function spaces. The material of this part is mostly taken from [16, 17, 24, 38, 48, 49, 64, 67].

Chapter 2 is devoted to the pioneering tool used in this approach, namely the Berezin transform. We collect various aspects of the Berezin transform and its main properties. Our source for such purpose are the following papers and monographs [1, 2, 14, 15, 51, 53, 57, 67]. Interested people are advised to consult them for details. In this chapter we have included one of our main contributions, namely Theorem 2.3.3 and the original proof of Lemma 2.4.4.

Chapter 3 includes the heart of the matter. In fact, the problem of products of Toeplitz operators on the Bergman space is well established there, and the pivot of this thesis, namely Brown-Halmos type theorems related to the case of bounded harmonic symbols, is studied in details from various points of view with sufficiently numerous comments. The zero product problem, as a corollary of the Brown-Halmos theorem, is also discussed. The results of this chapter hinge on the excellent work of Ž. Čučkovič and P. Ahern [1, 2], which is our source of inspiration. The generalization of the Brown-Halmos conjecture to the weighted case has been initiated here. Although the obtained results are far from being complete, but it is a promising contribution towards a weighted Brown-Halmos type theorem.

Chapter 4 mainly concerns the case of radial symbols. Although this situation is not completely studied, but Brown-Halmos type theorems related to some special cases are discussed through several examples. The main tool exploited in the investigation of this situation is the Mellin transform. The results of this chapter rely on the nice paper [4] by Čučkovič and Ahern.

Chapter 5 is devoted to the commutativity problem for Bergman space Toeplitz operators. It represents the second part of this thesis. The main references used in this chapter are: the excellent paper by S. Axler and Ž. Čučkovič [13], Čučkovič's dissertation [22], and the one by S. Axler, Ž. Čučkovič and N. P. Rao [12]. We have collected, in a well organized way, various results in this direction with detailed explanation. We contribute, equally, here by supplying different proofs to certain assertions in order to give insight on the use of the Berezin transform to tackle the commutativity problem.

We conclude our thesis by some remarks and perspectives in the form of questions for further research. Some of them are more or less affordable. The investigation of certain ones is in progress.

List of Symbols

$A(D)$, disk algebra

A^2_α, weighted Bergman space

$a_n(f)$, Fourier coefficient of f

$\mathbf{B}$, Bloch space of D

$\mathbf{B}_h$, harmonic Bloch space

$\mathbf{B}_0$, little Bloch space

$B(u)$, Berezin transform of u

$B_\alpha(u)$, weighted Berezin transform of u

BMO, space of functions on ∂D with bounded mean oscillation

$BMOA$, space of analytic functions in BMO

$\mathbb{C}$, complex plane

$\overline{\mathbb{C}}$, extended complex plane

$\mathbb{C}^n$, n-dimensional complex vector space.

$C(\Omega)$, space of all complex-valued continuous functions on a domain Ω

$C_0(D)$, continuous functions on D vanishing on the boundary

D, open unit disk in the complex plane

$\overline{D}$, closed unit disk in the complex plane

D_r, open Euclidian disk with center 0 and radius r

$D_{(z,R)}$, open Euclidian disk with center z and radius R

$\mathfrak{D}\left(\mathbb{R}^2\right)$, space of test functions on $\mathbb{R}^2$

$d\theta$, arc-length measure on ∂D

dA, normalized area measure on D

$dA\alpha$, weighted area measure on D

δ, the delta function of Dirac

$\oplus$, direct sum

$\ominus$, direct substraction

$\mathbb{E}$, algebra

F, numerical field

$\overline{f}$, conjugate of f

$f \otimes g$, rank one operator for f and g

f_I, mean of f over an interval I

$f * g$, convolution of f and g

$\widehat{f}$, Poisson extension of f

$\mathcal{F}(f)$, Fourier transform of f

$\widetilde{f}$, Mellin transform of f

$\bigtriangledown$, gradient

$\mathcal{H}$, Hilbert space

H^∞, space of bounded analytic functions on D or ∂D

H_u, Hankel operator with symbol u

H^p, Hardy space

$(H^p)^*$, dual of H^p

$H^\perp$, orthogonal complement of H

$\langle ., . \rangle$, inner product

$\widetilde{\Delta}$, invariant Laplacian

$\mathbf{K}$, ideal of all compact operators

$\ker T$, kernel of the mapping T

k_w, normalized kernel function in L_a^2

$k_w^{(\alpha)}$, normalized kernel function in A_α^2

K_w, kernel function in L_a^2

$K_w^{(\alpha)}$, kernel function in A_α^2

L^1, Lebesgue space of functions which are integrable with respect to dA

L^2, Lebesgue space of square integrable functions with respect to dA

L^∞, Lebesgue space of bounded functions

L_a^2, subspace of L^2 consisting of holomorphic functions (Bergman space)

L_h^2, subspace of L^2 consisting of harmonic functions

L_h^∞, subspace of harmonic functions in L^∞

$L^1(D, dA_\alpha)$, space of integrable functions on D with respect to dA_α

$L^2(D, dA_\alpha)$, space of square integrable functions on D with respect to dA_α

$\mathbf{L}(\mathcal{H})$, algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$

Δ, Laplacian

M, the maximal ideal space of H^∞

M_φ, the multiplicative operator with symbol φ

φ_a, Möbius transformation

$\|.\|$, norm

$\frac{\partial}{\partial \eta}$, normal derivative

P, orthogonal projection from L^2 onto L_a^2

P_1, orthogonal projection from L^1 onto L^1_a

P_S, Szegö projection

P_α, orthogonal projection from $L^2(D, dA_\alpha)$ onto A^2_α

Π, natural projection

$\mathbf{\Pi}$, the half plane $\{z : \mathcal{R}e(z) > 1\}$

$\wp$, the set of trigonometric polynomials

$\wp_+$, the set of analytic trigonometric polynomials

$rad(f)$, radialization of a function f

$Rad(A)$, radialization of an operator A

S_z, Szegö kernel

S', commutant of the operator S

$\mathbf{S}$, unilateral shift

T^*, adjoint of an operator T

T_u, Toeplitz operator with symbol u

$\mathcal{T}$, norm closed subalgebra of $\mathbf{L}(L^2_a)$ generated by Toeplitz operators

Chapter 1

Hardy Spaces, Bergman Spaces and Haplitz Operators

At the beginning of this chapter we give some basic aspects of real, complex and functional analysis in preparation to our study in the subsequent chapters. Then we talk about Hardy, Bergman, Weighted-Bergman spaces and operators defined on them, namely the multiplication (or Laurent), Toeplitz and Hankel operators. J. Peetre has matched both kinds of operators, (Toeplitz and Hankel), together in one curious name: Ha-plitz operators! The results exhibited in this chapter are rather classical and most of them are well-known with a few exceptions. The background on Hardy spaces and their operators is taken mostly from the famous paper by Brown and Halmos [16] and the common text books [24, 25, 32, 38, 49, 67]. Whereas the material on Bergman spaces and their operators is taken from various recent papers in addition to Axler's paper [9] and Stroethoff's nice notes [57] as well as well-known textbooks such as [40, 67]. The material on the Weighted-Bergman space is taken from [60, 51] as well as [67]. For the sake of completeness, we include a historical survey, where we describe recent development in the subject. The reader who is familiar to the basic concepts may skip this chapter.

First, recall some very basic facts from complex function theory, functional analysis and integral transforms. Such facts are directly involved in our analysis along the whole thesis. Then we supply some words connected to introductory preliminaries that help understanding the general framework of reproducing kernel Hilbert spaces and their operators.

1.1 Preliminaries

1.1.1 Some facts from complex function theory

Let $\mathbb{C}^n$ denote the n-dimensional complex vector space. A point in $\mathbb{C}^n$ is written as $z = (z_1, z_2, \cdots, z_{n-1}, z_n)$ where $z_j = x_j + iy_j$ are complex numbers. We write $\frac{\partial f}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})f$ for the complex partial derivatives of a function $f : \mathbb{C}^n \to \mathbb{C}$. We also use the notation for $m = (m_1, \ldots, m_2) \in \mathbb{N}^n, z \in \mathbb{C}^n$:

$$\frac{\partial^{|m|} f}{\partial z^m} = \frac{\partial^{m_1}}{(\partial z_1)^{m_1}} \cdots \frac{\partial^{m_n}}{(\partial z_n)^{m_n}} f(z), \quad |m| = m_1 + \cdots + m_n.$$

Definition 1.1.1. *[Holomorphic functions in several variables]*
Let $D \subset \mathbb{C}^n$ be an open set. A continuous function $f : D \to \mathbb{C}$ is called holomorphic in D if for all $z \in D$ and $1 \leq j \leq n$, the complex partial derivatives $\frac{\partial f}{\partial z_j}$ exist and are finite.

Remark 1.1.2. *In other words, a function $f : D \to \mathbb{C}$ is holomorphic if it is continuous and holomorphic in each of its variables.*
Actually, it is not necessary to assume that f is continuous, a theorem of Hartogs asserts that the continuity assumption is redundant in the definition of analyticity.

Lemma 1.1.3. *If f is holomorphic, then $\frac{\partial f}{\partial z_j}$ is holomorphic. Moreover, all partial derivatives $\frac{\partial^n f}{(\partial z)^n}$ exist.*

Definition 1.1.4. *Let $D \subset \mathbb{C}$. A function f on D is said to be radial if it depends only on $|z|$.*

Theorem 1.1.5. *[The principle of analytic continuation]*
Assume that f is analytic in an open connected subset of $\mathbb{C}^n$. If f vanishes on an open subset of D, then $f = 0$ on D.

Theorem 1.1.6. *[The maximum principle]*
Let Ω be a bounded domain in $\mathbb{C}^n$ and let f be analytic on Ω and continuous on $\overline{\Omega}$. Then $|f(z)|$ takes its maximum on the boundary of Ω.

Theorem 1.1.7. *[Liouville'sTheorem]*
If f is both entire and bounded, then f is a constant function.

Definition 1.1.8. *[19] [Möbius transformation]*
A Mobius transformation, or a bilinear transformation, is a rational function

$$T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$$

of the form

$$T(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ are fixed and $ad - bc \neq 0$. We write formally $T(-\frac{d}{c}) = \infty$ and $T(\infty) = \frac{a}{c}$.

Remark 1.1.9. *[19] The only one-to-one conformal mappings that map the unit disk D onto itself are the Möbius transformations of the form*

$$T(z) = \lambda \frac{z - a}{1 - \overline{a}z}$$

where $a \in D$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. They are called "disk automorphisms". Here $\lambda = e^{i\theta}$ is an extra rotation about the origin .

Definition 1.1.10. *[19] Let f be an entire function. The order of f is defined by*

$$\rho = \rho(f) = \lim_{r\to\infty} \sup \frac{\log(\log M(r))}{\log r}$$

where $M(r) = M(r, f) = \max\limits_{|z|=r} |f(z)|, 0 \leq r < R$.

Theorem 1.1.11. *[19] [Hadamard's factorization theorem]*
Let f be a non-constant entire function of finite order ρ. Then f admits a canonical product representation :

$$f(z) = z^m e^{q(z)} p(z)$$

where $m \geq 0$ is the multiplicity of the zero at the origin, q is a polynomial of degree $v \leq \rho$. and p is the canonical product associated to the zeros $\{a_k\}$ of f in $\mathbb{C}$.

Definition 1.1.12. *Let u be a continuously differentiable complex valued function in D, the gradient of u is the map $\mathbb{R}^2 \to \mathbb{C}^2$ defined by*

$$\nabla u = \left(\frac{\partial u(z)}{\partial x}, \frac{\partial u(z)}{\partial y} \right) = (\partial u(z) + \overline{\partial} u(z), i(\partial u(z) - \overline{\partial} u(z))).$$

Definition 1.1.13. *A sequence $\{c_j\}$ in D is called a Blaschke sequence if*

$$\sum_j (1 - |c_j|) < \infty.$$

Example 1.1.14. *The sequence $\{2k + 2\}$ is not Blaschke since*

$$\sum_k (1 - |2k + 2|) = \sum_k (-2k - 1)$$

is not convergent.

Remark 1.1.15. *We know that a Blaschke sequence gives rise to a function*

$$z^m \prod_{j=1}^{\infty} \left\{ -\frac{\overline{a_j}}{|a_j|} \frac{z - a_j}{1 - \overline{a_j} z} \right\} F(z), \tag{1.1.1}$$

with F is zero free and $a_j \neq 0$, which is analytic. So that, if a function f vanishes on a Blaschke sequence $\{c_j\}$, this does not mean that $f \equiv 0$, because it can have the form of Formula (1.1.1).

The following helpful proposition plays an important role in the proofs of many results. It is some kind of analytic continuation principle. To clarify things, first let us remark the following

Remark 1.1.16. *If f is holomorphic in a neighborhood of the origin in $\mathbb{C}^n$ then in the absolutely convergent power series $\sum_{\alpha} c_\alpha z^\alpha$ of $f(z)$ around 0, we may collect terms of the same degree*

$$f(z) = P_0(z) + P_1(z) + P_2(z) + \dots$$

where $P_j(z) = \sum_{|\alpha|=j} c_\alpha z^\alpha$ is homogeneous in $z_1, \dots, z_n$ of degree j.

Proposition 1.1.17. *If F is holomorphic in $D \times D$ and $F(z, \overline{z}) = 0, \forall z \in D$, then $F \equiv 0$.*

Proof: Let $0 < p < 1$, then F can be written in the following form

$$F = \sum_{m=0}^{\infty} f_m \quad \text{on } D_p \times D_p,$$

where each f_m is a homogeneous polynomial of degree m on $\mathbb{C} \times \mathbb{C}$, and D_p is the disk of radius p around the origin. For $-1 < t < 1, z \in D_p$ and $m \geq 0$ we have

$$f_m(tz, t\overline{z}) = t^m f_m(z, \overline{z}).$$

Thus

$$\sum_{m=0}^{\infty} f_m(z, \overline{z}) t^m = \sum_{m=0}^{\infty} f_m(tz, t\overline{z}) = F(tz, t\overline{z}) = 0.$$

Which means that $f_m(z, \overline{z}) = 0, \forall m \geq 0$, whence it suffices to prove the proposition for homogeneous polynomials of degree m on $\mathbb{C} \times \mathbb{C}$. Suppose that $f_m(z, w) = \sum_{k=0}^{m} a_k z^k w^{m-k}$ satisfies $f_m(z, \overline{z}) = 0, \forall z \in D_p$. Taking $0 < r < p$ and $\theta \in \mathbb{R}$ and putting $z = re^{i\theta}$, we obtain

$$\sum_{k=0}^{m} a_k r^{2k} e^{2ik\theta} e^{-im\theta} = 0.$$

Which implies that

$$\sum_{k=0}^{m} a_k e^{2ik\theta} = 0, \forall \theta \in \mathbb{R}.$$

Since the system $\{e^{2ik\theta}\}$ is linearly independent, we conclude that $a_0 = a_1 = \dots = a_m = 0$. Thus $f_m \equiv 0$ and the proof is complete. ■

1.1.2 Some basic facts about Banach algebras

Theorem 1.1.18. *[Hahn-Banach theorem]*
Let H be a normed space, M a subspace of H and f a bounded linear functional on M. Then there exists a bounded linear functional F on H such that

1. $F(x) = f(x), \forall x \in M$,

2. $\|F\| = \|f\|$.

In other words, there exists an extension F of f which is also bounded linear and preserves the norm.

Theorem 1.1.19. *[Riesz representation theorem]*
If f is a bounded linear functional on a Hilbert space $\mathcal{H}$, then there exists a unique vector $y \in \mathcal{H}$ such that $f(x) = \langle x, y\rangle, \forall x \in \mathcal{H}$ and $\|f\| = \|y\|$.

In the following we give some basic results on Banach algebras which are needed in the sequel.

Definition 1.1.20.

1. *A vector space $\mathbb{E}$ over the field $\mathbb{F}$ is called an algebra if for every pair $(x, y) \in \mathbb{E} \times \mathbb{E}$ a unique product xy is defined with the properties:*

   (a) $(xy)z = x(yz)$,
   (b) $x(y + z) = xy + xz$,
   (c) $(x + y)z = xz + yz$,
   (d) $\lambda(xy) = (\lambda x)y = x(\lambda y)$, *for all $x, y, z \in \mathbb{E}$ and scalar λ.*

2. *$\mathbb{E}$ is an algebra with identity if it contains an element e such that $ex = xe$, $\forall x \in \mathbb{E}$.*

3. *A normed algebra is a normed space which is an algebra such that*

$$\|xy\| \leq \|x\| \|y\|, \forall x, y \in \mathbb{E},$$

   and if $\mathbb{E}$ has an identity e, then $\|e\| = 1$.

4. *A Banach algebra is a normed algebra which is complete, considered as a normed space.*

5. *A Banach algebra $\mathbb{E}$ whose elements satisfy $xy = yx, \forall x, y \in \mathbb{E}$ is called a commutative Banach algebra.*

Definition 1.1.21. *The disk algebra $A(D)$ is the set of all continuous functions on the closure of the unit disk $\overline{D}$, which are analytic in D.*

Definition 1.1.22. *A mapping $x \to x^*$ of an algebra $\mathbb{E}$ into itself is called an involution if it has the following properties for all $x, y \in \mathbb{E}$ and all $\alpha, \beta \in \mathbb{F}$:*

1. $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$.

2. $(xy)^* = y^* x^*$.

3. $(x^*)^* = x$.

Definition 1.1.23. *A Banach algebra $\mathbb{E}$ with an involution satisfying the identity*

$$\|x^* x\| = \|x\|^2 \text{ for all } x \in \mathbb{E},$$

is called a C^*-algebra.

Example 1.1.24. *For a Hilbert space $\mathcal{H}$, $\mathbf{L}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$.*

Definition 1.1.25. *An ideal in a Banach algebra $\mathbb{E}$ is a closed linear subspace $J \subseteq \mathbb{E}$ such that $A \in J$ implies $AB \in J$, $\forall B \in \mathbb{E}$.*

1.1.3 Some facts from operator theory

Definition 1.1.26. *An operator $A : \mathcal{H} \to \mathcal{H}$ is called a diagonal operator if Ae_j is a scalar multiple of e_j, i.e. $Ae_j = \alpha_j e_j$ for all j. Here $\{e_j\}$ is the basis of the Hilbert space $\mathcal{H}$.*

Remark 1.1.27.

1. *The family* $\{\alpha_j\}$ *is called the diagonal of* A *=diag*$\langle \alpha_0, \alpha_1, \ldots \rangle$.
2. *The spectrum of a diagonal operator is the closure of the set of its diagonal terms.*
3. *The norm of a diagonal operator* A *is just* $\|A\| = \sup_j |\alpha_j|$.

Definition 1.1.28. *An isometry is a linear operator* U *on a Hilbert space* $\mathcal{H}$ *such that*

$$\|Uf\| = \|f\|, \text{ for all } f \in \mathcal{H}.$$

Theorem 1.1.29. *For a linear operator* U *on a Hilbert space* $\mathcal{H}$*, the following conditions are equivalent:*

1. $\|Uf\|^2 = \|f\|^2$, *for all* f.
2. $\langle U^*Uf, f\rangle = \langle f, f\rangle$, *for all* f.
3. $\langle U^*Uf, g\rangle = \langle f, g\rangle$, *for all* f *and* g.
4. $U^*U = I$.

Remark 1.1.30. *It is obvious from the previous theorem that a necessary and sufficient condition that a linear operator* U *be an isometry is that* $U^*U = I$.

Definition 1.1.31. *For* f *and* g *in* L^2*, define the rank one operator* $f \otimes g$ *by*

$$\begin{array}{rcl} f \otimes g : L^2 & \longrightarrow & L^2 \\ h & \longrightarrow & (f \otimes g)(h) = \langle h, g\rangle f. \end{array}$$

Remark 1.1.32. *Actually, we have a worth stressing observation regarding this definition: from the above definition we see that its range is at most one-dimensional. In particular, if* f *and* g *are in* L^2*, then* $\{f\}$ *constitutes a basis of the range of* $f \otimes g$*. Therefore, in our opinion, the original definition should be simply: a rank one operator is an operator with one-dimensional range. So that the above definition becomes a property which can be deduced from Riesz representation theorem as follows:*

denote this operator by μ and let the corresponding range be generated by f. Then for any $h \in L^2$, there exists some α_h satisfying $\mu(h) = \alpha_h f$. The correspondence $h \to \alpha_h$ is a bounded linear functional λ say, whence by Riesz representation theorem there exists some $g \in L^2$ satisfying $\lambda(h) = \alpha_h = \langle h, g\rangle$. Thus $\mu(h) = \alpha_h f = \langle h, g\rangle f$. So that the rank one operator is defined by two L^2-functions f, g and is denoted customarily by $(f \otimes g)(h) = \langle h, g\rangle f$.

Now, we introduce some basic results concerning fundamental solutions, in the sense of distributions, which are going to be used later. For further details we refer to [33].

Definition 1.1.33. *E is called the fundamental solution of the operator equation $Au = f$, where A is a differential operator, if $AE = \delta$ where δ is the delta function of Dirac.*

Definition 1.1.34. *We define the convolution of two L^1 functions f and g on $\mathbb{R}$ by*

$$f * g = \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

Lemma 1.1.35. *If A is a differential operator and E is its fundamental solution, then*

$$A(f * E) = f * AE.$$

Remark 1.1.36. *If A is a differential operator then the solution of the operator equation $Au = f$ is $f * E = u$. Indeed, since $AE = \delta$ and δ is the convolution algebra unit, we see that $A(f * E) = f * AE = f * \delta = f$.*

Example 1.1.37. *[6] The fundamental solution of the Cauchy-Riemann operator $\overline{\partial}$ in $\mathbb{C}$ is given by $\frac{1}{\pi z}$. Since $\frac{1}{|z|} = \frac{1}{r} \in L^1_{loc}(\mathbb{R}^2)$, $\frac{1}{z}$ defines a distribution in $\mathbb{R}^2$ of order 0. For any $\phi \in \mathfrak{D}(\mathbb{R}^2)$, where $\mathfrak{D}(\mathbb{R}^2)$ is the space of test functions on $\mathbb{R}^2$, (see [6])*

$$\left\langle \overline{\partial}\frac{1}{z}, \phi \right\rangle = -\left\langle \frac{1}{z}, \overline{\partial}\phi \right\rangle = -\frac{1}{2}\int_{\mathbb{R}^2} \frac{1}{x+iy}\left(\frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y}\right) dxdy.$$

Changing to polar coordinates, we see that

$$\begin{cases} \frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \\ \\ \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}. \end{cases}$$

Hence

$$\left\langle \overline{\partial}\frac{1}{z}, \phi \right\rangle = -\frac{1}{2}\int_0^{2\pi}\int_0^{\infty} \frac{1}{re^{i\theta}}\left[e^{i\theta}\frac{\partial \varphi}{\partial r} + \frac{i}{r}e^{i\theta}\frac{\partial \varphi}{\partial \theta}\right] r dr d\theta,$$

where $\varphi(r,\theta) = \phi(x,y)$. *By Fubini's theorem,*

$$\begin{aligned} \left\langle \overline{\partial}\frac{1}{z}, \phi \right\rangle &= -\frac{1}{2}\int_0^{2\pi}\int_0^{\infty} \frac{\partial \varphi}{\partial r} dr d\theta - \frac{1}{2} i \int_0^{\infty}\frac{1}{r}\int_0^{2\pi}\frac{\partial \varphi}{\partial \theta} d\theta dr \\ &= -\frac{1}{2}\left(-2\pi\varphi(0)\right) - 0, \ \textit{(since } \varphi(r,2\pi) = \varphi(r,0)\textit{)}, \\ &= \pi\phi(0). \end{aligned}$$

Therefore $\overline{\partial}\left(\frac{1}{\pi z}\right) = \delta$. *Moreover, any fundamental solution* E *of* $\overline{\partial}$ *in the space of distributions in* $\mathbb{R}^2$ *is of the form* $E(z) = \frac{1}{\pi z} + h(z)$, *where* h *is an entire function.*

1.1.4 Some facts about certain integral transforms

Definition 1.1.38. *[54] If* $f \in L^1$, *then*

$$\mathcal{F}(f)(t) = \int_{\mathbb{R}} f \ e_{-t} \ dm, \ t \in \mathbb{R}$$

is the Fourier transform of f, *where* $e_t(x) = e^{itx}$.

Theorem 1.1.39. *[54] [Plancherel's theorem]*
One can associate to each $f \in L^2$ *a function* $\mathcal{F}(f) \in L^2$ *so that the following properties hold:*

1. *If* $f \in L^1 \cap L^2$, *then* $\mathcal{F}(f)$*is the Fourier transform of* f.
2. *For every* $f \in L^2$, *we have* $\|\mathcal{F}(f)\|_2 = \|f\|_2$.
3. *The mapping* $f \to \mathcal{F}(f)$ *is a Hilbert space isomorphism of* L^2 *onto* L^2.
4. *The following symmetric relation exists between* f *and* $\mathcal{F}(f)$: *If*

   $$\Phi_A(t) = \int_{-A}^{A} f(x)e^{-ixt}dm(x) \quad \textit{and} \quad \Psi_A(x) = \int_{-A}^{A}\mathcal{F}(f)(t)e^{itx}dm(t),$$

   then, as $A \to \infty$, *we have*

   $$\|\Phi_A - \mathcal{F}(f)\|_2 \to 0 \quad \textit{and} \quad \|\Psi_A - \mathcal{F}(f)\|_2 \to 0.$$

Definition 1.1.40.

1. *If* $f \in L^1(0,1)$ *then its Mellin transform is defined to be*

$$\widetilde{f}(z) = \int_0^1 f(r) r^{z-1} dr.$$

2. *For* f *and* g *in* L^1, *the Mellin convolution of* f *and* g *is defined by*

$$(f * g)(t) = \int_t^1 f(s) g\left(\frac{t}{s}\right) \frac{ds}{s}.$$

Remark 1.1.41.

1. $\widetilde{f}$ *is bounded and holomorphic in the half plane* $\mathbf{\Pi} = \{z : \mathcal{R}e(z) > 1\}$ *as*

$$\begin{aligned} \left|\widetilde{f}(z)\right| &\leq \int_0^1 |f(r)| \left|r^{z-1}\right| dr \leq \sup_{r \in (0,1)} \left|r^{z-1}\right| \int_0^1 |f(r)| \, dr \\ &\leq \int_0^1 |f(r)| \, dr < \infty, \ (\text{as } f \in L^1(0,1)). \end{aligned}$$

2. $f * g \in L^1$. *Indeed, since* $\int_0^1 \int_0^t |f(s)| \left|g\left(\frac{t}{s}\right)\right| \frac{ds}{s} dt \geq 0$, *we see that*

$$\begin{aligned} \|f * g\|_1 &= \int_0^1 \left| \int_t^1 f(s) g\left(\frac{t}{s}\right) \frac{ds}{s} \right| dt \leq \int_0^1 \int_t^1 |f(s)| \left|g\left(\frac{t}{s}\right)\right| \frac{ds}{s} dt \\ &= \int_0^1 \int_0^1 |f(s)| \left|g\left(\frac{t}{s}\right)\right| \frac{ds}{s} dt - \int_0^1 \int_0^t |f(s)| \left|g\left(\frac{t}{s}\right)\right| \frac{ds}{s} dt \\ &\leq \int_0^1 \int_0^1 |f(s)| \left|g\left(\frac{t}{s}\right)\right| \frac{ds}{s} dt. \end{aligned}$$

Now, if we make the variable change $r = \frac{t}{s}$ *then* $dt = sdr$, *we see that*

$$\begin{aligned} \int_0^1 \int_0^1 |f(s)| \left|g\left(\frac{t}{s}\right)\right| \frac{ds}{s} dt &= \int_0^1 |f(s)| \left(\int_0^1 \left|g\left(\frac{t}{s}\right)\right| dt \right) \frac{ds}{s} \\ &= \int_0^1 |f(s)| \left(\int_0^s s \, |g(r)| \, dr \right) \frac{ds}{s} \\ &\leq \int_0^1 |f(s)| \left(\int_0^1 s \, |g(r)| \, dr \right) \frac{ds}{s} \\ &= \int_0^1 |g(r)| \, dr \int_0^1 |f(s)| \, ds \\ &= \|f\|_1 \, \|g\|_1 \, . \end{aligned}$$

Thus

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1 < \infty.$$

3. $\widetilde{(f * g)}(z) = \widetilde{f}(z)\widetilde{g}(z)$.

4. *The Mellin transform is one-to-one.*

1.1.5 Reproducing kernel Hilbert spaces

Let $\mathcal{H} := \{f : \Omega \subset \mathbb{C} \to \mathbb{C}\}$ be a Hilbert space of functions equipped with inner product $\langle ., . \rangle$. For each $z \in \Omega$ a map μ_z from $\mathcal{H}$ into $\mathbb{C}$ can be defined as follows:

$$\begin{aligned} \mu_z : \mathcal{H} &\longrightarrow \mathbb{C} \\ f &\longrightarrow \mu_z f = f(z). \end{aligned}$$

It is called the point evaluation map. If such a functional is bounded, by Riesz representation theorem, there exists some representative element, denoted $K_z \in \mathcal{H}$, satisfying

$$\mu_z(f) = f(z) = \langle f, K_z \rangle.$$

K_z is called the reproducing kernel of $\mathcal{H}$, (or kernel function), because it reproduces the value of the function f at the point z. Such a Hilbert space is called, in this case, a reproducing kernel Hilbert space, (or sometimes a functional Hilbert space). The reproducing kernel $K_z(.) = K(., z)$ satisfies some properties such as:

- For every $z \in \Omega$, one has $f(z) = \langle f, K_z \rangle$, $\forall f \in \mathcal{H}$.
- $K(z, w) = \overline{K(w, z)}$.
- $|K(w, z)| \leq \|K_w\| \|K_z\|$.
- If $\mathcal{H}$ admits a basis $\{e_n\}_n$, then $K(w, z) = \sum_n \overline{e_n(z)} e_n(w)$.
- $\sum_{i=1}^N \sum_{j=1}^N c_i \overline{c_j} K(z_i, z_j) \geq 0$, $\forall c_1, \dots, c_N \in \mathbb{C}$, $\forall z_1, \dots, z_N \in \Omega$.

An important concept providing a powerful tool in the investigation of functions as well as operators on $\mathcal{H}$ is the so called Berezin transform. Let $k_z = \frac{K_z}{\|K_z\|}$ be the normalized reproducing kernel. The Berezin transform of a function $\varphi \in \mathcal{H}$ is defined to be the function

$$\begin{aligned} B(\varphi) : \Omega &\longrightarrow \mathbb{C} \\ z &\longrightarrow B(\varphi)(z) := \langle \varphi k_z, k_z \rangle. \end{aligned}$$

For a bounded operator S on $\mathcal{H}$, the Berezin transform of S is defined to be the function

$$B(S) = \langle S k_z, k_z \rangle, z \in \Omega.$$

Now, let D be the unit disk in the complex plane $\mathbb{C}$ and let ∂D be its boundary (the unit circle). Denote by dA the Lebesgue area measure on D and by $d\theta$ the Lebesgue measure on ∂D. Recall the familiar Lebesgue space of square summable (class of) functions: $L^2(D, dA)$ and $L^2(\partial D)$; they are both known to be separable Hilbert spaces. An interesting class of such function spaces is the class of holomorphic functions. So, let us denote by H^2 the Hardy space which is a closed subspace of $L^2(\partial D)$ consisting of those functions having analytic extensions to D. Similarly, denote by L_a^2 the Bergman space which is the subspace of $L^2(D, dA)$ consisting of analytic functions in the unit disk. Fortunately, both spaces H^2 and L_a^2 are reproducing kernel Hilbert spaces. We shall give the explicit expression of their kernel functions in Sections 1.2 and 1.3 respectively. For simplicity let $\mathbb{H}$ denotes either L_a^2 or H^2 and let $\mathbb{L}$ denotes either $L^2(D, dA)$ or $L^2(\partial D)$, respectively.

On $\mathbb{L}$ a well-known operator is defined, namely the multiplication operator M_φ, (or

Laurent operator), with bounded symbol φ defined as follows:

$$\begin{aligned} M_\varphi : \mathbb{L} &\longrightarrow \mathbb{L} \\ f &\longrightarrow M_\varphi(f) := \varphi f. \end{aligned}$$

Since $\mathbb{H}$ is a closed subspace of $\mathbb{L}$, there exists an orthogonal projection P from $\mathbb{L}$ to $\mathbb{H}$. Denote by $\mathbb{H}^\perp$ the orthogonal complement of $\mathbb{H}$ in $\mathbb{L}$; it is also a closed subspace. Moreover, since $\mathbb{H}$ is a proper non-dense subspace of $\mathbb{L}$, $\mathbb{H}^\perp$ is non-trivial; and there is an orthogonal projection $I - P$ from $\mathbb{L}$ onto $\mathbb{H}^\perp$.

Now, if on $\mathbb{H}$ one considers a multiplication followed by a projection, a new operator T_φ is then defined. It is called customarily the Toeplitz operator, namely

$$\begin{aligned} T_\varphi : \mathbb{H} &\longrightarrow \mathbb{H} \\ f &\longrightarrow T_\varphi(f) := (PM_\varphi)(f) = P(\varphi f). \end{aligned}$$

In particular, the Berezin transform of the Toeplitz operator T_φ is given by

$$B(T_\varphi)(z) = \langle T_\varphi k_z, k_z \rangle = \langle P(\varphi k_z), k_z \rangle = \langle \varphi k_z, k_z \rangle = B(\varphi)(z).$$

Thus, the Berezin transform of a Toeplitz operator is the Berezin transform of its symbol.

By analogy, another operator H_φ can be defined as a multiplication on $\mathbb{H}$ followed by a projection onto $\mathbb{H}^\perp$. It is called the Hankel operator and it is defined by

$$\begin{aligned} H_\varphi : \mathbb{H} &\longrightarrow \mathbb{H}^\perp \\ f &\rightarrow H_\varphi(f) = (I - P)M_\varphi(f) = (I - P)(\varphi f). \end{aligned}$$

Now we turn to the most classical space, namely

1.2 Hardy spaces and their operators

Regarding the Hardy space, Subsection 1.2.1 is devoted to its main properties; for further details we refer to [24, 25, 38, 49, 67]. Hankel operators, (in the form of matrices), were considered quite earlier by the middle of the 19th century, by Hankel himself and Krönecker. In the fifties of the 20th century, Nehari and Hartman have done a great progress in the study of Hankel operators on the Hardy space. Since then, the theory of Hardy space Hankel operators has been in development. For more details, we refer to the widely used textbooks [49, 50, 52, 67]; see also Subsection 1.2.2. Also for the recent work, we refer to [43, 44, 45, 65, 66].

Toeplitz matrices, (that are constants along diagonals parallel to the main one), have been known for a while. Actually, they define a class of operators called Toeplitz operators. Such theory was less understood till 1963, when the famous paper by A. Brown and P. R. Halmos [16] appeared. In that paper, the modern theory of Hardy space Toeplitz operators was elaborated, namely they have given the most important properties of such operators and they have even established a strategy with. This is why that paper is acknowledged by specialists to be the pillar of this area. Hardy space Toeplitz operators are by now well understood and a large corresponding bibliography is available; we refer for example to [16, 24, 38, 49, 64, 67] for basics and to [11, 16, 34, 35, 38, 55, 63, 65, 66] for products and commutativity, as well as to [16, 24] for their spectral properties and to [24, 28] for C^*-algebras generated by them. Now, let us establish the main results obtained so far regarding the material just described.

1.2.1 Hardy spaces

Let $\mathbb{C}$ be the complex plane, and let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $\partial D = \{z \in \mathbb{C} : |z| = 1\}$ be the open unit disk and its boundary in $\mathbb{C}$, respectively, and $d\theta$ be the arc-length measure on ∂D. For $1 \le p < \infty$, $L^p(\partial D)$ denotes the Banach space of Lebesgue measurable functions f on ∂D with

$$\|f\|_p = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^p \, d\theta\right]^{\frac{1}{p}} < \infty.$$

$L^\infty(\partial D)$ denotes the Banach space of bounded measurable functions f on ∂D with

$$\|f\|_\infty = \text{ess.}\sup\{|f(\theta)|, \theta \in [0, 2\pi]\} < \infty.$$

Since $d\theta$ is $\sigma-$finite, $L^p(\partial D) \subset L^q(\partial D)$ for all $p \ge q \ge 1$. Given $f \in L^1(\partial D)$, the Fourier coefficients of f are given by

$$a_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta, \qquad n \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of all integers.

First, we give the definition of Hardy spaces.

Definition 1.2.1. *For $1 \le p \le \infty$, the Hardy space of ∂D, denoted by H^p, is the subspace of $L^p(\partial D)$ consisting of functions f with $a_n(f) = 0$ for all negative integers n.*

Proposition 1.2.2. *For $1 < p \le \infty$, H^p is a closed subspace of $L^p(\partial D)$.*

Remark 1.2.3.

1. *For $1 < p \le \infty$, H^p is a Banach space.*
2. *$L^2(\partial D)$ is a Hilbert space with the inner product*

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g}(\theta)d\theta, \quad \textit{for } f, g \in L^2(\partial D).$$

3. *In particular, H^2 is a closed subspace of the Hilbert space $L^2(\partial D)$, and we have*

$$L^2(\partial D) = H^2 \oplus \left(H^2\right)^{\perp}$$

where $(H^2)^{\perp}$ is the orthogonal complement of H^2.

4. *The set of trigonometric polynomials $\wp = \left\{\sum_{n=-N}^{N} \alpha_n z^n, \alpha_n \in \mathbb{C}\right\}$ is a self-adjoint algebra of $C(\partial D)$. The uniform closure of $\wp$ is $C(\partial D)$.*

   The set of analytic trigonometric polynomials $\wp_+ = \left\{\sum_{n=0}^{N} \alpha_n z^n, \alpha_n \in \mathbb{C}\right\}$ is dense in H^2. Accordingly H^∞ is dense in H^2.

5. *In fact there is an intimate relation between the spaces introduced so far and their analogs defined on the disk. This relation is confirmed by the well-known reverse procedures we are going to see later, namely the Poisson extension and radial limits.*

6. *$L^2(\partial D)$ is a separable Hilbert space and so is H^2. The corresponding generic orthonormal basis is given in the following proposition.*

Proposition 1.2.4. *Let $e_n(\theta) = e^{in\theta}$, where $\theta \in [0, 2\pi]$ and $n \in \mathbb{Z}$. Then, $\{e_n\}_{-\infty}^{\infty}$ is an orthonormal basis of $L^2(\partial D)$. In particular, $\{e_n\}$ for $n \in \{0, 1, 2, \ldots\}$ is an orthonormal basis of the Hardy space H^2.*

Now, we define the Hardy space of the unit disk.

Definition 1.2.5. *For $1 \leq p < \infty$, we shall let $H^p(D)$ denote the space of analytic functions f on D with*

$$\sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right) < \infty,$$

and $H^\infty(D)$ denotes the space of bounded analytic functions on D.

Remark 1.2.6. *For $1 \leq p < \infty$, by 3.2 in [49], we have that $H^p(D)$ is a normed space with norm defined by*

$$\|f\|_p = \left\{\sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right)\right\}^{\frac{1}{p}} < \infty,$$

and for $H^\infty(D)$, we have $\|f\|_\infty = \sup\limits_{z \in D} |f(z)|$.

The reproducing kernel of $H^2(D)$

Let us first verify that point evaluation is a bounded linear functional on $H^2(D)$, which means that it is a functional Hilbert space. The main tool we are using is Cauchy's formula. For $z \in D$ and $|z| < r < 1$, let Γ_r be the circle centered at the origin and with radius r. Then for any f in H^P we have

$$\begin{aligned} |f(z)| &= \left| \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \\ &= \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta} - z} rie^{i\theta} d\theta \right| \\ &\leq \int_0^{2\pi} \left| \frac{f(re^{i\theta})}{r - ze^{-i\theta}} \right| r \frac{d\theta}{2\pi}. \end{aligned}$$

Hölder's inequality implies that

$$|f(z)| \leq \|f_r\|_{L^p} \left\| \frac{r}{r - ze^{-i\theta}} \right\|_{L^q}.$$

For any $z \in D$, $\frac{r}{r-ze^{-i\theta}}$ converges uniformly, as $r \to 1$, to the bounded function $\frac{1}{1-ze^{-i\theta}}$ on ∂D. Thus the point evaluation is a bounded linear functional on $H^p(D)$. The boundedness of the point evaluation functional implies, (by Riesz representation theorem), the existence of a reproducing kernel at each point of the disk, which means that $H^2(D)$ is a reproducing kernel Hilbert space. In other words, for each $z \in D$ there exists a unique function K_z in $H^2(D)$ such that

$$f(z) = \langle f, K_z \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1 - ze^{-i\theta}} f(e^{i\theta}) d\theta.$$

This kernel is called the Szegö kernel and is given by

$$K_z(w) = \frac{1}{1 - \overline{z}w}.$$

Let us show that indeed this kernel function reproduces the value of an H^2-function f at a point z. Let $z \in D$ and consider the expansion

$$K_z(w) = \frac{1}{1-\overline{z}w} = \sum_{n=0}^{\infty} \overline{z}^n w^n.$$

If $f \in H^2(D)$ admits a Taylor series expansion, say $\sum_{n=0}^{\infty} a_n z^n$, we infer that

$$\langle f, K_z \rangle = \sum_{n=0}^{\infty} a_n \overline{\overline{z}}^n = \sum_{n=0}^{\infty} a_n z^n = f(z).$$

In order to normalize the reproducing kernel, we compute its norm

$$\|K_z\|_2^2 = \langle K_z, K_z \rangle = K_z(z) = \frac{1}{1-|z|^2}.$$

Therefore, the normalized reproducing kernel takes the form

$$k_z(w) = \frac{K_z(w)}{\|K_z\|} = \frac{\sqrt{1-|z|^2}}{1-\overline{z}w}.$$

Poisson extension

Given $f \in L^1(\partial D)$, the harmonic extension of f to D, denoted by $\widehat{f}(z)$, is defined by

$$\widehat{f}(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) P_z(\theta) d\theta, \;\; z \in D,$$

where

$$P_z(\theta) = \frac{1-|z|^2}{|1-\overline{z}e^{i\theta}|^2} = \mathcal{R}e \left(\frac{e^{i\theta}+z}{e^{i\theta}-z} \right),$$

is the Poisson kernel of D. Being the real part of an analytic function, $P_z(\theta)$ is harmonic in z for any fixed $\theta \in [0, 2\pi]$. It follows that $\widehat{f}(z)$ is harmonic in D. Clearly, by Poisson extension one can go from $H^p(\partial D)$ to $H^p(D)$. The reverse sense is also true but it is less trivial. In fact, in a non-trivial way it was proved that for $f \in H^p$ the radial limit

$$f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta}),$$

exists for almost all θ and defines a function from $H^p(D)$ into $H^p(\partial D)$.

Theorem 1.2.7. *For $1 \leq p \leq \infty$, there is an isometric isomorphism between $H^p(D)$ and H^p.*

Proof: See Theorem 3.4.1 in [49]. ■

In particular, $H^2(D)$ can be viewed as the Hilbert space of square summable analytic functions, which is a subspace of $L^2(D, dA)$ and one has

$$L^2(D, dA) = H^2(D) \oplus (H^2(D))^{\perp}.$$

Proposition 1.2.8. *The Banach space H^∞ is in fact a Banach algebra.*

Proof: See Section 6.3 of [24]. ■

Remark 1.2.9. *By Proposition 1.2.8 and from identifying $H^\infty(D)$ and H^∞, we see that $H^\infty(D)$ is a Banach algebra too. It is called the algebra of bounded analytic functions.*

Duals of H^p–Spaces and $BMOA$

Let $1 < p < \infty$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$, then the dual of the space H^p under the duality pairing

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\overline{g}(\theta)d\theta,$$

is given by

$$(H^p)^* \cong H^q.$$

With regard to the dual of H^1, let us first introduce the space $BMOA$.

For a function f and an interval I contained in ∂D, define the mean of f over I by

$$f_I = \frac{1}{|I|} \int_I f(\theta)d\theta,$$

where $|I|$ is the length of I. The space BMO, of functions with bounded mean oscillation, is defined by

$$BMO = \{f \in L^2(\partial D) : \|f\|_{BMO} < \infty\},$$

with norm

$$\|f\|_{BMO} = \sup_I \left(\frac{1}{|I|} \int_I |f(\theta) - f_I|^2 \, d\theta \right)^{\frac{1}{2}}.$$

Next, we define the space $BMOA$ to be $BMOA = BMO \cap H^2$. So that a function f is in $BMOA$ if and only if it is analytic and has bounded mean oscillation (i.e. $\|f\|_{BMO} < \infty$). It turns out that the dual of H^1 is the space just defined, in other words

$$(H^1)^* \cong BMOA.$$

Note also that $L^\infty(\partial D) \subset BMO$ and $BMO = BMOA + \overline{BMOA}$, that is, f is in BMO if and only if $f = f_1 + \overline{f_2}$ with f_1 and f_2 both in $BMOA$. Moreover, this decomposition is unique if we require that $f_2(0) = 0$.

1.2.2 Haplitz operators on the Hardy space

It is customary that the main component of operator theory on both Hardy and Bergman spaces is the orthogonal projection; so we might start with

The orthogonal projection on $H^2(\partial D)$

Since $H^2(\partial D)$ is a closed subspace of the Hilbert space $L^2(\partial D)$, there exists an orthogonal projection from $L^2(\partial D)$ onto $H^2(\partial D)$ denoted by P, i.e.

$$\begin{aligned} P : L^2 = H^2 \oplus (H^2)^\perp &\longrightarrow H^2 \\ f &\longrightarrow P(f) = P_S(f_1 + f_2) = f_1. \end{aligned}$$

Let $\widehat{P}$ be the composition of P with the harmonic extension, i.e.

$$\widehat{P}(f) = \widehat{Pf}, \ \forall f \in L^2(\partial D).$$

$\widehat{P}$ maps $L^2(\partial D)$ onto $H^2(D)$ and it is a projection in the sense that when applied to $f \in H^2(D)$ gives its Poisson extension. It is in fact an integral operator:

$$\widehat{P}f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)}{1 - ze^{-i\theta}} d\theta, \quad f \in L^2(\partial D).$$

It is called the Szegö projection and the underlying kernel, namely $\frac{1}{1-ze^{-i\theta}}$, is called the Szegö kernel.

The multiplication operator M_φ on $L^2(\partial D)$

For $\varphi \in L^\infty(\partial D)$, define the multiplication operator, (or Laurent operator according to [16]), M_φ as follows

$$\begin{aligned} M_\varphi : L^2(\partial D) &\longrightarrow L^2(\partial D) \\ f &\longrightarrow M_\varphi(f) = \varphi f. \end{aligned}$$

The multiplication here is to be understood in the obvious sense, namely it is the pointwise one: $\varphi f(e^{i\theta}) = \varphi(e^{i\theta})f(e^{i\theta})$, for all $\theta \in [0, 2\pi]$.

Remark 1.2.10. *Clearly the multiplication operator is bounded, since $\varphi \in L^\infty(\partial D)$, then*

$$\|M_\varphi(f)\|_2 = \|\varphi f\|_2 \leq \|\varphi\|_\infty \|f\|_2, \ \textit{for all } f \in L^2(\partial D).$$

Now we are in the position to introduce the concept of Toeplitz operators.

Toeplitz operators on $H^2(\partial D)$

Given $f \in L^2(\partial D)$, a linear operator T_f can be defined on $H^2(\partial D)$ as follows

$$\begin{aligned} T_f : H^2 &\longrightarrow H^2 \\ g &\longrightarrow T_f(g) = P(fg). \end{aligned}$$

This definition makes sense since T_f can be defined first on H^∞ which is dense in $H^2(\partial D)$. T_f is called the Toeplitz operator with symbol f. It is in fact a multiplication followed by projection.

Remark 1.2.11.

1. *For any $z \in D$, the vector*

$$k_z(t) = \frac{\sqrt{1-|z|^2}}{1-\overline{z}e^{it}},$$

is the normalized reproducing kernel of H^2.

2. *The harmonic extension of f to D is given also by*

$$\widehat{f}(z) = \langle T_f k_z, k_z\rangle, z \in D.$$

Which corresponds to the Berezin transform as we are going to see in Chapter 2. In other words, on H^2 the Berezin transform coincides with Poisson extension.

3. *Since P is bounded and has norm one, we clearly have that for $f \in L^\infty(\partial D)$ that*

$$\begin{aligned} \|T_f(g)\|_2 &= \|P(fg)\|_2 \leq \|P\| \, \|fg\|_2 \\ &\leq \|fg\|_2 \leq \|f\|_\infty \|g\|_2 \,, \text{ for all } g \in H^2. \end{aligned}$$

Thus $\|T_f\| \leq \|f\|_\infty$.

4. *If $\varphi \in H^\infty$, then T_φ coincides with the multiplication operator M_φ on H^2 and in this case it is called the analytic Toeplitz operator.*

The following result shows that the boundedness of f is necessary and sufficient for T_f to be bounded.

Proposition 1.2.12. *[67] The Toeplitz operator T_f is bounded if and only if f is in $L^\infty(\partial D)$. Moreover, $\|T_f\| = \|f\|_\infty$.*

The following result states that there is no non-zero compact Toeplitz operators on the Hardy space.

Theorem 1.2.13. *[67] Suppose that $\phi \in L^\infty(\partial D)$, Then T_ϕ is compact if and only if $\phi = 0$.*

Matrix of the Toeplitz operator on $H^2(\partial D)$

By a Toeplitz matrix it is meant a matrix which is constant along diagonals parallel to the main one. In fact, Toeplitz matrices were known for a while and extensive studies have been done on them; and only by the sixties, the operator theoretic approach was introduced by Brown and Halmos [16] to study them from another point of view. Since then people speak about Toeplitz operators rather than Toeplitz matrices. The relationship between such matrices and the modern theory of Toeplitz operators on the Hardy space is explained as follows: Let $\varphi \in L^\infty(\partial D)$, and recall that $\{e_n(\theta) = e^{in\theta}, n \in \{0, 1, 2, ...\}\}$ is the generic orthonormal basis of the Hardy space H^2. Since T_φ defined from $H^2(\partial D)$ into $H^2(\partial D)$ is a linear operator, then it has the form

$$T_\varphi e_n = \sum_{m=0}^{\infty} a_{mn} e_m = \sum_{m=0}^{\infty} \langle T_\varphi e_n, e_m \rangle e_m,$$

where (a_{mn}) is the matrix of T_φ.

Now, as $\varphi \in L^\infty(\partial D) \subseteq L^2(\partial D)$, we have also

$$\varphi = \sum_{k=-\infty}^{\infty} b_k e_k,$$

where $b_k \in \mathbb{C}$ and $\{e_n\}_{-\infty}^{\infty}$ is the orthonormal basis of $L^2(\partial D)$. Thus

$$T_\varphi e_n = P(\varphi e_n) = P\left(\sum_{k=-\infty}^{\infty} b_k e_k e_n\right) = \sum_{n+k=0}^{\infty} b_k e_{k+n}.$$

So that

$$a_{mn} = \langle T_\varphi e_n, e_m \rangle = \left\langle \sum_{n+k=0}^{\infty} b_k e_{k+n}, e_m \right\rangle = \sum_{n+k=0}^{\infty} b_k \langle e_{k+n}, e_m \rangle = b_{m-n}.$$

Hence $a_{mn} = b_{m-n}$, $a_{m+1,n+1} = b_{m-n}, \ldots\ldots\ldots\ldots$, for all $m, n \in \{0, 1, 2, \ldots\ldots\}$. Therefore, we infer that the matrix of T_φ reads as

$$(a_{mn}) = \begin{pmatrix} a_{00} & a_{01} & a_{02} & . & . \\ a_{10} & a_{11} & a_{12} & . & . \\ a_{20} & a_{21} & a_{22} & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix} = \begin{pmatrix} b_0 & b_{-1} & b_{-2} & . & . \\ b_1 & b_0 & b_{-1} & . & . \\ b_2 & b_1 & b_0 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix}.$$

Remark 1.2.14. *According to Theorem 4.1.4 of [49] we see that this matrix actually characterizes Toeplitz operators. In particular, if T is any linear operator on H^2 with the above matrix under the standard basis, then $T = T_f$ with*

$$f(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}.$$

Denote by $\mathbf{S}$ the Toeplitz operator T_z. It is called the unilateral shift operator and it plays a capital role in the theory of Toeplitz operators. For instance, besides the above matricial criterion, we have the following characterization of Hardy space Toeplitz operators

Theorem 1.2.15. *A bounded linear operator T on H^2 is a Toeplitz operator if and only if*

$$\mathbf{S}^* T \mathbf{S} = T.$$

Some immediate algebraic properties of Hardy space Toeplitz operators can be listed in the following

Proposition 1.2.16. *If f and g are bounded on ∂D and λ is a complex number. Then the Hardy space Toeplitz operator has the following properties:*

1. $T_0 = 0$ *and* $T_1 = I$.
2. $T_{f+\lambda g} = T_f + \lambda T_g$.
3. $T_f^* = T_{\overline{f}}$.
4. T_f *is self-adjoint if and only if* f *is real.*
5. $T_f = 0$ *if and only if* $f = 0$.
6. *If* f *is in* H^∞, *then* $T_g T_f = T_{gf}$.
7. *If* f *is in* H^∞, *then* $T_{\overline{f}} T_g = T_{\overline{f}g}$.

Products of Hardy space Toeplitz operators:

The set of all Toeplitz operators on H^2 is not commutative and not closed under multiplication. Indeed, the unilateral shift and its adjoint are Toeplitz operators

$$T_z = T_{e^{it}} = \mathbf{S} \text{ and } T_z^* = T_{e^{-it}} = \mathbf{S}^*.$$

The product $\mathbf{S}^*\mathbf{S} = T_1 = I$ is a Toeplitz operator. But $\mathbf{SS}^*$ is not a Toeplitz operator because

$$\mathbf{S}^*(\mathbf{SS}^*)\mathbf{S} = (\mathbf{S}^*\mathbf{S})(\mathbf{S}^*\mathbf{S}) = (I)\,(I) = I \neq \mathbf{SS}^*.$$

From one hand the underlying set is not closed under multiplication since $\mathbf{SS}^*$ is not Toeplitz, on the other hand it is not commutative since $\mathbf{SS}^* \neq \mathbf{S}^*\mathbf{S}$.

Accordingly, it will be interesting to know when the product of Toeplitz operators is a Toeplitz one and when a couple of Toeplitz operators commute. Because Brown-Halmos' approach to Toeplitz operators was based on matrices, most of their results based on them. In particular, the following "product matrix formula" was particularly important in their work [16].

Lemma 1.2.17. *Let T_φ and T_ψ be Toeplitz operators with matrices $(\alpha_{i-j})\,,(\beta_{i-j})$ respectively and (c_{i-j}) be the matrix of $T_\varphi T_\psi$, then we have the product matrix formula*

$$c_{(i+1)(j+1)} = c_{ij} + \alpha_{i+j}\beta_{-j-1}.$$

Proof: We have

$$c_{ij} = \sum_{k=0}^{\infty} \alpha_{i-k}\beta_{k-j}.$$

Thus

$$\begin{aligned} c_{(i+1)(j+1)} &= \sum_{k=0}^{\infty} \alpha_{i-k+1}\beta_{k-j-1} = \alpha_{i+1}\beta_{-j-1} + \sum_{k=1}^{\infty} \alpha_{i-k+1}\beta_{k-j-1} \\ &= \alpha_{i+1}\beta_{-j-1} + \sum_{k=0}^{\infty} \alpha_{i-k}\beta_{k-j} = c_{ij} + \alpha_{i+1}\beta_{-j-1}. \end{aligned}$$

Which ends the proof. ■

The following theorem for commuting Toeplitz operators on the Hardy space H^2 is due to Brown and Halmos [16]. The original proof is based on matrices. Modern proofs based on recent advanced techniques can also be given. But still we keep the original proof in order to exhibit various methods.

Theorem 1.2.18. *A necessary and sufficient condition that two Toeplitz operators commute is that either both be holomorphic, or both be co-holomorphic or one be a linear function of the other.*

Proof: *Sufficiency*: Let $f, g \in L^\infty(\partial D)$ and T_f,T_g be the corresponding Toeplitz operators. If f, g are holomorphic, then by (6) of Proposition 1.2.16, we have

$$T_g T_f = T_{fg} = T_f T_g.$$

Similarly, if f, g are co-holomorphic, then

$$T_g T_f = T_{fg} = T_f T_g.$$

If $f = \alpha g + \beta$, $\alpha,\beta \in \mathbb{C}$, then

$$T_f = T_{\alpha g+\beta} = \alpha T_g + \beta.$$

So that

$$T_f T_g = \alpha T_g^2 + \beta T_g = T_g T_f.$$

Necessity: regarding the necessity, it is less trivial. It relies on the above product matrix formula. Suppose that

$$T_f T_g = T_g T_f,$$

where $f, g \in L^\infty(\partial D)$, with $f = \sum_{-\infty}^{\infty} a_i e_i$ and $g = \sum_{-\infty}^{\infty} b_j e_j$. Recall the matrix product formula:

$$c_{i+1,j+1} = c_{ij} + a_{i+1} b_{-j-1},$$

with (c_{ij}) denotes the matrix of $T_f T_g$. Apply this formula for both $T_f T_g$ and $T_g T_f$, we obtain via the identity $T_f T_g = T_g T_f$:

$$a_{i+1} b_{-j-1} = b_{i+1} a_{-j-1}, \tag{1.2.1}$$

whenever $i, j \geq 0$. We distinguish several cases

1. If $f = 0$ (or $g = 0$), the assertion is trivially satisfied.

2. If $b_{-j-1} = a_{-j-1} = 0$ for all $j \geq 0$, then f and g are holomorphic functions.

3. If $a_{i+1} = b_{i+1} = 0$ for all $i \geq 0$, then f and g are co-holomorphic.

4. If non of the above cases occurs, then there exist non-negative integers i_o and j_o, such that $a_{i_o+1} \neq 0$ and $b_{-j_0-1} \neq 0$. Then put

$$\frac{b_{-j_0-1}}{a_{j_o-1}} = \frac{b_{i_o+1}}{a_{i_o+1}} = \lambda.$$

Thus by Equation (1.2.1), we obtain $b_{-j-1} = \lambda a_{-j-1}$ for $j \geq 0$, and $b_{i+1} = \lambda a_{i+1}$ for $i \geq 0$. Hence for all $k \neq 0$, we have $b_k = \lambda a_k$. And g becomes

$$g = \sum_{i=0}^{\infty} b_i e_i = b_0 e_0 + \lambda \sum_{i=1}^{\infty} a_i ei + \lambda a_0 - \lambda a_0.$$

Therefore

$$g = \lambda f + (b_0 - \lambda a_0),$$

i.e., there are constants $\alpha, \beta \in \mathbb{C}$ such tat $g = \alpha f + \beta$. In other words:

$$T_g - b_0 = \lambda \left(T_f - a_0\right).$$

and the proof is complete ■

Now, we give one of the most important results in the theory of Toeplitz operators on the Hardy space. This theorem is due to Brown-Halmos [16]. Since then, any theorem of this kind is called a Brown-Halmos type theorem.

Definition 1.2.19. *A theorem is called a Brown-Halmos type theorem if it answers the following question: when is the product of two Toeplitz operators a Toeplitz operator?*

The original Brown-Halmos theorem is the following:

Theorem 1.2.20. *Let $\varphi, \psi \in L^{\infty}(\partial D)$. A necessary and sufficient condition that the product $T_\varphi T_\psi$ of two Toeplitz operators be a Toeplitz operator is that either φ be co-holomorphic or ψ be holomorphic, in which case $T_\varphi T_\psi = T_{\varphi\psi}$.*

Proof: *The sufficiency:* if ψ is holomorphic, then for any $f \in H^2$ we have

$$T_\varphi T_\psi(f) = T_\varphi(P(\psi f)) = T_\varphi(\psi f) = P(\varphi\psi f) = P((\varphi\psi)f) = T_{\varphi\psi}(f).$$

Thus $T_\varphi T_\psi = T_{\varphi\psi}$.

If φ is co-holomorphic, then

$$T_\varphi T_\psi = (T_\psi^* T_\varphi^*)^* = (T_{\overline{\psi}} T_{\overline{\varphi}})^* = (T_{\overline{\varphi\psi}})^* = T_{\varphi\psi}.$$

The necessity: suppose $T_\varphi T_\psi$ is Toeplitz operator, then the matrix (c_{ij}) of $T_\varphi T_\psi$ is the Toeplitz matrix, i.e. ($c_{ij} = c_{(i+1)(j+1)}$). Let (α_{ij}) and (β_{ij}) be the Toeplitz matrices of T_φ and T_ψ respectively, then from Lemma 1.2.17 we have

$$c_{(i+1)(j+1)} = c_{ij} + \alpha_{1+j}\beta_{-j-1},$$

thus $\alpha_{i+1}\beta_{-j-1} = 0$, for all $i, j \geq 0$. Then $\alpha_{i+1} = 0$ for all $i \geq 0$, or $\beta_{-j-1} = 0$ for all $j \geq 0$ (since if there is $k \geq 0$, such that $\alpha_{k+1} \neq 0$, then $\beta_{-j-1} = 0$ for all $j \geq 0$, as $\alpha_{k+1}\beta_{-j-1} = 0$ for all $j \geq 0$). Thus

$$\varphi = \sum_{-\infty}^{i=0} \alpha_i e_i \quad \text{or} \quad \psi = \sum_{i=0}^{\infty} \beta_i e_i.$$

Hence φ is co-holomorphic or ψ is holomorphic. ■

Corollary 1.2.21. *A necessary and sufficient condition that the product $T_\varphi T_\psi$ of two Toeplitz operators on the Hardy space H^2 be zero is that at least one factor be zero. In other words, among the class of Toeplitz operators there are no zero divisors.*

Proof: Clearly if $T_\varphi = 0$ or $T_\psi = 0$, then $T_\varphi T_\psi = 0$. Now, suppose $T_\varphi T_\psi = 0$, then $T_\varphi T_\psi$ is a Toeplitz operator as its matrix is a Toeplitz matrix. And by Theorem 1.2.20 we see that either φ is co-holomorphic or ψ is holomorphic and $T_\varphi T_\psi = T_{\varphi\psi} = 0$. Thus $\varphi\psi = 0$ a.e. on D. We have several cases:

1. If ψ or φ is zero, then the problem is solved.

2. If $\psi \neq 0$ and $\varphi \neq 0$, then

   (a) If $\varphi\psi = 0$ and ψ is holomorphic, then $\psi = 0$.

(b) If $\varphi\psi = 0$ and φ is co-holomorphic, then $\overline{\psi}\overline{\varphi} = 0$ and $\overline{\varphi}$ is holomorphic. Thus $\overline{\varphi} = 0$, that is $\varphi = 0$.

Hence we infer that either $T_\varphi = 0$ or $T_\psi = 0$. ■

Yet, two more corollaries of Brown-Halmos theorem

Corollary 1.2.22. *A Toeplitz operator T_f is an isometry if and only if f is constant of modulus* 1.

Proof: If T_f is an isometry, then Remark 1.1.30 tells us that $T_f^* T_f = I$, i.e.

$$T_{\overline{f}} T_f = T_f T_{\overline{f}} = T_1.$$

This implies, by Theorem 1.2.20, that both f and $\overline{f}$ are holomorphic. Hence f is constant and $\overline{f} f = 1$, i.e.

$$|f|^2 = 1.$$

Hence

$$|f| = 1.$$

Conversely if f is a constant function of modulus 1 then it is clear that

$$T_{\overline{f}} T_f = T_{\overline{f} f} = T_{|f|^2} = T_1 = I,$$

and hence, by Remark 1.1.30, T_f is an isometry and the proof is complete. ■

Corollary 1.2.23. *The only idempotent Toeplitz operators are the trivial ones, i.e.* 0 *and* I.

Proof: If $T_f^2 = T_f$, then

$$T_f^2 - T_f = T_f(T_f - I) = T_f(T_f - T_1) = T_f T_{f-1} = 0.$$

So by Corollary 1.2.21, we have $T_f = 0$ or $T_{f-1} = 0$. Thus $T_f = 0$ or $T_f = T_1 = I$. ■

Corollary 1.2.24. *If a Toeplitz operator T_f is invertible, then T_f^{-1} is a Toeplitz operator if and only if f is analytic or f is co-analytic.*

Proof: If f is analytic then, by Theorem 1.2.20, we have

$$\mathbf{S}T_f = T_f\mathbf{S},$$

which implies that

$$T_f^{-1}(\mathbf{S}T_f)T_f^{-1} = T_f^{-1}(T_f\mathbf{S})T_f^{-1}.$$

Hence, we have

$$T_f^{-1}\mathbf{S} = \mathbf{S}T_f^{-1},$$

and we conclude, from Theorem 1.2.15, that T_f^{-1} is a Toeplitz operator. If f is co-analytic then, by Theorem 1.2.20, we have

$$\mathbf{S}^*T_f = T_f\mathbf{S}^*,$$

which implies that

$$T_f^{-1}\mathbf{S}^* = \mathbf{S}^*T_f^{-1},$$

and we conclude, from Theorem 1.2.15, that T_f^{-1} is a Toeplitz operator. For the other direction, suppose that T_f^{-1} is a Toeplitz operator say T_g. Since

$$T_f^{-1}T_f = T_gT_f = I = T_1,$$

which is a Toeplitz operator, Theorem 1.2.20 implies that either f or $\overline{g}$ is analytic. On the other hand

$$T_fT_f^{-1} = T_fT_g = I = T_1,$$

which implies, again by Theorem 1.2.20, that either g or $\overline{f}$ is analytic. Now, if $\overline{f}$ is analytic then the prove is complete. But if $\overline{f}$ is not analytic then g must be analytic and non-constant (because if g is constant then $T_g = T_f^{-1} = cI$ which means that $T_f = \frac{1}{c}I$. i.e. $f = \frac{1}{c}$ and $\overline{f} = \frac{1}{\overline{c}}$ which is analytic). Thus $\overline{g}$ is not analytic and hence f is analytic which completes the proof. ■

Hankel operators for the Hardy space

For $f \in L^2(\partial D)$, define the Hankel operator as follows

$$\begin{aligned} H_f : H^2 &\longrightarrow (H^2)^{\perp} \\ g &\longrightarrow H_f(g) = (I - P)(fg), \end{aligned}$$

where P is the orthogonal projection from $L^2(\partial D)$ onto H^2 and $(H^2)^{\perp}$ is the orthogonal complement of H^2 in $L^2(\partial D)$, i.e. the closed subspace spanned by the basis vectors of negative index $e_n(e^{it}) = e^{int}, n < 0$.

Remark 1.2.25.

1. *If f is bounded on ∂D, then H_f is bounded as well and $\|H_f\| \leq \|f\|_{\infty}$.*
2. *The operator H_f is densely defined and its domain contains H^{∞}. So the above definition makes sense.*
3. *$H_f(g) = (I - P)M_f(g)$, that is $M_f = T_f + H_f$ in the obvious sense.*

1.3 Bergman spaces and their operators

While the Hardy space case is well understood, the Bergman space situation seems to be more involved. So many questions related to Bergman space operators are still very open. Nevertheless, plenty of results in the former case can be generalized somehow to the latter one. Let us describe certain cases to some extent:

Hankel operators on the Bergman space are by now more transparent and a rich corresponding bibliography is available. Unfortunately, we will not study them in details here since they are not directly involved in our progress. Nevertheless, the interested reader is advised to consult for example [8, 9, 10, 31, 42, 58, 67].

Probably the most elementary question in operator theory to be asked during dealing with a specific operator is whether it is bounded. Unlike the Hardy space case, a non-bounded symbol can give rise to a bounded, (or even compact), Toeplitz operator on the Bergman space. The great paper by Cima and Čučković [17] confirms such pretences. A little bit earlier, Axler and Zheng in [14] showed, in particular, that a Bergman space Toeplitz operator with bounded symbol is compact if and only if the Berezin transform of its symbol tends to zero when approaching the boundary (the unit circle). Miao and Zheng in [48] have generalized such results to unbounded symbols. They established also a sufficient conditions on the symbols to produce a bounded Toeplitz operator, see Subsection 1.3.2. Actually the results of [14, 48] don't concern only a single operator, but includes even products of several operators.

The celebrated Brown-Halmos type theorem for the Bergman space is still in progress. Ahern and Čučković [2] are the leaders of this direction. In case of harmonic symbols a lot is done by them. In fact, a series or recent excellent papers become already very popular; we mention for example two of them [1, 2].

Sarason's conjecture [55] on the boundedness of Toeplitz products has been considered by Stroethoff and Zheng [58]. Very recently, in [59] they have studied the invertibility of such products. Also more recently they addressed the several variable case [61]. The zero product problem seems to be curiously resistant; it occurs for the case of two operators as a corollary from Brown-Halmos theorem [16] in the Hardy space setting. We learned that S. Axler has proved it for three operators by the same method as in [16], but unfortunately we ignore the exact reference. In the Hardy space case, some progress has been done by C. Gu in [34] for five operators. K. Y. Guo in [35] has generalized the results of the latter to the product of six operators. For arbitrary products, the conjecture is believed to be true but without any satisfactory proof till now. The Bergman space case is certainly terribly complicated. The case of two Toeplitz operators with harmonic symbols is a simple corollary of the celebrated Brown-Halmos type theorem of Ahern-Čučkovič [2]. While for general symbols, the matter is very delicate; even the case of two operators is still open. Toeplitz algebras are studied by many authors [18, 28, 47, 62] and so many deep results have been established. The spectral theory of Bergman space Toeplitz operators is less investigated. Only very few results are known [41, 47]. So this problem remains very open.

1.3.1 Bergman spaces

In this subsection we introduce the Bergman spaces and focus on their general aspects. This brief account on these spaces concerns their structure, basis and reproducing kernels.

Let D be the open unit disk in $\mathbb{C}$ and dA denote the normalized usual two-dimensional normalized area measure on D. In rectangular and polar coordinates, it reads as

$$dA(z) = \frac{dxdy}{\pi} = \frac{rdrd\theta}{\pi}.$$

For $1 \leq p < \infty$, the Lebesgue space $L^p(D, dA)$, which will be denoted by L^p throughout this thesis, is the Banach space of Lebesgue measurable functions f on D satisfying

$$\|f\|_p = \left(\int_D |f(z)|^p \, dA(z))^{\frac{1}{p}} \right) < \infty,$$

while $L^\infty(D, dA)$, which will be denoted by L^∞, is defined similarly with

$$\|f\|_\infty = \text{ess.} \sup\{|f(z)|, z \in D\} < \infty.$$

Definition 1.3.1. *For $1 \leq p < \infty$, the Bergman space L_a^p is the subspace of L^p consisting of analytic functions on D.*

We have the following immediate crucial result.

Theorem 1.3.2. *For $1 \leq p < \infty$, L_a^p is a closed subspace of L^p.*

Proof: See for example Proposition 4.1.3 of [67]. ■

Remark 1.3.3.

1. *For $1 \leq p < \infty$, L_a^p is a Banach space.*
2. *The set of analytic polynomials, and thus $H^\infty(D)$ as well, is dense in L^p.*
3. *The set of polynomials in z and $\overline{z}$ is dense in L^p.*
4. *Recall that L^2 is a separable Hilbert space with inner product*

$$\langle f, g\rangle = \int_D f(z)\overline{g}(z)dA(z), \ \forall f, g \in L^2.$$

5. *In particular, L_a^2 is a closed subspace of the Hilbert space L^2. Then*

$$L^2 = L_a^2 \oplus (L_a^2)^\perp,$$

where $(L_a^2)^\perp$ is the orthogonal complement of L_a^2.

Since L^2 is a separable Hilbert space, L_a^2 has this property as well and an orthonormal basis is given in the following assertion.

Lemma 1.3.4. *The functions $e_n(z) = \sqrt{n+1}z^n, n = 0, 1, 2, \ldots$ form the standard orthonormal basis of L_a^2, where $z \in D$.*

Proof: First for $n \neq m$ compute $\langle z^n, z^m \rangle$:

$$\begin{aligned} \langle z^n, z^m \rangle &= \frac{1}{\pi} \int_D z^n \overline{z}^m dxdy \\ &= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r^{n+m} e^{i\theta(n-m)} r dr d\theta = 0. \end{aligned}$$

So that $\{ \ z_n \ \}_{n=0}^{\infty}$ is an orthogonal system. Let us normalize it, put $\lambda_n = \|z^n\|$, then

$$\begin{aligned} \lambda_n &= \left(\frac{1}{\pi} \int_D |z^n| \, dxdy \right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{\pi}} \left(\int_0^1 \int_0^{2\pi} r^{2n+1} dr d\theta \right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{\pi}} \left(\frac{1}{2n+2} \int_0^{2\pi} d\theta \right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{\pi}} \left(\frac{2\pi}{2\,(n+1)} \right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{n+1}}. \end{aligned}$$

So let $e_n = \frac{z^n}{\|z^n\|} = \sqrt{n+1} z^n$. Then we obtain an orthonormal system. To see that this system is complete, let $f \in L_a^2$ with Taylor series $\sum\limits_{n=0}^{\infty} \alpha_n z^n$, then

$$\langle f, e_n \rangle = \frac{1}{\sqrt{n+1}} \alpha_n.$$

Suppose that $\langle f, e_n \rangle = 0$ for all $n \in \{0, 1, ...\}$, Then $\alpha_n = 0$ for all $n \in \{0, 1, ...\}$. Which implies that $f = 0$. Hence $\left\{ \sqrt{n+1} z^n \right\}_{n=0}^{\infty}$ is a complete orthonormal system, i.e. it forms an orthonormal basis of L_a^2. ■

Certainly, the most important fact in this theory is that L_a^2 is a reproducing kernel Hilbert space. Its kernel function has a very friendly form.

The reproducing kernel for the Bergman space L_a^2

First, let us show that the point evaluation functional is bounded on the Bergman space. For $f \in L_a^2$, $z \in D$ and $0 < r < R$, where $R = dist(z, \partial D)$ is the distance from

z to the boundary of D, Cauchy's formula yields

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{-\pi}^{\pi} f(z+re^{i\theta})r^{-n}e^{-n\theta}d\theta.$$

Taking the absolute value, multiplying by r^{n+1} and then integrating the latter from 0 to R, we obtain

$$\frac{R^{n+2}}{n+2}\left|f^{(n)}(z)\right| \leq \frac{n!}{2\pi}\int_{D(z,R)} |f(z)|\, dxdy.$$

Applying Cauchy-Schwarz inequality to the right hand side, we obtain

$$\begin{aligned}\left|f^{(n)}(z)\right| &\leq \frac{n+2}{2}\frac{n!}{\sqrt{\pi}R^{n+1}}\left(\int_{D(z,R)} |f(z)|^2\, dxdy\right)^{\frac{1}{2}} \\ &\leq \frac{n!(n+2)}{2}\frac{1}{(dist(z,\partial D))^{n+1}}\left(\int_D \left(|f(z)|^2\, dA(z)\right)\right)^{\frac{1}{2}}.\end{aligned}$$

Thus we get

$$\left|f^{(n)}(z)\right| \leq \frac{n!(n+2)\left\|f\right\|_2}{2(dist(z,\partial D))^{n+1}}.$$

Hence for $n=0$, we obtain

$$|f(z)| \leq \frac{1}{(dist(z,\partial D))}\left\|f\right\|_2,$$

which means that the point evaluation is bounded on L_a^2.

Now, the boundedness of the point evaluation functional implies, (by Riesz representation theorem), the existence of a reproducing kernel for the Bergman space. In other words for fixed $w \in D$, there is a function $K_w \in L_a^2$ such that $f(w) = \langle f, K_w\rangle$ for all $f \in L_a^2$. To establish a formula for K_w, consider the power series expansion:

$$K_w(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Fixing m and taking $f(z) = z^m$, we obtain

$$\begin{aligned} w^m &= \langle z^m, K_w \rangle = \int_D z^m \left(\sum_{n=0}^{\infty} \overline{b_n z^n} \right) dA(z) \\ &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r^{m+1} e^{im\theta} \left(\sum_{n=0}^{\infty} \overline{b_n r^n e^{in\theta}} \right) dr d\theta \\ &= \frac{\overline{b_m}}{m+1}. \end{aligned}$$

Then $b_m = (m+1)\, \overline{w^m}$, and hence

$$K_w(z) = \sum_{n=0}^{\infty} (n+1)(\overline{w}z)^n = \frac{1}{(1-\overline{w}z)^2}.$$

So that the Bergman space L_a^2 has the reproducing kernel K_w given by

$$K_w(z) = \frac{1}{(1-\overline{w}z)^2}, \qquad w, z \in D. \tag{1.3.1}$$

In particular

$$\|K_w\|_2 = \langle K_w, K_w \rangle^{\frac{1}{2}} = [K_w(w)]^{\frac{1}{2}} = \frac{1}{1-|w|^2}.$$

Thus

$$k_w(z) = \frac{1-|w|^2}{(1-\overline{w}z)^2}, \tag{1.3.2}$$

is the normalized reproducing kernel of L_a^2.

Proposition 1.3.5. *Let $f \in L_a^2$ and $w \in D$. Then*

$$f(w) = \langle f, K_w \rangle = \int_D \frac{f(z)}{(1-w\overline{z})^2} dA(z).$$

Remark 1.3.6. *There is another kernel function that will rise in our arguments later in Chapter 2 which is given by*

$$k_w^1(z) = \frac{2(1-|z|^2)}{(1-\overline{w}z)^3}. \tag{1.3.3}$$

It is well-known that the operator P_1 associated to k_w^1 maps L^1 to L_a^1. For further details we refer to Section 7.1 in [53].

The following result is needed in the sequel.

Remark 1.3.7. *A crucial decomposition of L^2 is given by*

$$L^2 = \bigoplus_{k \in Z} e^{ik\theta} \mathcal{R},$$

where $\mathcal{R}$=$\{u : D \to \mathbb{C}$ *radial and* $\int_0^1 r\,|u(r)|^2\,dr < \infty\}$. *Thus every* $f \in L^2$ *can be written componentwise in its polar form as follows*

$$f\left(re^{i\theta}\right) = \sum_{k=-\infty}^{\infty} f_k(r) e^{ik\theta}, \;\; f_k \in \mathcal{R}.$$

Duals of Bergman spaces and the Bloch space

For $1 < p < \infty$, the dual of L_a^p can be identified with L_a^q, where $\frac{1}{p} + \frac{1}{q} = 1$. In other words $(L_a^p)^* \cong L_a^q$.

More precisely every bounded linear functional on L_a^p is of the form:

$$f \longrightarrow \int_D f\overline{g}dA, \text{ for some unique } g \in L_a^q.$$

Furthermore, the norm of the linear functional on L_a^p induced by $g \in L_a^q$ is comparable to $\|g\|_q$.

The Bloch space $\mathbf{B}$ of D is defined to be the space of analytic functions f on D such that

$$\|f\|_{\mathbf{B}} = \sup\{(1 - |z|^2)\,|f'(z)|\,, z \in D\} < \infty.$$

$\|.\|_{\mathbf{B}}$ is a complete semi-norm on $\mathbf{B}$. Moreover, $\mathbf{B}$ can be made into a Banach space by introducing the norm

$$\|f\| = |f(0)| + \|f\|_{\mathbf{B}}\,.$$

Similarly, we can define the little Bloch space denoted by $\mathbf{B}_0$. It is the subspace of $\mathbf{B}$ consisting of functions f on D for which

$$(1 - |z|^2)f'(z) \to 0 \text{ as } |z| \to 1.$$

The dual of L_a^1 can be identified with the Bloch space, i.e. $(L_a^1)^* \cong \mathbf{B}$. While the dual of $\mathbf{B}_0$ can be identified to L_a^1, i.e. $(\mathbf{B}_0)^* \cong L_a^1$.

Besides, H^∞ is contained in $\mathbf{B}$ and $\|f\|_{\mathbf{B}} \leq \|f\|_\infty$ for all f in H^∞. The harmonic Bergman space L_h^2 is defined to be the subspace of L^2 consisting of harmonic functions on D. Moreover, it admits the decomposition $L_h^2 = L_a^2 \oplus \overline{zL_a^2}$; or equivalently $L_h^2 = zL_a^2 \oplus \overline{L_a^2}$.

Similarly, the harmonic Bloch space is the set of harmonic functions on D such that

$$\|u\|_{\mathbf{B}_h} = \sup_D \{(1-|z|^2)\,|\nabla u(z)|\} < \infty,$$

where

$$\nabla u(z) = (\partial u(z) + \overline{\partial} u(z), i(\partial u(z) - \overline{\partial} u(z))), \tag{1.3.4}$$

is the gradient of u.

The harmonic Bloch space $\mathbf{B}_h$ can be written as a sum of $\mathbf{B}$ and $\overline{\mathbf{B}}$. Namely, for every $f \in \mathbf{B}_h$, there exist f_1, $f_2 \in \mathbf{B}$ such that $f = f_1 + \overline{f_2}$. This decomposition becomes unique if we require $f_2(0) = 0$.

Since we have $H^\infty \subset \mathbf{B} \subset L_a^2$ and $L_h^\infty \subset \mathbf{B}_h \subset L_h^2$, we conclude that if $f \in L_h^\infty$, (i.e. a bounded harmonic function), then there exist f_1, $f_2 \in \mathbf{B}$ such that $f = f_1 + \overline{f_2}$. However, if we require that $f_2(0) = 0$ then this decomposition is unique.

Before we turn to another topic, we notice the following:

Remark 1.3.8. *For $z \in D$, we consider the modulus of the gradient:*

$$|\nabla u(z)|^2 = \left|\partial u(z) + \overline{\partial} u(z)\right|^2 + \left|\partial u(z) - \overline{\partial} u(z)\right|^2$$

and we deduce the estimates:

$$|\partial u(z)| \leq \frac{1}{\sqrt{2}} |\nabla u(z)| \text{ and } \left|\overline{\partial} u(z)\right| \leq \frac{1}{\sqrt{2}} |\nabla u(z)|,$$

as well as

$$\frac{1}{\sqrt{2}} \left(|\partial u(z)| + \left|\overline{\partial} u(z)\right| \right) \leq |\nabla u(z)| \leq \sqrt{2}(|\partial u(z)| + \left|\overline{\partial} u(z)\right|\right).$$

1.3.2 Haplitz operators on the Bergman space

In this subsection, we are interested in the analog of Hardy space Toeplitz operators. Since the basis of the Bergman space is of weighted character, we cannot suspect that such operators will possess friendly matrices. In fact, it turns out that Bergman space Toeplitz operators do not correspond to Toeplitz matrices anymore. So we substitute matrix theory by complex function theory in addition to the Berezin transform. Since a Toeplitz operator is a multiplication followed by a projection, we start with such concepts

The orthogonal projection on L^2_a

Since L^2_a is a closed subspace of the Hilbert space L^2, there exists an orthogonal projection, called Bergman projection, from L^2 onto L^2_a denoted by P and is defined as follows

$$\begin{aligned} P : L^2 = L^2_a \oplus (L^2_a)^{\perp} &\longrightarrow L^2_a \\ f &\longrightarrow P(f) = P(f_1 + f_2) = f_1. \end{aligned}$$

Remark 1.3.9.

1. *For $f \in L^2$, and $w \in D$ we use the reproducing kernel K_w to give an explicit integral formula for Pf, namely:*
$$(Pf)(w) = \langle Pf, K_w \rangle = \langle f, K_w \rangle = \int_D \frac{f(z)}{(1 - w\overline{z})^2} dA(z).$$
2. *From the above characterization, the projection P is an integral operator. So that the integral makes sense whenever $f \in L^1$ and so P can be extended to $f \in L^1$:*
$$(Pf)(w) = \int_D \frac{f(z)}{(1 - w\overline{z})^2}, \ f \in L^1, w \in D.$$
Moreover, since we can differentiate under the integral sign, clearly Pf is analytic on D for each $f \in L^1$. Besides, since K_w is the reproducing kernel of L^1_a, we have that $Pf = f$, $\forall f \in L^1_a$.

Besides, we have the following assertion whose proof can be found in [10].

Theorem 1.3.10. *Let $1 < p < \infty$. Then, P is a bounded projection of L^p onto L^p_a.*

The multiplication operator M_φ on L^2

For $\varphi \in L^\infty$, define the multiplication operator M_φ as follows

$$\begin{aligned} M_\varphi : \; L^2 &\longrightarrow L^2 \\ f &\longrightarrow M_\varphi(f) = \varphi f. \end{aligned}$$

The multiplication here is the pointwise one, i.e. $\varphi f(z) = \varphi(z) f(z)$, for all $z \in D$.

Remark 1.3.11. *Note that the multiplication operator is bounded. Indeed, since $\varphi \in L^\infty$ then*

$$\|M_\varphi(f)\|_2 = \|\varphi f\|_2 \leq \|\varphi\|_\infty \|f\|_2, \quad \forall f \in L^2.$$

Now, we are in the position to introduce the concept of Bergman space Toeplitz operators:

Toeplitz operators on L^2_a

For $\varphi \in L^2$, the Toeplitz operator with symbol φ is the operator T_φ from L^2_a to L^2_a defined by

$$T_\varphi(f) = P(\varphi f) = \int_D \frac{\varphi(w) f(w)}{(1 - z\overline{w})^2} dA(w), \quad \forall f \in L^2_a.$$

Actually we first define T_φ on the set of analytic polynomials $\wp_a$ and for symbols in L^2. Then we observe that the integral representation of T_φ makes sense for a larger class of symbols, namely L^1, and that $\wp_a$ is dense in L^2. So that, the above definition makes sense.

Remark 1.3.12.

1. *For* $\varphi \in L^\infty$, *it is clear that* T_φ *is bounded. In fact,*

$$\begin{aligned} \|T_\varphi(f)\|_2 &= \|P(\varphi f)\|_2 \leq \|P\| \, \|\varphi f\|_2 \\ &\leq \|\varphi f\|_2 \leq \|\varphi\|_\infty \|f\|_2 \,, \text{ for all } f \in L_a^2. \end{aligned}$$

2. *Note that for* $\varphi \in H^\infty$, *the Toeplitz operator* T_φ *coincides with the multiplication operator* M_φ *on* L_a^2; *and in this case it is called the analytic Toeplitz operator* T_φ.

Proposition 1.3.13. *Suppose that* a *and* b *are complex numbers,* φ *and* ψ *are* L^2 *functions. Then we have the following basic algebraic properties of Toeplitz operators:*

1. $T_{a\varphi+b\psi} = aT_\varphi + bT_\psi$.
2. $T_\varphi^* = T_{\overline{\varphi}}$.

Proof: (1) For $h \in L_a^2$ we have

$$\begin{aligned} (aT_\varphi + bT_\psi)(h) &= aT_\varphi(h) + bT_\psi(h) \\ &= aP(\varphi h) + bP(\psi h) = P(a\varphi h) + P(b\psi h) \\ &= P((a\varphi + b\psi)h) = T_{a\varphi+b\psi}(h). \end{aligned}$$

Hence $aT_\varphi + bT_\psi = T_{a\varphi+b\psi}$.

(2) For any $h, f \in L_a^2$,

$$\begin{aligned} \langle T_\varphi^* h, f\rangle &= \langle h, T_\varphi f\rangle \\ &= \langle h, P(\varphi f)\rangle = \langle h, \varphi f\rangle \\ &= \langle \overline{\varphi} h, f\rangle = \langle P(\overline{\varphi} h), f\rangle \\ &= \langle T_{\overline{\varphi}} h, f\rangle. \end{aligned}$$

Hence $T_\varphi^* = T_{\overline{\varphi}}$. ■

Theorem 1.3.14. *If $\varphi \in L^1$ is harmonic, then T_φ is bounded if and only if φ is bounded. Besides, T_φ is compact if and only if $\varphi \equiv 0$.*

Proof: See for example Corollary 6.1.5 of [67] ■

The problem of boundedness of Toeplitz operators, with more general symbols, on the Bergman space is still open. However, very recently J. Miao and D. Zheng have presented some progress in that direction. The problem of compactness and boundedness of such operators relies heavily on the concept of the Berezin transform, which will be studied in details in Chapter 2.

Products of Bergman space Toeplitz operators

We know from Brown-Halmos theorem that a necessary and sufficient condition for the product $T_\varphi T_\psi$ of two Hardy space Toeplitz operators to be a Toeplitz operator is that either φ is co-analytic or ψ is analytic. If the condition is satisfied then $T_\varphi T_\psi = T_{\varphi\psi}$. We conclude that among the Hardy space Toeplitz operators there are no zero divisors.

Throughout this thesis we consider the problem of determining when the product of two Bergman space Toeplitz operators is a Toeplitz operator. We show that the Brown-Halmos theorem fails for general symbols but when we restrict ourselves to the case of bounded harmonic or radial symbols, we do have theorems of Brown-Halmos type.

Hankel operators on L_a^2

For $\varphi \in L^\infty$, the Hankel operator with symbol φ, denoted by H_φ, is the operator from L_a^2 to $(L_a^2)^\perp$ defined by

$$H_\varphi f = (I - P)(\varphi f), \quad \forall f \in L_a^2,$$

where $I - P$ is the orthogonal projection from L^2 onto $(L_a^2)^\perp$, while P is the Bergman projection introduced in Subsection 1.3.2.

The following result about Hankel operators is needed in the sequel.

Lemma 1.3.15. *Let $f \in L^1$. Then*

1. *$H_{\overline{f}}$ is bounded on L_a^2 if and only if $f \in \mathbf{B}$.*
2. *$H_{\overline{f}}$ is compact on L_a^2 if and only if $f \in \mathbf{B}_0$.*

In the next lemma, we introduce a well-known identity that relates Toeplitz and Hankel operators together.

Lemma 1.3.16. $T_\varphi T_z - T_z T_\varphi = T_{z\varphi} - T_z T_\varphi = H_{\overline{z}}^* H_\varphi$, *where* $\varphi \in L^\infty$.

Proof: Since z is an analytic function, we have that $T_\varphi T_z - T_z T_\varphi = T_{z\varphi} - T_z T_\varphi$. But

$$T_{z\varphi} - T_z T_\varphi = P(M_{z\varphi}) - (PM_z)(PM_\varphi),$$

where M_φ is the multiplication operator with symbol φ. Thus

$$\begin{aligned} T_{z\varphi} - T_z T_\varphi &= P(M_z M_\varphi) - (PM_z)(PM_\varphi) \\ &= PM_z(M_\varphi - PM_\varphi) \\ &= PM_z((I-P)M_\varphi). \end{aligned}$$

But it is not difficult to show that $H_{\overline{z}}^* = PM_z$. Thus we have

$$T_{z\varphi} - T_z T_\varphi = H_{\overline{z}}^* H_\varphi,$$

which completes the proof. ■

1.4 Weighted Bergman spaces and their operators

The (weighted) Bergman space A^2_α is the Hilbert space consisting of analytic functions on D which are square integrable with respect to the weighted measure

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z),$$

where dA denotes the normalized Lebesgue area measure on D and $\alpha > -1$.
The inner product of f and g in A^2_α is

$$\langle f, g\rangle_\alpha = \int_D f(z)\overline{g}(z)dA_\alpha(z).$$

The norm of f in A^2_α is

$$\|f\|_\alpha = (\langle f, f\rangle_\alpha)^{\frac{1}{2}} = \left(\int_D |f|^2 \, dA_\alpha\right)^{\frac{1}{2}}.$$

Remark 1.4.1.

1. *The monomials form an orthogonal basis for A^2_α.[51].*
2. *The set of all polynomials is dense in A^2_α.([67], page 120).*
3. *The space A^2_α is a reproducing kernel Hilbert space on D with reproducing kernel given by*
$$K^{(\alpha)}_w(z) = \frac{1}{(1-\overline{w}z)^{2+\alpha}}.$$

Proposition 1.4.2. *For every $h \in A^2_\alpha$ and $w \in D$, we have $h(w) = \langle h, K^{(\alpha)}_w\rangle_\alpha$.*

Orthogonal projection on A^2_α

The orthogonal projection P_α of $L^2(D, dA_\alpha)$ onto A^2_α is given by

$$(P_\alpha g)(w) = \langle g, K^{(\alpha)}_w\rangle_\alpha = \int_D \frac{g(z)}{(1-\overline{z}w)^{\alpha+2}} dA_\alpha(z), \text{ for } g \in L^2(D, dA_\alpha) \text{ and } w \in D.$$

Toeplitz operator on A^2_α

For $f \in L^\infty(D)$, the Toeplitz operator T_f is defined on A^2_α by

$$T_f h = P_\alpha(fh).$$

More explicitly, it reads as

$$(T_f h)(w) = \int_D \frac{f(z)h(z)}{(1-\overline{z}w)^{2+\alpha}} dA_\alpha(z), \text{ for } h \in A^2_\alpha \text{ and } w \in D.$$

Chapter 2

The Berezin Transform and the Invariant Laplacian

The reproducing property of the Bergman space yields a pioneering tool in the theory of operators on L_a^2, namely the Berezin transform. It plays the same role played by Poisson extension in the Hardy space case. Actually, according to the definition below Poisson extension is nothing else but the Hardy space Berezin transform. This concept is, in our opinion, the most useful one in the theory, especially that it has an intimate relation with harmonic functions. In this chapter, we introduce the concept of Berezin transform and we exhibit its main properties. Probably the most deep result is the fact that the only functions invariant under its action are the harmonic ones. This result was proved independently by Ahern, Flores and Rudin [5] and by Engliš [29]. Other results on the Berezin transform can be found in Engliš thesis [28], Ahern's recent paper [1], Stroethoff's valuable notes [57], Ahern and Čučković marvelous work [2] and the paper of Axler and Zheng [15] as well as the indispensable textbooks by Zhu et al. [40, 67]. Another important result in this chapter is Theorem 2.4.5, which was proved in 2004 by P. Ahern [1].

2.1 Definition and main properties

Definition 2.1.1. *Let H be a reproducing kernel functional Hilbert space on an open subset Ω of $\mathbb{C}$. If S is a bounded linear operator on H, then the Berezin transform of S is defined by*

$$B(S)(w) = \langle Sk_w, k_w \rangle, \text{ for } w \in \Omega,$$

where k_w is the normalized kernel function of H.
So, in particular if H is the Bergman space L_a^2 then for any function $f \in L^1$ and any $z \in D$, the Berezin transform of f reads as

$$B(f)(z) = \langle fk_z, k_z \rangle = \int_D f(w) |k_z(w)|^2 \, dA(w) = \int_D \frac{(1 - |z|^2)^2}{|1 - \overline{z}w|^4} f(w) dA(w). \quad (2.1.1)$$

In terms of disk automorphisms it has another form.

Proposition 2.1.2. *The Berezin transform of the function $f \in L^1$ is given by*

$$B(f)(z) = \int_D (f \circ \varphi_z)(w) dA(w), \; z \in D. \quad (2.1.2)$$

Where φ_z is the Möbius transformation.

Proof: Recall the famous Möbius transformation

$$\varphi_z(w) = \frac{z - w}{1 - \overline{z}w}, \; w \in D.$$

It is a disk automorphism and moreover it is equal to its own inverse, i.e. $\varphi_z^{-1} = \varphi_z$. Indeed, put $\zeta = \varphi_z(w) = \frac{z-w}{1-\overline{z}w}$. Then, we obtain $w(1 - \zeta\overline{z}) = z - \zeta$, whence

$$w = \frac{z - \zeta}{1 - \overline{z}\zeta} = \varphi_z(\zeta).$$

Now, for any $w \in D$, we have

$$\varphi_z'(w) = \frac{-(1 - \overline{z}w) + (z - w)(\overline{z})}{(1 - \overline{z}w)^2} = \frac{|z|^2 - 1}{(1 - \overline{z}w)^2}.$$

So that the change of variable $\zeta = \varphi_z(w)$ has real Jacobian equals to

$$\varphi_z'(w)\overline{\varphi_z'(w)} = |\varphi_z'(w)|^2 = \frac{(1 - |z|^2)^2}{|1 - \overline{z}w|^4}.$$

Thus $dA(w) = |\varphi_z'(w)|^2 \, dA(\zeta)$, whence

$$\int_D (f \circ \varphi_z)(w) dA(w) = \int_D f(\zeta) \, |\varphi_z'(\zeta)|^2 \, dA(\zeta).$$

So that by Formula (2.1.1), the crucial identity (2.1.2) holds, namely

$$B(f)\,(z) = \int_D (f \circ \varphi_z)(w) dA(w),$$

which completes the proof. ■

Now, we give some properties of Berezin transform whose proofs can be found in [67]. For an extensive study of Berezin transform and its applications on spaces of analytic functions, we refer to the excellent notes of Stroethoff [57] as well as [40]. A first property of the Berezin transform asserts that it commutes with the Möbius group and we will give our own explicit proof of it that depends on direct calculations:

Proposition 2.1.3. *For any* $u \in L^1$ *and any* $a \in D : B(u \circ \varphi_a) = (B(u)) \circ \varphi_a$.

Proof: We have to show that $B(u \circ \varphi_a) = (B(u)) \circ \varphi_a$, $\forall \varphi_a \in Aut(D)$. Any disk automorphism φ_a of $Aut(D)$ has the form

$$\varphi_a(z) \;=\; e^{i\theta} \frac{a - z}{1 - \overline{a}z}.$$

Using Formula (2.1.1), we get

$$\begin{aligned} B(u \circ \varphi_a)(z) &= (1 - |z|^2)^2 \int_D \frac{(u \circ \varphi_a)(\zeta)}{\left|1 - z\overline{\zeta}\right|^4} dA(\zeta) \\ &= (1 - |z|^2)^2 \int_D \frac{u\left(e^{i\theta} \frac{a-\zeta}{1-\overline{a}\zeta}\right)}{\left|1 - z\overline{\zeta}\right|^4} dA(\zeta), \end{aligned}$$

and

$$\begin{aligned} \left(B(u) \circ \varphi_a\right)(z) &= B(u) \left(e^{i\theta} \frac{a - z}{1 - \overline{a}z}\right) \\ &= \left(1 - \left|e^{i\theta} \frac{a - z}{1 - \overline{a}z}\right|^2\right)^2 \int_D \frac{u(w)}{\left|1 - \overline{w} e^{i\theta} \frac{a-z}{1-\overline{a}z}\right|^4} dA(w). \end{aligned}$$

Let $w = \varphi_a(\zeta) = e^{i\theta}\frac{a-\zeta}{1-\overline{a}\zeta}$, then we get $dA(w) = \frac{(1-|a|^2)^2}{|1-\overline{a}\zeta|^4}dA(\zeta)$, and we have that

$$\left|e^{i\theta}\frac{a-z}{1-\overline{a}z}\right|^2 = \left|\frac{a-z}{1-\overline{a}z}\right|^2 = \frac{|a|^2 - 2\mathcal{R}e(\overline{a}z) + |z|^2}{1 - 2\mathcal{R}e(\overline{a}z) + |a|^2|z|^2}.$$

This means that

$$1 - \left|e^{i\theta}\frac{a-z}{1-\overline{a}z}\right|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2}.$$

On the other hand, we have

$$\begin{aligned} 1 - \overline{w}e^{i\theta}\frac{a-z}{1-\overline{a}z} &= 1 - \frac{\overline{a}-\overline{\zeta}}{1-a\overline{\zeta}}\frac{a-z}{1-\overline{a}z} \\ &= \frac{1-\overline{a}z - a\overline{\zeta} + |a|^2\overline{\zeta}z - |a|^2 + \overline{a}z + a\overline{\zeta} - \overline{\zeta}z}{(1-a\overline{\zeta})(1-\overline{a}z)} \\ &= \frac{(1-|a|^2)(1-\overline{\zeta}z)}{(1-a\overline{\zeta})(1-\overline{a}z)}. \end{aligned}$$

Hence, we arrive at

$$\begin{aligned} (B(u)\circ\varphi_a)(z) &= \frac{(1-|z|^2)^2(1-|a|^2)^2}{|1-\overline{a}z|^4}\int_D \frac{u(e^{i\theta}\frac{a-\zeta}{1-\overline{a}\zeta})}{\frac{(1-|a|^2)^4|1-\overline{\zeta}z|^4}{|1-a\overline{\zeta}|^4|1-\overline{a}z|^4}}\frac{(1-|a|^2)^2}{|1-\overline{a}\zeta|^4}dA(\zeta) \\ &= (1-|z|^2)^2\int_D \frac{u(e^{i\theta}\frac{a-\zeta}{1-\overline{a}\zeta})}{|1-\overline{\zeta}z|^4}dA(\zeta) \\ &= B(u\circ\varphi_a)(z), \text{ for all } z \in D. \end{aligned}$$

Thus we see that $(B(u)\circ\varphi_a = B(u\circ\varphi_a)$. ■

The following assertion concerns the injectivity of this transform and it will be proved in a more general situation later in this chapter, (see Proposition 2.3.1).

Proposition 2.1.4. *The Berezin transform is injective on* L^1.

Probably the deepest result about the Berezin transform is the fact that the only invariant functions under the Berezin transform are the harmonic ones. The necessity is easier and it was known for a while, see for instance [28]. However, the sufficiency is extremely subtle. It was proved independently by Engliš in [29] and by Ahern, Flores and Rudin in [5]. A good exposition of this result can be found in [40].

Theorem 2.1.5. *Let $f \in L^1$. Then f is harmonic if and only if $B(f) = f$.*

The details of the following remark can be found in [2, 67]:

Remark 2.1.6.

1. *Note that the function $B(S)$ is bounded on Ω and is called the Berezin symbol of the operator S.*
2. *$B(f)$ is an infinitely differentiable function on D.*
3. *The Berezin transform is not a projection onto the harmonic functions, that is $B(u)$ is not always harmonic. Even rather more is true, namely $B(u)$can not be harmonic unless u is. In fact, if $v = B(u)$ is harmonic, then $B(u) = v = B(v)$ as B reproduces harmonic functions. Therefore $B(u-v) = 0$. By the injectivity of B, we infer that $u = v$. In other words, $B(u)$ is harmonic if and only if u is harmonic.*
4. *$B(\overline{f})(z) = \overline{B(f)}(z)$. Indeed, we have*
$$\overline{B(f)}(z) = \overline{\langle f k_z, k_z \rangle} = \overline{\langle k_z, \overline{f} k_z \rangle} = \langle \overline{f} k_z, k_z \rangle = B(\overline{f})(z).$$
5. *For a Toeplitz operator T_f, we have that $B(T_f) = B(f)$.*

2.2 The Laplacian and the invariant Laplacian

In dealing with harmonic functions on the unit disk, we find it more convenient to use the invariant Laplacian $\widetilde{\Delta}$ instead of the usual Laplacian Δ. We shall use the operator
$$\Delta = \frac{\partial^2}{\partial z \partial \overline{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$
where $z = x + iy$, to denote the "complex" Laplacian (this is quarter of the standard Laplacian). This normalization has the advantage that certain formulae assume a

particularly attractive form; for instance, if f is a holomorphic function, then $\Delta |f|^2 = |f'|^2$.

Now we define the invariant Laplacian.

Definition 2.2.1. *The invariant Laplacian* $\widetilde{\Delta}$ *is defined by*

$$\widetilde{\Delta}_z = (1-|z|^2)^2\Delta,$$

where $\Delta = \frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}$ *is the usual Laplacian. In other words, the invariant Laplacian is defined by*

$$\widetilde{\Delta} f(z) = (1-|z|^2)^2\Delta f(z).$$

As its name suggests, the invariant Laplacian is Möbius invariant, namely we have the following characterization of it. Note that, this property is well-known. However we were not able to find a corresponding proof. Here we propose our own elegant direct proof of it.

Lemma 2.2.2. *Let* f *be in* L^2 *and let* φ_a *be the Möbius transformation given in Remark 1.1.9. Then we have*

$$\widetilde{\Delta}(f\circ\varphi_a)(z) = (\widetilde{\Delta} f)(\varphi_a(z)).$$

Proof: By definition 2.2.1, we have

$$\widetilde{\Delta}(f\circ\varphi_a)(z) = (1-|z|^2)^2\Delta(f\circ\varphi_a)(z), \tag{2.1.3}$$

and

$$(\widetilde{\Delta} f)(\varphi_a(z)) = \left(1-|\varphi_a(z)|^2\right)^2 \Delta f(\varphi_a(z)). \tag{2.1.4}$$

From one hand, we have

$$\begin{aligned}\left(1-|\varphi_a(z)|^2\right)^2 &= \left(1-\frac{a-z}{1-\overline{a}z}\frac{\overline{a}-\overline{z}}{1-a\overline{z}}\right)^2\\ &= \left(1-\frac{|a|^2-2\mathcal{R}e(\overline{a}z)+|z|^2}{|1-\overline{a}z|^2}\right)^2\\ &= \frac{(1-|a|^2)^2(1-|z|^2)^2}{|1-\overline{a}z|^4}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Delta(f\circ\varphi_a)(z) &= \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\left(f\left(e^{i\theta}\frac{a-z}{1-\overline{a}z}\right)\right)\\
&= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \overline{z}}\left(e^{i\theta}\frac{a-z}{1-\overline{a}z}\right)\frac{|a|^2-1}{|1-a\overline{z}|^2}\right)\\
&= \frac{|a|^2-1}{|1-\overline{a}z|^2}\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \overline{z}}\left(e^{i\theta}\frac{a-z}{1-\overline{a}z}\right)\right)+0\\
&= \frac{|a|^2-1}{|1-\overline{a}z|^2}\frac{\partial^2 f}{\partial z\partial \overline{z}}\left(e^{i\theta}\frac{a-z}{1-\overline{a}z}\right)\frac{|a|^2-1}{|1-\overline{a}z|^2}\\
&= \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}\Delta f(\varphi_a(z)).
\end{aligned}$$

Hence, from Equations (2.1.3) and (2.1.4), we have that

$$\begin{aligned}
\widetilde{\Delta}(f\circ\varphi_a)(z) &= \frac{(1-|a|^2)^2(1-|z|^2)^2}{|1-\overline{a}z|^4}\Delta f(\varphi_a(z))\\
&= \left(1-|\varphi_a(z)|^2\right)^2\Delta f(\varphi_a(z))\\
&= (\widetilde{\Delta}f)(\varphi_a(z)).
\end{aligned}$$

Which completes the proof ■

We end this subsection by a powerful result which shows that the Berezin transform commutes with the invariant Laplacian.

Lemma 2.2.3. *Suppose that u is twice continuously differentiable in D and suppose that both u and $\widetilde{\Delta}u$ are in L^1. Then*

$$\widetilde{\Delta}Bu = B(\widetilde{\Delta}u).$$

Proof: We fix $0<r<1$ and $z\in D$ and consider the integral

$$\int_{D_r}\frac{(r^2-|w|^2)^2\Delta u(w)}{|1-\overline{w}z|^4}dA(w),$$

where D_r is the disk of radius r centered at the origin. By Green's theorem we have that

$$\int_{D_r}\left\{\frac{(r^2-|w|^2)^2}{|1-\overline{w}z|^4}\Delta u(w)-u(w)\Delta_w\frac{(r^2-|w|^2)^2}{|1-\overline{w}z|^4}\right\}dA(w)$$
$$=\int_{\partial D_r}\left\{\frac{(r^2-|w|^2)^2}{|1-\overline{w}z|^4}\mid_{\partial D_r}\frac{\partial u}{\partial\eta}-u\frac{\partial}{\partial\eta_w}\frac{(r^2-|w|^2)^2}{|1-\overline{w}z|^4}\mid_{\partial D_r}\right\}ds.$$

Concerning the boundary values in the last integrand of the right hand side of the latter, we have

$$\frac{(r^2-|w|^2)^2}{|1-\overline{w}z|^4}\to 0 \quad\text{as}\quad w\to\partial D_r,$$

and also

$$\frac{\partial}{\partial\eta_w}\frac{(r^2-|w|^2)^2}{|1-\overline{w}z|^4}\to 0 \quad\text{as}\quad w\to\partial D_r.$$

So that

$$\int_{D_r}\frac{(r^2-|w|^2)^2}{|1-\overline{w}z|^4}\Delta u(w)dA(w)=\int_{D_r}u(w)\Delta_w\frac{(r^2-|w|^2)^2}{|1-\overline{w}z|^4}dA(w).$$

Now, observe that

$$\left|(r^2-|w|^2)^2\Delta u\right|\leq\left|(1-|w|^2)^2\Delta u\right| \text{ in } D_r,$$

so we may take the limit as $r\to 1$ under the integral sign in the first integral. For each fixed $z\in D$, $\Delta_w\left(\frac{(r^2-|w|^2)^2}{|1-\overline{w}z|^4}\right)$ converges pointwise and boundedly to $\Delta_w\left(\frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\right)$ as $r\to 1$. So we obtain

$$\int_D u(w)\Delta_w\frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}dA(w)=\int_D\frac{\widetilde{\Delta}u(w)}{|1-\overline{w}z|^4}dA(w). \tag{2.1.5}$$

In the first integral we use the symmety identity

$$\Delta_w\frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}=\Delta_z\frac{(1-|z|^2)^2}{|1-\overline{w}z|^4}.$$

If we multiply (2.1.5) by $(1-|z|^2)^2$, we obtain

$$(1-|z|^2)^2\int_D u(w)\Delta_w \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}dA(w) = (1-|z|^2)^2\int_D \frac{\widetilde{\Delta}u(w)}{|1-\overline{w}z|^4}dA(w).$$

Hence we infer that

$$(1-|z|^2)^2\int_D u(w)\Delta_z \frac{(1-|z|^2)^2}{|1-\overline{w}z|^4}dA(w) = B(\widetilde{\Delta}u)\,(z)\,.$$

This implies that

$$(1-|z|^2)^2\Delta_z\left\{(1-|z|^2)^2\int_D \frac{u(w)}{|1-\overline{w}z|^4}dA(w)\right\} = B(\widetilde{\Delta}u)\,(z)\,.$$

Hence we conclude that $\widetilde{\Delta}(Bu) = B(\widetilde{\Delta}u)$ and the lemma is proved. ■

2.3 The weighted Berezin transform

For a function $u \in L^1(D, dA_\alpha)$, the Berezin transform is the function on D defined by

$$B_\alpha[u](w) = \int_D u(z)\frac{(1-|w|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}}dA_\alpha(z).$$

The Berezin transform of a bounded linear operator S on A^2_α is the function $B_\alpha[S]$ which is defined on D by

$$B_\alpha[S](w) = \langle Sk_w^{(\alpha)}, k_w^{(\alpha)}\rangle, \text{ for } w \in D,$$

where

$$k_w^{(\alpha)}(z) = \frac{(1-|w|^2)^{2+\alpha}}{(1-\overline{w}z)^{\alpha+2}},$$

is the normalized reproducing kernel in A^2_α.

The following result concerns the injectivity of the weighted Berezin transform. i.e. the analog of Proposition 2.1.4 and as we promised we include a corresponding proof.

Proposition 2.3.1. *[40] For each α with $-1 < \alpha < \infty$, the operator B_α is one-to-one on the space $L^1(D, dA_\alpha)$.*

Proof: Suppose that $f \in L^1(D, dA_\alpha)$ and $B_\alpha f = 0$. Let

$$F(z) = \int_D \frac{f(w)dA_\alpha(w)}{(1-z\overline{w})^{2+\alpha}(1-\overline{z}w)^{2+\alpha}}, \; z \in D.$$

Since

$$F(z) = \frac{B_\alpha f(z)}{(1-|z|^2)^{2+\alpha}},$$

we have $F(z) = 0$ throughout D, and hence

$$\frac{\partial^{n+m} F}{\partial z^n \partial \overline{z}^m}(0) = 0,$$

for all nonnegative integers n and m. Differentiating under the integral sign, we find that

$$\int_D \overline{w}^n w^m f(w) dA_\alpha(w) = 0,$$

for all nonnegative integers n and m. This clearly implies that $f = 0$. ■

In the following proposition we exhibit some properties of the weighted Berezin transform and we refer to [40] for details.

Proposition 2.3.2.

1. *If $-1 < \alpha < \infty$ and φ is a disk automorphism, then for every $f \in L^1(D, dA_\alpha)$, we have*
$$(B_\alpha f) \circ \varphi = B_\alpha(f \circ \varphi).$$
2. *If $-1 < \alpha < \infty$, $-1 < p < \infty$, and $\beta \in \mathbb{R}$, then B_α is bounded on $L^p(D, dA_\beta)$ if and only if $-(\alpha+2)p < \beta + 1 < (\alpha+1)p$*
3. *If $-1 < \alpha < \infty$ and $f \in C(\overline{D})$, then we have $B_\alpha f \in C(\overline{D})$ and $f - B_\alpha f \in C_0(D)$.*
4. *If $-1 < \beta < \alpha < \infty$, then $B_\alpha B_\beta = B_\beta B_\alpha$ on $L^1(D, dA_\beta)$.*
5. *If $-1 < \alpha < \infty$ and $f \in L^1(D, dA_\alpha)$, then $B_\beta f \to f$ in $L^1(D, dA_\alpha)$ as $\beta \to \infty$.*

Among our main contributions in this thesis the following result which is the weighted Bergman space analog of Lemma 2.2.3.

Theorem 2.3.3. *Suppose that $u \in C^2(D)$ and that both of u and $\widetilde{\Delta} u$ are in $L^1(D, dA_\alpha)$. Then the invariant Laplacian $\widetilde{\Delta} u$ commutes with the weighted Berezin transform $B_\alpha u$, i.e.*

$$\widetilde{\Delta} B_\alpha u = B_\alpha \widetilde{\Delta} u.$$

Proof: Fix $0 < r < 1$ and $z \in D$ and consider the integral

$$\int_{D_r} \frac{(r^2 - |w|^2)^{2+\alpha} \Delta u(w)}{|1 - \overline{w}z|^{4+2\alpha}} dA(w),$$

where D_r is the disk of center at the origin and radius r. Green's formula yields

$$\int_{D_r} \left\{ \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \Delta u(w) - u(w) \Delta_w \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \right\} dA(w)$$

$$= \int_{\partial D_r} \left\{ \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \mid_{\partial D_r} \frac{\partial u}{\partial \eta} - u \mid_{\partial D_r} \frac{\partial}{\partial \eta_w} \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \right\} ds.$$

Concerning the boundary values in the integrand of the right hand side of the latter, we have:

$$\frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \to 0 \quad \text{as} \quad w \to \partial D_r.$$

This is clear since $w \to \partial D_r$ means that $|w| \to r$, whence $(r^2 - |w|^2)^{2+\alpha} \to 0$. With regard to the normal derivative

$$\frac{\partial}{\partial \eta_w} \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}}, \tag{2.1.6}$$

the things are less trivial. Let us compute this quantity: we know that

$$\frac{\partial}{\partial \eta} u = \overrightarrow{\nabla} u \cdot \overrightarrow{\eta} = \nabla u \cdot \eta.$$

The normal vector on ∂D_r is the normalized radius vector, i.e.

$$\eta = \frac{z}{r}.$$

On the other hand, as we have seen in Definition 1.1.12, the gradient of a function $u(z)$ is given by:

$$\nabla u(z) = \left(\partial u + \overline{\partial} u, i(\partial u - \overline{\partial} u)\right),$$

with $\partial = \frac{\partial}{\partial z}$, $\overline{\partial} = \frac{\partial}{\partial \overline{z}}$. Hence the normal derivative of $u(z)$ is:

$$\begin{aligned}
\frac{\partial u}{\partial \eta}(z) &= \nabla u \cdot \eta = (\partial u + \overline{\partial} u, i(\partial u - \overline{\partial} u)) \cdot \left(\frac{x}{r}, \frac{y}{r}\right) \\
&= \frac{x \partial u}{r} + \frac{x \overline{\partial} u}{r} + i \frac{y \partial u}{r} - i \frac{y \overline{\partial u}}{r} \\
&= \frac{1}{r} \partial u (x + iy) + \frac{1}{r} \overline{\partial} u (x - iy) \\
&= \frac{z}{r} \partial u + \frac{\overline{z}}{r} \overline{\partial} u.
\end{aligned}$$

Thus the normal derivative in this case is given by

$$\frac{\partial}{\partial \eta} = \frac{z}{r} \frac{\partial}{\partial z} + \frac{\overline{z}}{r} \frac{\partial}{\partial \overline{z}}.$$

In particular on ∂D, we have:

$$\frac{\partial}{\partial \eta} = z \frac{\partial}{\partial z} + \overline{z} \frac{\partial}{\partial \overline{z}}.$$

In order to compute the normal derivative in (2.1.6), we must first compute:

$$\begin{aligned}
\frac{\partial}{\partial w} \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w} z|^{4+2\alpha}} &= \frac{\partial}{\partial w} \frac{(r^2 - w\overline{w})^{2+\alpha}}{(1 - \overline{w} z)^{2+\alpha} (1 - w \overline{z})^{2+\alpha}} \\
&= \frac{(2+\alpha)(-\overline{w})(r^2 - w\overline{w})^{1+\alpha} |1 - \overline{w} z|^{4+2\alpha}}{|1 - \overline{w} z|^{8+4\alpha}} \\
&\quad + \frac{-(2+\alpha)(-\overline{z})(1 - \overline{w} z)^{2+\alpha} (1 - w\overline{z})^{1+\alpha} (r^2 - w\overline{w})^{2+\alpha}}{|1 - \overline{w} z|^{8+4\alpha}} \\
&= \frac{(2+\alpha)(r^2 - |w|^2)^{1+\alpha} \{\overline{z}(1 - \overline{w} z)(r^2 - |w|^2) - \overline{w} |1 - \overline{w} z|^2\}}{|1 - \overline{w} z|^{6+2\alpha}},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \overline{w}} \frac{(r^2-|w|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} &= \frac{\partial}{\partial \overline{w}} \frac{(r^2-w\overline{w})^{2+\alpha}}{(1-\overline{w}z)^{2+\alpha}(1-w\overline{z})^{2+\alpha}} \\
&= \frac{(2+\alpha)(-w)(r^2-|w|^2)^{1+\alpha}\,|1-\overline{w}z|^{4+2\alpha}}{|1-\overline{w}z|^{8+4\alpha}} \\
&\quad + \frac{-(2+\alpha)(-z)(1-\overline{w}z)^{1+\alpha}(1-w\overline{z})^{2+\alpha}(r^2-|w|^2)^{2+\alpha}}{|1-\overline{w}z|^{8+4\alpha}} \\
&= \frac{(2+\alpha)(r^2-|w|^2)^{1+\alpha}\{z(1-w\overline{z})(r^2-|w|^2)-w\,|1-\overline{w}z|^2\}}{|1-\overline{w}z|^{6+2\alpha}}.
\end{aligned}$$

Thus,

$$\frac{\partial}{\partial \eta_w} \frac{(r^2-|w|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} = \frac{(2+\alpha)(r^2-|w|^2)^{1+\alpha}}{|1-\overline{w}z|^{6+2\alpha}} \left\{ \begin{array}{c} w\overline{z}(1-\overline{w}z)(r^2-|w|^2)-|w|^2\,|1-\overline{w}z|^2 \\ +\overline{w}z(1-w\overline{z})(r^2-|w|^2)-|w|^2\,|1-\overline{w}z|^2 \end{array} \right\}. \tag{2.1.7}$$

Now, clearly

$$\frac{\partial}{\partial \eta_w} \frac{(r^2-|w|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} \to 0 \quad \text{as} \quad w \to \partial D_r,$$

provided that

$$\frac{w\overline{z}(1-\overline{w}z)(r^2-|w|^2)-|w|^2\,|1-\overline{w}z|^2+\overline{w}z(1-w\overline{z})(r^2-|w|^2)-|w|^2\,|1-\overline{w}z|^2}{|1-\overline{w}z|^{4+2\alpha}}, \tag{2.1.8}$$

is bounded. Since $|w\overline{z}| \le |w|\,|\overline{z}| \le r$, $(r^2-|w|^2) \le r^2$, we have

$$0 < 1-r \le 1-|\overline{w}z| \le |1-\overline{w}z| \le 1+|\overline{w}z| \le 1+r < 2.$$

Hence, we get

$$\frac{1}{|1-\overline{w}z|^{6+2\alpha}} \le \frac{1}{(1-r)^{6+2\alpha}}.$$

Thus

$$\left| \frac{w\overline{z}(1-\overline{w}z)(r^2-|w|^2)+\overline{w}z(1-w\overline{z})(r^2-|w|^2)-2\,|w|^2\,|1-\overline{w}z|^2}{|1-\overline{w}z|^{4+2\alpha}} \right| \le \frac{2r^3+8r^2}{(1-r)^{6+2\alpha}}.$$

Whence (2.1.8) is bounded. Hence, from (2.1.7), we see that

$$\frac{\partial}{\partial \eta_w} \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \to 0 \text{ as } w \to \partial D_r.$$

Now, we must show that

$$\Delta_w \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \;\to\; \Delta_w \frac{(1 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \quad \text{as} \quad r \to 1.$$

We know that

$$\Delta_w \frac{(1 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} = \frac{(2+\alpha)(1 - |w|^2)^{\alpha} \left\{(2+\alpha)(|w|^2 + |z|^2 - w\overline{z} - z\overline{w}) - |1 - \overline{z}w|^2\right\}}{|1 - \overline{z}w|^{6+2\alpha}}.$$

While

$$\begin{aligned}
\Delta_w \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} &= \frac{(2+\alpha)(r^2 - |w|^2)^{1+\alpha}}{|1 - \overline{w}z|^{6+2\alpha}} \left\{ \begin{array}{c} -|1 - \overline{w}z|^2 + (2+\alpha)|w|^2 |1 - \overline{w}z|^2 \\ -(2+\alpha)w\overline{z}(1 - \overline{w}z)(r^2 - |w|^2) \\ +(2+\alpha)|z|^2 (r^2 - |w|^2)^2 \\ -(2+\alpha)z\overline{w}(1 - w\overline{z})(r^2 - |w|^2) \end{array} \right\} \\
&= \frac{(2+\alpha)(r^2 - |w|^2)^{\alpha}\{(2+\alpha)(|w|^2 + r^4|z|^2 - r^2 z\overline{w} - r^2 w\overline{z}) - |1 - \overline{w}z|^2\}}{|1 - \overline{w}z|^{6+2\alpha}}.
\end{aligned}$$

For each z, as $r \to 1$, the latter expression goes to

$$\frac{(2+\alpha)(1 - |w|^2)^{\alpha}\{(2+\alpha)(|w|^2 + |z|^2 - w\overline{z} - z\overline{w}) - |1 - \overline{z}w|^2\}}{|1 - \overline{z}w|^{6+2\alpha}} = \Delta_w \frac{(1 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}}.$$

Thus we obtain:

$$\int_{D_r} \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \Delta u(w)\, dA(w) = \int_{D_r} u(w) \Delta_w \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}}\, dA(w). \qquad (2.1.9)$$

Now, by the uniform boundedness, (since $r < 1$), we have

$$\left|(r^2 - |w|^2)^{2+\alpha} \Delta u(w)\right| \le \left|(1 - |w|^2)^{2+\alpha} \Delta u\right| \text{ in } D_r.$$

Therefore, we are able to pass to the limit as $r \to 1$ under the integral sign in the left hand side of Equation (2.1.9). On the other hand for fixed $z \in D$, we see that

$$\Delta_w \frac{(r^2 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \to \Delta_w \frac{(1 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \text{ as } r \to 1.$$

So, we obtain from Equation (2.1.9) that

$$\int_D u(w)\Delta_w \frac{(1 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} dA(w) = \int_D \frac{(1 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} \Delta u(w) dA(w).$$

Multiplying both sides by $(\alpha + 1)(1 - |z|^2)^{2+\alpha}$, we obtain for the second integral:

$$\begin{aligned} & (\alpha+1)(1-|z|^2)^{2+\alpha} \int_D \frac{(1-|w|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} \Delta u(w) dA(w) \\ = & \int_D \frac{(1-|z|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} (1-|w|^2)^2 \Delta u(w) (\alpha+1)(1-|w|^2)^{\alpha} dA(w) \\ = & \int_D \frac{(1-|z|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} \widetilde{\Delta} u(w) dA_\alpha(w) = B_\alpha(\widetilde{\Delta} u)(z). \end{aligned}$$

Whereas concerning the first integral, we use the identity

$$(1 - |z|^2)^\alpha \Delta_w \frac{(1 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}} = (1 - |w|^2)^\alpha \Delta_z \frac{(1 - |z|^2)^{2+\alpha}}{|1 - \overline{w}z|^{4+2\alpha}},$$

and we obtain

$$\begin{aligned} & (\alpha+1)(1-|z|^2)^{2+\alpha} \int_D u(w) \Delta_w \frac{(1-|w|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} dA(w) \\ = & \int_D u(w)(1-|z|^2)^2 \left\{ (1-|z|^2)^\alpha \Delta_w \frac{(1-|w|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} \right\} (\alpha+1) dA(w) \\ = & \int_D u(w)(1-|z|^2)^2 \left\{ (1-|w|^2)^\alpha \Delta_z \frac{(1-|z|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} \right\} (\alpha+1) dA(w). \quad (2.3.1) \end{aligned}$$

But the right hand side of Equation (2.1.10) can be written as

$$\begin{aligned} & \int_D u(w)(1-|z|^2)^2 \Delta_z \frac{(1-|z|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} (\alpha+1)(1-|w|^2)^\alpha dA(w) \\ = & \int_D u(w) \widetilde{\Delta}_z \frac{(1-|z|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} dA_\alpha(w) = \widetilde{\Delta}_z \int_D \frac{(1-|z|^2)^{2+\alpha}}{|1-\overline{w}z|^{4+2\alpha}} u(w) dA_\alpha(w) \\ = & \widetilde{\Delta}_z B_\alpha u = \widetilde{\Delta}(B_\alpha u)(z). \end{aligned}$$

Thus, we obtain

$$B_\alpha(\widetilde{\Delta} u) = \widetilde{\Delta}(B_\alpha u).$$

This ends the proof ■

2.4 Remarks to the range of the Berezin transform

In this section we introduce the following question: Can we characterize all triples (f, g, u) where f and g are non-constant holomorphic functions on the unit disc D and u is integrable on D such that $f\overline{g} = Bu$? The answer was given recently in 2004 by Patrick Ahern as we are going to see in Theorem 2.4.5.

Definition 2.4.1. *For holomorphic functions f and g on D and $u \in L^1$, we say that $f\overline{g} = B(u)$ holds in a non-trivial way if neither f nor g is constant.*

Non-trivial examples will be given later, Example 2.4.3 as well as Lemma 2.4.4. First of all, we show that the area measure is rotation invariant. This property is certainly known and widely used by specialists. However, we cannot localize any reference giving a proof. So we have decided to provide our own proof.

Lemma 2.4.2. *The area measure is rotation invariant. i.e. if $w = e^{i\theta} z$ then $dA(z) = dA(w)$.*

Proof: Let $z = x + iy$ and let $w = ze^{i\theta} = u + iv$. We want to show that $dA(z) = dA(w)$, i.e. we need to show that $\frac{dxdy}{\pi} = \frac{dudv}{\pi}$. We have that

$$\begin{aligned} w &= ze^{i\theta} = z(\cos\theta + i\sin\theta) = z\cos\theta + iz\sin\theta \\ &= (x+iy)\cos\theta + i(x+iy)\sin\theta \\ &= (x\cos\theta - y\sin\theta) + i(y\cos\theta + x\sin\theta). \end{aligned}$$

Hence $u = x\cos\theta - y\sin\theta$ and $v = y\cos\theta + x\sin\theta$. By differentiation, we obtain

$$du = \cos\theta dx - \sin\theta dy \text{ and } dv = \sin\theta dx + \cos\theta dy.$$

Such equations can be rewritten in terms of a matrix system as follows:

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

But this transformation has real Jacobian equals to

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1.$$

Thus $dudv = dxdy$, which implies that $\frac{dudv}{\pi} = \frac{dxdy}{\pi}$ and the lemma is proved. ■

Probably, the simplest question one could ask here is: Does there exist a function $u \in L^1$ such that $z\overline{z} = B(u)(z)$? It turns out that the answer is yes, as can be seen through the following

Lemma 2.4.3. *The Berezin transform of the function*

$$u(\zeta) = 1 - \log\frac{1}{|\zeta|^2}$$

is given by

$$B(u)(z) = z\overline{z}.$$

Proof: To see this we need to show that if $v(\zeta) = \log\frac{1}{|\zeta|^2}$, then $B(v)(z) = 1 - |z|^2$. First we have that

$$B(\log|z|^2) = (1-|z|^2)^2 \int_D \frac{\log|\zeta|^2}{|1-\overline{\zeta}z|^4} dA(\zeta).$$

And since

$$\begin{aligned} \frac{1}{|1-\overline{\zeta}z|^4} &= \frac{1}{(1-\overline{\zeta}z)^2}\frac{1}{(1-\zeta\overline{z})^2} \\ &= \sum_n (n+1)(\overline{\zeta}z)^n \sum_k (k+1)(\zeta\overline{z})^k \\ &= \sum_{n,k} (n+1)(k+1)(\overline{\zeta}z)^n(\zeta\overline{z})^k, \end{aligned}$$

we see that

$$\int_D \frac{\log|\zeta|^2}{\left|1-\overline{\zeta}z\right|^4} dA(\zeta) = \int_D \log|\zeta|^2 \left(\sum_{n,k} (n+1)(k+1)\overline{\zeta}^n \zeta^k z^n \overline{z}^k \right) dA(\zeta).$$

But

$$\begin{aligned}\int_D \overline{\zeta}^n \zeta^k \log|\zeta|^2 \, dA(\zeta) &= \int_0^1 \int_0^{2\pi} r r^{n+k} e^{i\theta(k-n)} \log r^2 \frac{dr d\theta}{\pi} \;; "\zeta = re^{i\theta}" \\ &= \int_0^1 r r^{n+k} \log r^2 dr \int_0^{2\pi} e^{i\theta(k-n)} \frac{d\theta}{\pi},\end{aligned}$$

and

$$\int_0^{2\pi} e^{i\theta(k-n)} \frac{d\theta}{\pi} = \begin{cases} 0, & \text{if } k \neq n, \\ 2, & \text{if } k = n. \end{cases}$$

That is to say

$$\int_D \overline{\zeta}^n \zeta^k \log|\zeta|^2 \, dA(\zeta) = \begin{cases} 0, & \text{if } k \neq n, \\ \int_D |\zeta|^{2n} \log|\zeta|^2 \, dA(\zeta), & \text{if } k = n. \end{cases}$$

Hence

$$\int_D \frac{\log|\zeta|^2}{\left|1-\overline{\zeta}z\right|^4} dA(\zeta) = \sum_n (n+1)^2 \int_D |\zeta|^{2n} \log|\zeta|^2 \, dA(\zeta) \, |z|^{2n}. \tag{2.2.1}$$

Now

$$\int_D |\zeta|^{2n} \log|\zeta|^2 \, dA(\zeta) = \int_0^1 \int_0^{2\pi} r^{2n} \log r^2 \frac{r dr d\theta}{\pi} = 2\int_0^1 r^{2n} \log r^2 r dr.$$

Let $u = r^2$, then $du = 2rdr$. So

$$\begin{aligned}\int_D |\zeta|^{2n} \log|\zeta|^2 \, dA(\zeta) &= \int_0^1 u^n \log u \; du = \lim_{a\to 0} \left(\left[\log u \frac{u^{n+1}}{n+1} \right]_a^1 \right) - \int_0^1 \frac{u^n}{n+1} du \\ &= \left[-\frac{u^{n+1}}{(n+1)2} \right]_0^1 = -\frac{1}{(n+1)^2}.\end{aligned}$$

Thus, the right hand side of Equation (2.2.1) becomes

$$\sum_n -\frac{(n+1)^2}{(n+1)^2} |z|^{2n} = -\sum_n |z|^{2n} = -\left(\frac{1}{1-|z|^2} \right).$$

Hence

$$(1-|z|^2)^2\int_D \frac{\log|\zeta|^2}{\left|1-\overline{\zeta}z\right|^4}dA(\zeta) = -(1-|z|^2) = |z|^2-1,$$

and

$$\begin{aligned} B(v)(z) &= (1-|z|^2)^2\int_D \frac{\log\frac{1}{|\zeta|^2}}{\left|1-\overline{\zeta}z\right|^4}dA(\zeta) \\ &= -(1-|z|^2)^2\int_D \frac{\log|\zeta|^2}{\left|1-\overline{\zeta}z\right|^4}dA(\zeta) = -(|z|^2-1) = 1-|z|^2. \end{aligned}$$

and

$$u(\zeta) = 1-v(\zeta) \Rightarrow B(u) = 1-B(v) = 1-(1-|z|^2) = |z|^2 = z\overline{z}.$$

This ends the proof. ■

P. Ahern in [1] has proved that the image under the Berezin transform of the function $2\overline{\zeta}-\frac{1}{\zeta}$ is $z\overline{z}^2$. His elegant proof uses some facts from distribution theory together with the Formula (2.1.2) of the Berezin transform. He mentioned that a direct calculation using Formula (2.1.1) could so work. We have done such hard calculation by our own methods, and we have obtained the same results. For convenience we give the two fairly different proofs of this result.

Lemma 2.4.4. *The Berezin transform of the function*

$$u(\zeta) = 2\overline{\zeta}-\frac{1}{\zeta},$$

is given by

$$B(u)(z) = z\overline{z}^2.$$

Proof: *The first proof:* [using the direct calculation]

Let us first observe that $2\overline{\zeta}$ is a harmonic function. Since the Berezin transform reproduces harmonic functions as we have seen in Theorem 2.1.5, we infer that

$$B(2\overline{\zeta})(z) = 2\overline{z}. \tag{2.2.2}$$

On the other hand, in order to calculate $B\left(\frac{1}{\zeta}\right)$, we make use of the Formula (2.1.1). We have

$$\begin{aligned} B\left(\frac{1}{\zeta}\right)(z) &= \left(1-|z|^2\right)^2 \int_D \frac{1}{\zeta\left|1-z\overline{\zeta}\right|^4} dA(\zeta) \\ &= \frac{1}{\pi}\left(1-|z|^2\right)^2 \int_0^1 \int_0^{2\pi} \frac{1}{re^{i\theta}\left|1-zre^{-i\theta}\right|^4} r\,dr\,d\theta \\ &= \frac{1}{\pi}\left(1-|z|^2\right)^2 \int_0^1 \left(\int_0^{2\pi} \frac{d\theta}{e^{i\theta}\left|1-zre^{-i\theta}\right|^4}\right) dr. \end{aligned}$$

Using the substitution $t = e^{i\theta}$ in the latter integral we obtain

$$B\left(\frac{1}{\zeta}\right)(z) = \frac{-i}{\pi}\left(1-|z|^2\right)^2 \int_0^1 \int_{|t|=1} \frac{dt}{t^2\left|t-zr\right|^4}. \tag{2.2.3}$$

Let us denote the inside integral by

$$I_2 = \int_{|t|=1} \frac{dt}{t^2\left|t-zr\right|^4},$$

and notice that $|t-zr|^4 = \left(|t-zr|^2\right)^2 = \left((t-zr)\left(\overline{t}-\overline{z}r\right)\right)^2 = \left(1-zr\overline{t}-\overline{z}rt+|z|^2 r^2\right)^2$.

Now, setting $a = \overline{z}rt$ in I_2, we obtain

$$\begin{aligned} I_2 &= \int_{|a|=|rz|} \frac{\overline{z}r\,da}{a^2(1-(a+\overline{a})+r^2|z|^2)^2} \\ &= \overline{z}r\int_{|a|=|rz|} \frac{da}{a^2(1-(a+\overline{a})+r^2|z|^2)^2}. \end{aligned}$$

In order to evaluate I_2, set

$$I_3 = \int_{|a|=|rz|} \frac{da}{a^2(1-(a+\overline{a})+r^2|z|^2)^2}.$$

Using the substitution $a = r|z|e^{i\theta}$, $0 \le \theta \le 2\pi$, we see that $da = ia\,d\theta$ and $a+\overline{a} = 2r|z|\cos\theta$. Thus, we obtain

$$I_3 = i\int_0^{2\pi} \frac{r|z|e^{i\theta}d\theta}{r^2|z|^2 e^{2i\theta}\left(1-2r|z|\cos\theta+r^2|z|^2\right)^2} = \frac{i}{r|z|}\int_0^{2\pi} \frac{d\theta}{e^{i\theta}\left(1-2r|z|\cos\theta+r^2|z|^2\right)^2}.$$

Again, put $w = e^{i\theta}$, then $\cos\theta = \frac{1}{2}\left(w + \frac{1}{w}\right)$. Using either residues or Cauchy's theorem, I_3 reduces to:

$$\begin{aligned} I_3 &= \frac{1}{r|z|}\int_{|w|=1} \frac{dw}{w^2\left(1 - r|z|\left(w + \frac{1}{w}\right) + r^2|z|^2\right)^2} \\ &= \frac{1}{r|z|}\int_{|w|=1} \frac{dw}{\left(w - r|z|w^2 - r|z| + r^2|z|^2 w\right)^2} \\ &= \frac{1}{r|z|}\int_{|w|=1} \frac{dw}{\left(-r|z|(w - r|z|)\left(w - \frac{1}{r|z|}\right)\right)^2} \\ &= \frac{1}{r^3|z|^3}\int_{|w|=1} \frac{dw}{(w - r|z|)^2\left(w - \frac{1}{r|z|}\right)^2} \\ &= \frac{2\pi i}{r^3|z|^3}\lim_{w\to r|z|}\left(\frac{d}{dw}\frac{1}{\left(w - \frac{1}{r|z|}\right)2}\right). \end{aligned}$$

Thus, we get

$$I_3 = \frac{2\pi i}{r^3|z|^3}\frac{-2}{\left(r|z| - \frac{1}{r|z|}\right)^3} = \frac{4\pi i}{\left(1 - r^2|z|^2\right)^3}.$$

We conclude that

$$I_2 = \frac{4\pi i\overline{z}r}{\left(1 - r^2|z|^2\right)^3}. \tag{2.2.4}$$

Substituting (2.2.4) in (2.2.3), we see that

$$\begin{aligned} B\left(\frac{1}{\zeta}\right)(z) &= 4\overline{z}\left(1 - |z|^2\right)^2\int_0^1 \frac{r}{\left(1 - r^2|z|^2\right)^3}dr \\ &= \frac{4\overline{z}}{2|z|^2}\left(1 - |z|^2\right)^2\int_0^{|z|^2}\frac{du}{(1-u)^3}, \text{ where } u = r^2|z|^2, \\ &= \frac{2}{z}\left(1 - |z|^2\right)^2\left(\frac{1}{2\left(1 - |z|^2\right)^2} - \frac{1}{2}\right) \\ &= \frac{\left(1 - |z|^2\right)^2}{z}\frac{1 - 1 + 2|z|^2 - |z|^4}{\left(1 - |z|^2\right)^2} \\ &= \frac{|z|^2\left(2 - |z|^2\right)}{z} = 2\overline{z} - z\overline{z}^2. \end{aligned}$$

Combining the latter with (2.2.2), we infer that

$$B\left(2\overline{\zeta}-\frac{1}{\zeta}\right)(z)=2\overline{z}-2\overline{z}+z\overline{z}^{2}.$$

This ends the first proof.■

The second proof: [due to P. Ahern [1]]. According to P. Ahern in [1], this lemma has a more interesting proof if we use Proposition 2.1.2 to calculate $B(v)(z)$, where

$$v(\zeta)=\frac{1}{\zeta}.$$

First recall that the fundamental solution of $\overline{\partial}$ operator is given by $\frac{1}{\pi z}$, as we have seen in Example 1.1.37. By Lemma 1.1.35, we have

$$\frac{\partial}{\partial\overline{z}}\left(1*\frac{1}{\pi z}\right)=1*\frac{\partial}{\partial\overline{z}}\left(\frac{1}{\pi z}\right)=1*\delta=1.$$

Denote again by 1 the constant function 1. So we have

$$\begin{aligned}1*\frac{1}{\pi z} &= \iint_{(x,y)\in D\times D}\frac{1}{\pi(z-\zeta)}1(\zeta)dxdy\\ &= \iint_{(x,y)\in D\times D}\frac{1}{z-\zeta}\frac{dxdy}{\pi}=\int_D\frac{1}{z-\zeta}dA(\zeta).\end{aligned}$$

Hence, we see that

$$\frac{\partial}{\partial\overline{z}}\left(1*\frac{1}{\pi z}\right)=\frac{\partial}{\partial\overline{z}}\int_D\frac{1}{z-\zeta}dA(\zeta)=1,\ \text{for all } z\in D.$$

Integrating both sides of the latter with respect to $\overline{z}$ we get

$$\int_D\frac{1}{z-\zeta}dA(\zeta)=\overline{z}+h(z)$$

where h is holomorphic in D.

Now, remark that $\frac{1}{z}$ is a locally integrable function in $\mathbb{C}$. And consider the bounded function with compact support, namely

$$1(z) = \begin{cases} 1, & \text{if } z \in \overline{D}, \\ 0, & \text{if } z \in \mathbb{C} \setminus \overline{D}. \end{cases}$$

The function $\int_D \frac{1}{z-\zeta} dA(\zeta)$ is the convolution of the latter two functions. Then it is continuous on the entire plane. Hence h is continuous up to the boundary of D. So we have

$$\int_D \frac{1}{e^{i\theta} - \zeta} dA(\zeta) = e^{-i\theta} + h(e^{i\theta}).$$

Now, by Lemma 2.4.2, the area measure is rotation invariant, thus we have

$$\int_D \frac{1}{e^{i\theta} - \zeta} dA(\zeta) = e^{-i\theta} \int_D \frac{1}{1 - e^{-i\theta}\zeta} dA(\zeta) = e^{-i\theta} \int_D \frac{1}{1-\zeta} dA(\zeta).$$

But the integral on the R.H.S. of the latter is finite, i.e. $c = \int_D \frac{1}{1-\zeta} dA(\zeta) < \infty$, so we infer that $e^{-i\theta}c = e^{-i\theta} + h(e^{i\theta})$ for some constant c. In other words $c = 1 + e^{i\theta}h(e^{i\theta})$, i.e. $1 + zh(z) - c = 0$ on ∂D. Hence by the maximum principle we see that. $1 + zh(z) - c = 0$ for all $z \in D$. Letting $z = 0$ in the latter, we see that $c = 1$ and hence $h(z) \equiv 0$. We conclude that

$$\int_D \frac{1}{z - \zeta} dA(\zeta) = \overline{z} \quad \text{for } z \in D.$$

Now,

$$\begin{aligned}
B(v)(z) &= \int_D v\left(\frac{z-\zeta}{1-\overline{z}\zeta}\right) dA(\zeta) = \int_D \left(\frac{1-\overline{z}\zeta}{z-\zeta}\right) dA(\zeta) \\
&= \int_D \frac{1}{z-\zeta} dA(\zeta) - \overline{z} \int_D \frac{\zeta}{z-\zeta} dA(\zeta) \\
&= \overline{z} - \overline{z} \int_D \left(-1 + \frac{z}{z-\zeta}\right) dA(\zeta) \\
&= \overline{z} + \overline{z} \int_D dA(\zeta) - \overline{z}z \int_D \frac{1}{z-\zeta} dA(\zeta) \\
&= \overline{z} + \overline{z} \int_0^1 \int_0^{2\pi} \frac{r dr d\theta}{\pi} - z\overline{z}^2 = 2\overline{z} - z\overline{z}^2.
\end{aligned}$$

Finally, we obtain

$$B\left(2\overline{\zeta}-\frac{1}{\zeta}\right)(z)=2\overline{z}-2\overline{z}+z\overline{z}^2=z\overline{z}^2.$$

which completes the second proof. ■

Now we state and prove the main result of this section. It characterizes all triples (f,g,u) where f and g are non-constant and $u\in L^1$ such that $f\overline{g}=B(u)$. This significantly deep result is due to P. Ahern [1]. The proof is hard and long, so that we present it in several steps. This nice piece of work has prodigious applications in establishing more general Brown-Halmos type theorems as we are going to see in the next chapter.

Theorem 2.4.5. *If f and g are holomorphic in D and neither is constant and $f\overline{g}=B(u)$ for some $u\in L^1$, then there are non-constant polynomials p and q with $\deg(pq)\le 3$ and an $a\in D$ such that $f=p\circ\phi_a$ and $g=q\circ\phi_a$, where $\phi_a(z)=\frac{a-z}{1-\overline{a}z}$.*

Proof:

Step 1: Starting with $f\overline{g}=B(u)$, taking the Laplacian of both sides, we see that

$$\Delta_z f\overline{g}=\Delta_z(B(u)).$$

Using the definition of Δ_z and the differentiation rules, we obtain

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial\overline{z}}\overline{g}+\frac{\partial\overline{g}}{\partial\overline{z}}f\right)=\Delta_z\int_D\frac{(1-|z|^2)^2u(\zeta)}{\left|1-z\overline{\zeta}\right|^4}dA(\zeta).$$

Since f is holomorphic, we infer that

$$\frac{\partial}{\partial z}\left(\frac{\partial\overline{g}}{\partial\overline{z}}f\right)=\Delta_z\int_D\frac{(1-|z|^2)^2u(\zeta)}{\left|1-z\overline{\zeta}\right|^4}dA(\zeta).$$

Hence we obtain

$$f(\Delta_z g)+\frac{\partial f}{\partial z}\frac{\partial\overline{g}}{\partial\overline{z}}=\int_D u(\zeta)\Delta_z\frac{(1-|z|^2)^2}{\left|1-\overline{\zeta}z\right|^4}dA(\zeta).$$

But $\Delta_z g = 0$, as g is holomorphic, and if we use the symmetry identity

$$\Delta_z \frac{(1-|z|^2)^2}{\left|1-\overline{\zeta}z\right|^4} = \Delta_\zeta \frac{(1-|\zeta|^2)^2}{\left|1-\overline{\zeta}z\right|^4},$$

we arrive at

$$f'(z)\overline{g'}(z) = \int_D u(\zeta)\Delta_\zeta \frac{(1-|\zeta|^2)^2}{\left|1-\overline{\zeta}z\right|^4} dA(\zeta). \tag{2.2.5}$$

Now, we complexify the above equation to get

$$f'(z)\overline{g'}(\overline{w}) = \int_D u(\zeta)\Delta_\zeta \frac{(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2(1-\zeta w)^2} dA(\zeta). \tag{2.2.6}$$

Note that the functions on either sides of Equation (2.2.6) are holomorphic in the bidisk $\{(z,w) : |z| < 1, |w| < 1\}$ and by (2.2.5) they are equal on the subset $\{(z,\overline{z}) : |z| < 1\}$. Hence by Proposition 1.1.17, they are equal on the whole bidisk.

Now, differentiating both sides of Equation (2.2.6) with respect to w and then letting $w = 0$, we get

$$\frac{\partial}{\partial w}\overline{g'}(\overline{w})f'(z)|_{w=0} = \left(\int_D u(\zeta)\Delta_\zeta \left(\frac{(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2}\frac{2\zeta(1-\zeta w)}{(1-\zeta w)^4}\right) dA(\zeta)\right)|_{w=0}.$$

Hence we obtain

$$c_1 f'(z) = \int_D u(\zeta)\Delta_\zeta \frac{\zeta(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} dA(\zeta).$$

Differentiate again with respect to w and then let $w = 0$, we get

$$c_2 f'(z) = \left(\int_D u(\zeta)\Delta_\zeta \left(\frac{\zeta(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2}\frac{\zeta(1-\zeta w)^2}{(1-\zeta w)^6}\right) dA(\zeta)\right)|_{w=0}.$$

Thus, we get

$$c_2 f'(z) = \int_D u(\zeta)\Delta_\zeta \frac{\zeta^2(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} dA(\zeta).$$

Continuing in this manner, we arrive at

$$c_k f'(z) = \int_D u(\zeta)\Delta_\zeta \frac{\zeta^k(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} dA(\zeta), \tag{2.2.7}$$

for some constants $c_k, k = 1, 2, \cdots$.

Step 2: If p is a bounded holomorphic function in D, then we have

$$
\begin{aligned}
\Delta_\zeta \frac{p(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} &= \frac{\partial}{\partial\zeta}\left[\frac{\partial}{\partial\overline{\zeta}}\frac{p(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2}\right] \\
&= \frac{\partial}{\partial\zeta}\left[\frac{(1-\overline{\zeta}z)^2(2p(\zeta)(1-|\zeta|^2)(-\zeta)) - p(\zeta)(1-|\zeta|^2)^2(-2z(1-\overline{\zeta}z))}{(1-\overline{\zeta}z)^4}\right] \\
&= \frac{\partial}{\partial\zeta}\left[\frac{-2\zeta(1-\overline{\zeta}z)p(\zeta)(1-|\zeta|^2) + 2zp(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^3}\right] \\
&= \frac{\partial}{\partial\zeta}\left\{p(\zeta)\left[\frac{(-2\zeta+2\zeta\overline{\zeta}z)(1-|\zeta|^2) + 2z(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^3}\right]\right\} \\
&= \frac{p'(\zeta)\left[(-2\zeta+2\zeta\overline{\zeta}z)\left(1-|\zeta|^2\right) + 2z\left(1-|\zeta|^2\right)^2\right]}{(1-\overline{\zeta}z)^3} \\
&\quad + \frac{p(\zeta)\left[\left(-2+2\overline{\zeta}z\right)\left(1-|\zeta|^2\right) + (-z\zeta+2\zeta\overline{\zeta}z)(-\overline{\zeta}) - 4\overline{\zeta}z\left(1-|\zeta|^2\right)\right]}{(1-\overline{\zeta}z)^3} \\
&= \frac{-2\zeta p'(\zeta)(1-\overline{\zeta}z)\left(1-|\zeta|^2\right) + 2zp'(\zeta)\left(1-|\zeta|^2\right)^2}{(1-\overline{\zeta}z)^3} \\
&\quad + \frac{-2p(\zeta)(1-\overline{\zeta}z)\left(1-|\zeta|^2\right) + 2|\zeta|^2 p(\zeta)(1-\overline{\zeta}z) - 4\overline{\zeta}zp(\zeta)\left(1-|\zeta|^2\right)}{(1-\overline{\zeta}z)^3} \\
&= \frac{-2\zeta p'(\zeta)\left(1-|\zeta|^2\right) - 2p(\zeta)(1-|\zeta|^2) + 2|\zeta|^2 p(\zeta)}{(1-\overline{\zeta}z)^2} \\
&\quad + \frac{2zp'(\zeta)(1-|\zeta|^2)^2 - 4\overline{\zeta}zp(\zeta)(1-|\zeta|^{2)}}{(1-\overline{\zeta}z)^3} \\
&= \frac{-2\zeta p'(\zeta)(1-|\zeta|^2) + 2p(\zeta)(-1+2|\zeta|^2)}{(1-\overline{\zeta}z)^2} + \frac{(1-|\zeta|^2)}{(1-\overline{\zeta}z)^3}(2z(p'(\zeta)(1-|\zeta|^2) - 2\overline{\zeta}p(\zeta)).
\end{aligned}
$$

Now if p is a bounded holomorphic function in D, then it is a Bloch function as $H^\infty \subseteq \mathbf{B}$, whence

$$\operatorname*{Sup}_D |p'(\zeta)|(1-|\zeta|^2) \leq c \operatorname*{Sup}_D |p(\zeta)|,$$

where c is independent of p. So we have

$$\Delta_\zeta \frac{p(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} = b_1(\zeta)K_\zeta(z) + zb_2(\zeta)k_\zeta^1(z),$$

where

$$b_1(\zeta) = -2\zeta p'(\zeta)(1-|\zeta|^2) + 2p(\zeta)(2|\zeta|^2-1),$$

and

$$b_2(\zeta) = p'(\zeta)(1-|\zeta|^2) - 2\overline{\zeta}p(\zeta).$$

Now again since $H^\infty \subset \mathbf{B}$, we see that:

$$\begin{aligned} \|b_1\|_\infty &= Sup\,|b_1(\zeta)| \leq Sup\left|-2\zeta p'(\zeta)(1-|\zeta|^2)\right| + 2Sup\left|p(\zeta)(2|\zeta|^2-1)\right| \\ &\leq c_1 Sup\left|p'(\zeta)(1-|\zeta|^2)\right| + c_2 Sup\,|p(\zeta)|\,, \text{ for } \zeta \in D. \\ &\leq cSup\,|p(\zeta)| \\ &= c\,\|p\|_\infty\,. \end{aligned}$$

By a similar way we obtain

$$\|b_2\|_\infty \leq c\,\|p\|_\infty\,.$$

We conclude that

$$\|b_j\|_\infty \leq c\,\|p\|_\infty\,,$$

for $j = 1, 2$, and for some constant c independent of p. Now, we have

$$\Delta_\zeta \frac{p(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} = b_1(\zeta)K_\zeta(z) + zb_2(\zeta)k_\zeta^1(z).$$

Then, we get

$$\begin{aligned} \int_D u(\zeta)\Delta_\zeta \frac{p(\zeta)\left(1-|\zeta|^2\right)^2}{(1-\overline{\zeta}z)^2}dA(\zeta) &= \int_D u(\zeta)\,b_1(\zeta)K_\zeta(z)dA(\zeta) + \int_D zu(\zeta)\,b_2(\zeta)k_\zeta^1(z)dA(\zeta) \\ &= F_1(z) + zF_2(z), \end{aligned}$$

where $F_1 = Pv_1$, $v_1 = ub_1 \in L^1$ and $F_2 = P_1 v_2$, $v_2 = ub_2 \in L^1$. Moreover, we have

$$\|v_i\|_1 = \|ub_i\|_1 \le \|u\|_1 \|b_i\|_\infty \le c\,\|p\|_\infty \|u\|_1 .$$

It follows that $F_2 = P_1 v_2$ and hence $F_3 = zF_2$ lie in L^1_a. In other words we now see that

$$\int_D u(\zeta)\Delta_\zeta \frac{p(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} dA(\zeta) = PH(z),$$

where $H(z) = v_1(z) + F_3(z) \in L^1$. For simplicity let us denote the integral in the L.H.S. of the latter by

$$\int_D u(\zeta)\Delta_\zeta \frac{p(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} dA(\zeta) = S_u\left(p(z)\right).$$

Step 3: Since we are assuming that g is not constant it follows from Equation (2.2.7) that not all c_k are 0. According to (2.2.7), this means that f' is a constant times $S_u(\zeta^k)$ for some k. Hence $f' = c_k S_u(\zeta^k) = PH$, for some $H \in L^1$. It now follows that for any holomorphic polynomial p, $p(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k$, we have

$$\begin{aligned}
S_u(p(\zeta)) &= S_u\left(\sum_{k=0}^{\infty} a_k \zeta^k\right) \\
&= \int_D u(\zeta)\Delta_\zeta \frac{\sum_{k=0}^{\infty} a_k \zeta^k (1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} dA(\zeta) \\
&= \int_D u(\zeta) \sum_{k=0}^{\infty} a_k \Delta_\zeta \frac{\zeta^k(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} dA(\zeta) \\
&= \sum_{k=0}^{\infty} a_k S_u(\zeta^k) \\
&= \sum_{k=0}^{\infty} a_k c_k f'(z).
\end{aligned}$$

Hence $S_u p = L(p) f'$ where L is a linear mapping from the set of all holomorphic polynomials into the field of complex numbers, that is given by

$$L(p(\zeta)) = L\left(\sum_{k=0}^{\infty} a_k \zeta^k\right) = \sum_{k=0}^{\infty} a_k c_k.$$

<u>**Step 4:**</u> Since we are assuming that f is not constant, f' is not identically zero. We pick $z \in D$ with $f'(z)$ is not identically zero, then we have

$$L(p)f'(z) = F_1(z) + zF_2(z).$$

Hence, since $F_1 = Pv_1$ and $F_2 = P_1v_2$, we obtain

$$\begin{aligned} |L(p)f'(z)| &\leq |F_1(z)| + |F_2(z)| \\ &\leq c\left(\|v_1\|_1 + \|v_2\|_1\right) \\ &\leq c\,\|u\|_1\,\|p\|_\infty\,. \end{aligned}$$

Hence

$$|L(p)| \leq c\frac{\|u\|_1}{|f'(z)|}\,\|p\|_\infty\,,$$

since $f'(z)$ is not identically zero. That is to say $|L(p)| \leq c\,\|p\|_\infty$ for some constant c that depends on z but not on p; whence L is bounded. Now Hahn-Banach Theorem 1.1.18 implies that L can be extended to the disk algebra $A(D)$ and that we have

$$S_u p(z) = L(p)f'(z).$$

Since L is continuous on $A(D)$, Riesz representation Theorem 1.1.19 implies that there is a unique $\lambda \in A(D)$ such that

$$L(p) = \langle p, \lambda\rangle = \int p(\zeta)\lambda(\zeta)d(\zeta), \text{ for all } p \in A(D),$$

In other words, there is a finite Borel measure μ on $\overline{D}$, with $d_\mu(\zeta) = \lambda(\zeta)d(\zeta)$, such that

$$L(p) = \int p(\zeta)d_\mu(\zeta), \text{ for all } p \in A(D).$$

From now on, we will denote f' by F. So $F = PH$ where $H \in L^1$.

Step 5: We will need some information about the Taylor coefficients of the projection of an L^1 function. Accordingly, let us observe that

$$\begin{aligned} F(z) &= \langle F, K_z\rangle = \langle PH, K_z\rangle = \langle H, K_z\rangle \\ &= \int_D \frac{H(\zeta)}{(1-\overline{\zeta}z)^2} dA(\zeta) = \sum_{n=0}^{\infty}\left((n+1)\int_D H(\zeta)\overline{\zeta}^n dA(\zeta)\right) z^n. \end{aligned}$$

A necessary condition for the convergence of this series reads as

$$\lim_{n\to\infty}(n+1)\int_D \overline{\zeta}^n H(\zeta)dA(\zeta) = 0,$$

which yields that

$$\lim_{n\to\infty}\int_D \overline{\zeta}^n H(\zeta)dA(\zeta) = 0.$$

In other words, if $F(z) = \sum_{n=0}^{\infty} F_n z^n$ then $F_n = o(n)$. From this information we conclude that F is not in L^1 but $\lim\limits_{r\to 1}\int_{D_r} F(z)\overline{G}(z)dA(z)$ exists for some $G \in H^\infty$. Here

$$D_r = \{z : |z| < r\}.$$

In fact, if $G(z) = \sum_{n=0}^{\infty} G_n z^n$ then we have

$$\begin{aligned} \lim_{r\to 1}\int_{D_r} F(z)\overline{G}(z)dA(z) &= \int_{D_r}\left(\sum_{n=0}^{\infty} F_n z^n\right)\left(\sum_{n=0}^{\infty}\overline{G_n} z^n\right) dA(z) \\ &= \int_0^r\int_0^{2\pi}\sum_{n=0}^{\infty} F_n\overline{G_n}a^{2n}\frac{a\,da\,d\theta}{\pi}, \quad \text{where } z = ae^{i\theta}, \\ &= 2\sum_{n=0}^{\infty} F_n\overline{G_n}\int_0^r a^{2n+1}da \\ &= \sum_{n=0}^{\infty} F_n\overline{G_n}\frac{r^{2n+2}}{n+1}. \end{aligned}$$

Since $F_n = o(n)$, $\left(as\frac{F_n}{n+1} \to 0\right)$, this integral have limit

$$\sum_{n=0}^{\infty}\frac{F_n\overline{G_n}}{n+1} \quad \text{as } r \to 1,$$

if $\sum_{n=0}^{\infty}|G_n|<\infty$. In fact we will only use G which are entire.

Step 6: Now, consider the equality

$$L(p)F(z)=\int_D u(\zeta)\Delta_\zeta\frac{p(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2}dA(\zeta).$$

Multiplying by $\overline{G}(z)$, we obtain

$$L(p)F(z)\overline{G}(z)=\overline{G}(z)\int_D u(\zeta)\Delta_\zeta\frac{p(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2}dA(\zeta).$$

Then we integrate over D_r, we obtain

$$\int_{D_r}L(p)F(z)\overline{G}(z)dA(z)=\int_{D_r}\left(\int_D u(\zeta)\Delta_\zeta\frac{p(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2}dA(\zeta)\right)\overline{G}(z)dA(z).$$

According to Step 5, this can be rewritten as

$$L(p)\sum_{n=0}^{\infty}F_n\overline{G_n}\frac{r^{2n+2}}{n+1}=\int_{D_r}\left(\int_D u(\zeta)\Delta_\zeta\frac{p(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2}dA(\zeta)\right)\overline{G}(z)dA(z).$$

Now, consider the integral of the R.H.S. of the latter. Since $r<1$ everything but $u(\zeta)$ is bounded and so we can interchange the order of integration to obtain

$$\int_D u(\zeta)\Delta_\zeta\left(p(\zeta)(1-|\zeta|^2)^2\int_{D_r}\frac{\overline{G}(z)}{(1-\overline{\zeta}z)^2}dA(z)\right)dA(\zeta). \tag{2.2.8}$$

In the inner integral let $z=rw$. If $z=x+iy$ and $w=u+iv$ then $rw=ru+irv$ and $dA(z)=\frac{dxdy}{\pi}=\frac{(rdu)(rdv)}{\pi}=\frac{r^2dudv}{\pi}=r^2dA(w)$. Thus,

$$\int_{D_r}\frac{\overline{G}(z)}{(1-\overline{\zeta}z)^2}dA(z)=r^2\int_{D_r}\frac{\overline{G}(rw)}{\left(1-(\overline{\zeta}r)w\right)^2}dA(w)=r^2\overline{G}(r(r\zeta))=r^2\overline{G}(r^2\zeta).$$

So that (2.2.8) reduces to

$$\int_D u(\zeta)\Delta_\zeta\left(p(\zeta)r^2\overline{G}(r^2\zeta)(1-|\zeta|^2)^2\right)dA(\zeta).$$

From this point we will always assume that p and G are entire functions. Relying on the fact that G is continuous, (as it is entire), we clearly see that $\Delta_\zeta\left(p(\zeta)r^2\overline{G}(r^2\zeta)(1-|\zeta|^2)^2\right)$

converges uniformly to $\Delta_\zeta(p(\zeta)\overline{G}(\zeta)(1-|\zeta|^2)^2)$ as $r \to 1$. Since $u \in L^1$, we can take the limit under the integral sign and finally obtain

$$L(p)\sum_{n=0}^{\infty}\frac{F_n\overline{G_n}}{n+1} = \int_D u(\zeta)\Delta_\zeta\left(p(\zeta)\overline{G}(\zeta)(1-|\zeta|^2)^2\right)dA(\zeta).$$

Step 7: At this stage we apply the discussion in the previous steps to the functions $p(\zeta)$=$e^{w\zeta}$ and $G(\zeta) = e^{-w\zeta}$. First, observe that the sum $\sum_{n=0}^{\infty}\frac{F_n\overline{G_n}}{n+1}$ becomes

$$\sum_{n=0}^{\infty}\frac{F_n}{n+1}\frac{(-\overline{w})^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n F_n \overline{w}^n}{(n+1)!}.$$

Let

$$J(w) = \sum_{n=0}^{\infty}\frac{(-1)^n \overline{F_n} w^n}{(n+1)!}.$$

Since $F_n = o(n)$ we see that J is an entire function. Moreover, since $F_n = o(n)$, we see that $\frac{|F_n|}{n+1} \to 0$, and hence we have the estimate

$$|J(w)| \le \sum_{n=0}^{\infty}\frac{|F_n|}{(n+1)}\frac{|w|^n}{n!} \le c\sum_{n=0}^{\infty}\frac{|w|^n}{n!} = ce^{|w|},$$

for some constant c.

Note that $L(p) = \int e^{w\zeta}d_\mu(\zeta) = K(w)$ is also an entire function. Since we deduce that K satisfies the estimate $|K(w)| \le ce^{|w|}$ for some constant c, as$\int e^{w\zeta}d_\mu(\zeta) = \sum_{n=0}^{\infty}\frac{w^n}{n!}\int \zeta^n d_\mu(\zeta)$. This implies that

$$\left|\int e^{w\zeta}d_\mu(\zeta)\right| = |K(w)| \le c\sum_{n=0}^{\infty}\frac{|w|^n}{n!} = ce^{|w|}.$$

Now we handle the integral

$$\int_D u(\zeta)\Delta_\zeta\left(p(\zeta)\overline{G}(\zeta)(1-|\zeta|^2)^2\right)dA(\zeta).$$

Calculate the Laplacian of the function that is under the integral sign:

$$
\begin{aligned}
\Delta_\zeta \left(e^{w\zeta-\overline{w\zeta}}(1-|\zeta|^2)^2\right) &= \frac{\partial}{\partial \zeta}\left[-\overline{w}e^{w\zeta}e^{-\overline{w\zeta}}(1-|\zeta|^2)^2 - 2\zeta e^{w\zeta-\overline{w\zeta}}(1-|\zeta|^2)\right] \\
&= -|w|^2 e^{-\overline{w\zeta}}e^{w\zeta}(1-|\zeta|^2)^2 + 2\overline{w\zeta}e^{w\zeta}e^{-\overline{w\zeta}}(1-|\zeta|^2) \\
&\quad + e^{-\overline{w\zeta}}\left(-2e^{w\zeta}-2w\zeta e^{w\zeta}\right)(1-|\zeta|^2) - 2\zeta e^{w\zeta-\overline{w\zeta}}(-\overline{\zeta}) \\
&= -|w|^2 e^{-\overline{w\zeta}}e^{w\zeta}(1-|\zeta|^2)^2 + 2\overline{w\zeta}e^{w\zeta-\overline{w\zeta}}(1-|\zeta|^2) \\
&\quad + 2|\zeta|^2 e^{w\zeta-\overline{w\zeta}} - 2e^{w\zeta-\overline{w\zeta}}(1-|\zeta|^2) - 2w\zeta e^{w\zeta-\overline{w\zeta}}(1-|\zeta|^2) \\
&= e^{w\zeta-\overline{w\zeta}}\left[\begin{array}{c} -|w|^2(1-|\zeta|^2)^2 + 2\overline{w\zeta}(1-|\zeta|^2) \\ -2w\zeta(1-|\zeta|^2) + 2(2|\zeta|^2-1) \end{array}\right].
\end{aligned}
$$

Now if $w = u + iv$ and $\zeta = x + iy$, then

$$w\zeta - \overline{w\zeta} = 2i(xv + yu).$$

The Fourier transform in $\mathbb{R}^2$ is given by:

$$\mathcal{F}(f)(x,y) = \int_{\mathbb{R}^2} e^{-i(x,y)(z,w)} f(z,w)\,d\Omega, \quad \text{where } f \in L^1,$$

with the inner product

$$(x,y)(z,w) = xz + yw.$$

Hence we obtain

$$\int_D e^{w\zeta-\overline{w\zeta}}\sigma(\zeta)\,dA(\zeta) = \mathcal{F}(\sigma)(2v,2u), \text{ for } \sigma \in L^1.$$

So from the fact that $\mathbb{C} \approx \mathbb{R}^2$, we might assume $w = u + iv \approx (u,v)$ and $\zeta = x + iy \approx (x,y)$ which implies that

$$w\zeta - \overline{w\zeta} = (u+iv)(x+iy) - (u-iv)(x-iy) = 2i(xv+yu) = 2i(v,u)(x,y).$$

In particular for the function σ, we have

$$\begin{aligned} \mathcal{F}(\sigma)(v,u) &= \int_D e^{-i(v,u)(x,y)}\sigma(x,y)dA(\zeta) \\ &= \int_D e^{-\frac{1}{2}(w\zeta-\overline{w}\overline{\zeta})}\sigma(\zeta)dA(\zeta) \\ &= \int_D e^{-\frac{w}{2}\zeta}e^{\overline{\frac{w}{2}}\overline{\zeta}}\sigma(\zeta)dA(\zeta). \end{aligned}$$

Thus we get

$$\begin{aligned} K(w)\overline{J}(w) &= L(p)\sum_{n=0}^{\infty}\frac{F_n\overline{G_n}}{n+1} \\ &= \int_D u(\zeta)\Delta_\zeta\left(e^{w\zeta-\overline{w}\overline{\zeta}}(1-|\zeta|^2)^2\right)dA(\zeta) \\ &= \int_D u(\zeta)e^{w\zeta-\overline{w}\overline{\zeta}}\left[\begin{array}{c} -|w|^2(1-|\zeta|^2)^2-2w\zeta(1-|\zeta|^2) \\ +2\overline{w}\overline{\zeta}(1-|\zeta|^2)+2(2|\zeta|^2-1) \end{array}\right]dA(\zeta) \\ &= -|w|^2\mathcal{F}(\sigma_1)(2v,2u)+w\mathcal{F}(\sigma_2)(2v,2u)+\overline{w}\mathcal{F}(\sigma_3)(2v,2u)+\mathcal{F}(\sigma_4)(2v,2u), \end{aligned}$$

where $\sigma_1(\zeta) = u(\zeta)(1-|\zeta|^2)^2$, $\sigma_2(\zeta) = -2\zeta u(\zeta)(1-|\zeta|^2)$, $\sigma_3(\zeta) = 2\overline{\zeta}u(\zeta)(1-|\zeta|^2)$ and $\sigma_4(\zeta) = 2u(\zeta)(2|\zeta|^2-1)$. Here clearly we have $\sigma_i \in L^1$, for all $i = 1,2,3,4$. From this we obtain

$$\begin{aligned} |K(w)J(w)| &= \left|K(w)\overline{J}(w)\right| \\ &= \left|w\mathcal{F}(\sigma_2)(2v,2u)-|w|^2\mathcal{F}(\sigma_1)(2v,2u)+\overline{w}\mathcal{F}(\sigma_3)(2v,2u)+\mathcal{F}(\sigma_4)(2v,2u)\right|. \end{aligned}$$

Since the Fourier transform of an L^1 function tends to zero at infinity, we conclude that $\frac{|K(w)J(w)|}{|w|^2} \to 0$ if $w \to \infty$, whence

$$|K(w)J(w)| = o(|w|^2).$$

Note that KJ is entire. So, it can be written as a power series. But since $|K(w)J(w)| = o(|w|^2)$, we see that $K(w)J(w) = Aw + B$ for some constants A and B. This means that one of the two functions K and J is $Aw + B$ times a non-vanishing function and that the other is non-vanishing. It follows from Hadamard's factorization theorem 1.1.11 that a non-vanishing function of order 1 is of the form $\gamma e^{q(w)}$, where q is a polynomial of degree≤ 1, i.e. a non-vanishing function of order 1 is of the form γe^{aw}. So we may conclude that one of these two functions, say K, is of the form

$$K(w) = (\alpha w + b)e^{aw},$$

and the other one takes the form

$$J(w) = \gamma e^{-aw}.$$

First, we turn our attention to J. On one hand $J(w) = \gamma e^{-aw}$ and on the other hand it is written as $\sum_{n=0}^{\infty} \frac{(-1)^n \overline{F_n} w^n}{(n+1)!}$. So we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n \overline{F_n} w^n}{(n+1)!} = \gamma e^{-aw} = \sum_{n=0}^{\infty} \frac{\gamma (-1)^n a^n w^n}{n!}.$$

Equating coefficients we see that $\overline{F_n} = \gamma(n+1)a^n$, i.e. $F_n = \overline{\gamma}(n+1)\overline{a}^n$. Now, recall that $F = f'$ and that we are assuming that f' is not identically zero, which means that F_n is not identically zero. This implies that $\gamma \neq 0$. Also, we have seen that $F_n = o(n)$ and since $\sum a^n \to 0$, we see that $|a| < 1$. Hence

$$F(w) = \sum_{n=0}^{\infty} F_n w^n = \overline{\gamma} \sum_{n=0}^{\infty} (n+1)\overline{a}^n w^n = \frac{\overline{\gamma}}{(1 - \overline{a}w)^2},$$

with $\gamma \neq 0$ and $|a| < 1$.

Now, since

$$\begin{aligned}
\frac{d}{dw}\left(\frac{-\overline{\gamma}}{(1-|a|^2)}\frac{a-w}{1-\overline{a}w}\right) &= \frac{-\overline{\gamma}}{(1-|a|^2)}\frac{-(1-\overline{a}w)+\overline{a}(a-w)}{(1-\overline{a}w)^2}\\
&= \frac{-\overline{\gamma}}{(1-|a|^2)}\frac{-1+\overline{a}w+|a|^2-\overline{a}w}{(1-\overline{a}w)^2}\\
&= \frac{-\overline{\gamma}}{(1-|a|^2)}\frac{-(1-|a|^2)}{(1-\overline{a}w)^2}\\
&= \frac{\overline{\gamma}}{(1-\overline{a}w)^2}\\
&= F(w) = f'(w),
\end{aligned}$$

we can write this as

$$f'(w) = \frac{d}{dw}\left(\frac{-\overline{\gamma}}{(1-|a|^2)}\frac{a-w}{1-\overline{a}w}\right). \tag{2.2.9}$$

Integrating both sides of Equation (2.2.9) with respect to w, we get

$$f(w) = c_1\frac{a-w}{1-\overline{a}w} + c_2,$$

where c_1 and c_2 are constants. We conclude that $f = p\circ\phi_a$, where $\phi_a(w) = \frac{a-w}{1-\overline{a}w}$ and $p(w) = c_1w + c_2$.

Step 8: Now, we turn our attention to K, which reads as

$$K(w) = \int e^{w\zeta}d_\mu(\zeta) = (\alpha w+\beta)e^{aw}.$$

Differentiating this identity with respect to w and then letting $w = 0$, we see that

$$\int \zeta e^{w\zeta}d_\mu(\zeta)\mid_{w=0} = ae^{aw}(\alpha w+\beta) + \alpha e^{aw}\mid_{w=0}.$$

Hence, we obtain

$$\int \zeta d_\mu(\zeta) = \beta a + \alpha.$$

Now differentiating again with respect to w and letting $w = 0$, we have

$$\frac{d^2}{dw^2}\int e^{w\zeta}d_\mu(\zeta) = \frac{d^2}{dw^2}\left((\alpha w+\beta)e^{aw}\right),$$

and then

$$\left(\int \zeta^2 e^{w\zeta} d_\mu(\zeta)\right)|_{w=0} = \left(a^2 e^{aw}(\alpha w + \beta) + a\alpha e^{aw} + a\alpha e^{aw}\right)|_{w=0}.$$

Hence, we obtain

$$\int \zeta^2 d_\mu(\zeta) = \beta a^2 + 2\alpha a.$$

Continuing in this manner, we get

$$\int \zeta^k d_\mu(\zeta) = \beta a^k + \alpha k a^{k-1}.$$

Now, note that if $z \in D$ and if we put $p_z(\zeta) = \frac{1}{(1-\overline{z}\zeta)^2}$, then $p_z \in A(D)$ and as above we obtain

$$\begin{aligned} S_u p_z(z) &= \int_D u(\zeta)\Delta_\zeta \frac{p_z(\zeta)(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} dA(\zeta) \\ &= \int_D u(\zeta)\Delta_\zeta \frac{(1-|\zeta|^2)^2}{|1-\overline{\zeta}z|^4} dA(\zeta) = f'(z)\overline{g'}(z), \end{aligned}$$

So that, since $L(p_z)f'(z) = S_u(p(z))$, we have that

$$f'(z)\overline{g'}(z) = L(p_z)f'(z),$$

and, since f' is not identically zero, we see that

$$\overline{g'}(z) = \int p_z(\zeta) d_\mu(\zeta),$$

where $L(p) = \int p(\zeta) d_\mu(\zeta)$.

From this we may conclude that

$$\begin{aligned}
\overline{g'}(z) &= \int \frac{1}{(1-\overline{z}\zeta)^2} d_\mu(\zeta) \\
&= \int \left(\sum_{n=0}^{\infty} (n+1)\zeta^n \overline{z}^n \right) d_\mu(\zeta) \\
&= \sum_{n=0}^{\infty} (n+1) \int \zeta^n d_\mu(\zeta) \overline{z}^n = \sum_{n=0}^{\infty} (n+1)(\beta a^n + \alpha n a^{n-1}) \overline{z}^n \\
&= \sum_{n=0}^{\infty} \beta (n+1) a^n \overline{z}^n + \alpha \sum_{n=1}^{\infty} n(n+1) a^{n-1} \overline{z}^n \\
&= \frac{\beta}{(1-a\overline{z})^2} + \alpha \overline{z} \sum_{n=1}^{\infty} n(n+1) a^{n-1} \overline{z}^{n-1} \\
&= \frac{\beta}{(1-a\overline{z})^2} + \frac{2\alpha \overline{z}}{(1-a\overline{z})^3}.
\end{aligned}$$

Note that in this series of equalities we have used the fact that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \; \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n \text{ and } \frac{2}{(1-x)^3} = \sum_{n=1}^{\infty} n(n+1) x^{n-1}.$$

Finally, we conclude that

$$g'(z) = \frac{\overline{\beta}}{(1-\overline{a}z)^2} + \frac{2\overline{\alpha} z}{(1-\overline{a}z)^3}.$$

Now it is easy to see that g has the form required by Theorem 2.4.5, as well. To conclude the proof, note that in principle we must consider the case that $J(w) = (\alpha w + \beta) e^{aw}$ and $K(w) = \gamma e^{-\alpha z}$, But this case is very similar to the one just discussed. This achieves the proof. ■

Chapter 3

Theorems of Brown-Halmos Type for Bergman Space Toeplitz Operators

The original Brown-Halmos theorem [16] asserts that for two Toeplitz operators T_u, T_v on the Hardy space, the product T_uT_v is again a Toeplitz operator if and only if either $\overline{u}$ or v is holomorphic and in such a case $T_uT_v = T_{uv}$. An interesting corollary is the so-called the zero product problem asserts that $T_uT_v = 0$ if and only if u or v vanishes identically. Surprisingly, the Bergman space analog of the celebrated Brown-Halmos theorem appeared only very recently, but only in some particular cases. In a series of excellent papers, [1, 2, 4], P. Ahern and Ž. Čučkovič has proved that in the case of bounded harmonic symbols a Brown-Halmos type theorem holds. They have discussed also the case of radial symbols. Unfortunately, for general symbols it fails. In this chapter we investigate the case of bounded harmonic symbols in the light of the paper [2]. We believe that a weighted Bergman space analog should be equally true. Although we were enable to prove it completely, nevertheless we have made some progress in this direction.

3.1 A theorem of Brown-Halmos type

Definition 3.1.1. *If f, g and h are functions, we say that $T_f T_g = T_h$ in a non-trivial way if neither $\overline{f}$ nor g is holomorphic.*

Remark 3.1.2. *Example **3.2.2** illustrates that such triples exist.*

Now, we start with the following technical results. Such results are known for experts. However, for the sake of completeness, we provide our own corresponding proofs. Notice that the first property was stated explicitly for a first time in [3] as follows: If $T_u = 0$ then u is orthogonal to all polynomials in z and $\overline{z}$. By the Stone Weirstrass theorem such polynomials are dense in L^2. Hence $u = 0$ almost every where. Here we even prove that exactly $T_u = 0$ implies $u = 0$ by a very short and convincing proof.

Proposition 3.1.3. *Bergman space Toeplitz operators have the following properties:*

1. *If $T_u = 0$, then $u = 0$.*
2. *If f is holomorphic, then $T_u T_f = T_{uf}$ and $T_{\overline{f}} T_u = T_{\overline{f}u}$ for any u.*
3. *If f is holomorphic and is not identically zero, then T_f is one-to-one.*
4. *If $g \in L_a^2$ and $w \in D$, then $P(\overline{g}K_w) = \overline{g}(w)K_w$.*

Proof: (1) If $T_u = 0$ then $T_u k_w = 0$ and so we have that

$$\langle T_u k_w, k_w \rangle = B(T_u) = B(u) = 0$$

By Proposition 2.3.1 the Berezin transform is injective, hence we infer that $u = 0$.

(2) If f is holomorphic, then for any $g \in L_a^2$ we have:

(i) $T_u T_f(g) = T_u(P(fg)) = T_u(fg) = P(ufg) = T_{uf}(g)$. Hence $T_u T_f = T_{uf}$.

(ii) $T_{\overline{f}} T_u = (T_{\overline{u}} T_f)^* = (T_{\overline{u}f})^* = T_{\overline{f}u}$.

(3) Let $u \in \ker T_f$. Then, since f is holomorphic, we see that $T_f(u) = fu = 0$. But the product of two analytic functions can be zero only in the case when one of them is zero. But f is not identically zero, whence $u = 0$. Thus we conclude that $\ker T_f = \{0\}$.

(4) If $g \in L_a^2$, $w \in D$, then

$$\begin{aligned} P(\overline{g}K_w)(z) &= \langle P(\overline{g}K_w), K_z \rangle \ , z \in D \\ &= \langle \overline{g}K_w, K_z \rangle = \langle K_w, gK_z \rangle \\ &= \overline{\langle gK_z, K_w \rangle} = \overline{gK_z}(w) \\ &= \overline{g}(w)K_w(z). \end{aligned}$$

Hence $P(\overline{g}K_w) = \overline{g}(w)K_w$. ■

Definition 3.1.4. *If F is an L^1 function which can be written in the form $F_1 + F_2$ where F_1 is a function with compact support, (namely $|z| \leq s$), and F_2 is a bounded function on $\{z : s < |z| < 1\}$ for some $0 < s < 1$, then F is said to be nearly bounded.*

The following easy and useful fact was used in [3]. Since we could not localize any corresponding reference, we propose our own proof, which is very close in spirit to the idea we have used to prove the first fact of Proposition 3.1.3.

Lemma 3.1.5. *If f, g and h are L^2 functions. Then we have the following easy and useful fact:*

$$T_f T_g = T_h \quad \text{if and only if} \quad T_f T_g K_w = T_h K_w, \forall w \in D.$$

Proof: It is clear that $T_f T_g = T_h$ implies that $T_f T_g K_w = T_h K_w, \forall w \in D$. With regard to the other implication, suppose that

$$T_f T_g K_w = T_h K_w, \forall w \in D.$$

Then, we see that

$$\langle T_f T_g K_w, K_w \rangle = \langle T_h K_w, K_w \rangle,$$

which can be rewritten as

$$\langle (T_f T_g - T_h)\, k_w, k_w \rangle = 0.$$

In terms of the Berezin transform it reads as

$$B(T_f T_g - T_h) = 0.$$

By the injectivity of the Berezin transform, Proposition 2.3.1, we infer that

$$T_f T_g = T_h.$$

Hence the conclusion of the Lemma follows. ■

Proposition 3.1.6. *Suppose that $f = f_1 + \overline{f_2}$, $g = g_1 + \overline{g_2}$ are bounded harmonic functions with f_i, g_i holomorphic and h is in L^2. Then the following are equivalent :*

1. $T_f T_g = T_h$.
2. $f_1(z)g_1(z) + \bar{f}_2(z)\bar{g}_2(z) + f_1(z)\bar{g_2}(z) = B(h - \bar{f}_2 g_1)(z)$, *for all* $z \in D$.
3. *For all* $(z,w) \in D \times D$, *we have*

$$f_1(z)g_1(z) + \bar{f}_2(\overline{w})\bar{g}_2(\overline{w}) + f_1(z)\bar{g}_2(\overline{w}) = (1 - zw)^2 \int_D \frac{h(\xi) - \bar{f}_2(\zeta)g_1(\zeta)}{(1 - z\bar{\zeta})^2(1 - w\zeta)^2} dA(\zeta).$$

Proof: First, let us prove that (1)⇔(3): using part (4) of Proposition 3.1.3, we see that

$$T_g K_w = P(g_1 K_w + \overline{g_2} K_w) = g_1 K_w + \overline{g_2}(w) K_w.$$

Another application of part (4) of Proposition 3.1.3 shows that

$$\begin{aligned}
T_f T_g K_w &= P(f(g_1 K_w + \overline{g_2}(w) K_w)) \\
&= P((f_1 + \overline{f_2})(g_1 K_w + \overline{g_2}(w) K_w) \\
&= P(f_1 g_1 K_w + f_1 \overline{g_2}(w) K_w + \overline{f_2} g_1 K_w + \overline{f_2}\,\overline{g_2}(w) K_w) \\
&= f_1 g_1 K_w + \overline{g_2}(w) f_1 K_w + P(\overline{f_2} g_1 K_w) + \overline{g_2}(w) P(\overline{f_2} K_w) \\
&= f_1 g_1 K_w + \overline{g_2}(w) f_1 K_w + P(\overline{f_2} g_1 K_w) + \overline{g_2}(w) \overline{f_2}(w) K_w.
\end{aligned}$$

Using Lemma 3.1.5, we see that $T_fT_g = T_h$ is equivalent to

$$f_1g_1K_w + \overline{g_2}(w)f_1K_w + \overline{g_2}(w)\overline{f_2}(w)K_w + P(\overline{f_2}g_1K_w) = P(hK_w), \forall w \in D.$$

Dividing the latter by K_w, we obtain for all $(z, w) \in D \times D$

$$f_1(z)g_1(z) + f_1(z)\overline{g_2}(w) + \overline{g_2}(w)\overline{f_2}(w) + \frac{1}{K_w(z)}P(\overline{f_2}g_1K_w)(z) = \frac{1}{K_w(z)}P(hK_w)(z).$$

Using the form of the reproducing kernel, we get for all $(z, w) \in D \times D$:

$$f_1(z)g_1(z)+f_1(z)\overline{g_2}(w)+\overline{g_2}(w)\overline{f_2}(w)+(1-\overline{w}z)^2\langle P(\overline{f_2}g_1K_w), K_z\rangle = (1-\overline{w}z)^2\langle P(hK_w), K_z\rangle.$$

Since the range of P lies in L_a^2, we see that for all $(z, w) \in D \times D$

$$f_1(z)g_1(z) + f_1(z)\overline{g_2}(w) + \overline{g_2}(w)\overline{f_2}(w) + (1-\overline{w}z)^2\langle \overline{f_2}g_1K_w, K_z\rangle = (1-\overline{w}z)^2\langle hK_w, K_z\rangle,$$

which can be rewritten as

$$\begin{aligned} f_1(z)g_1(z) + f_1(z)\overline{g_2}(w) + \overline{g_2}(w)\overline{f_2}(w) + (1-z\bar{w})^2\int_D \frac{\bar{f}_2(\zeta)g_1(\zeta)}{(1-z\bar{\zeta})^2(1-\bar{w}\xi)^2}dA(\zeta) \\ = (1-z\bar{w})^2\int_D \frac{h(\xi)}{(1-z\bar{\zeta})^2(1-\bar{w}\xi)^2}dA(\xi), \forall (z,w) \in D \times D, \end{aligned}$$

which is just equation (3) with w replaced by $\overline{w}$.

Now, we show that (2)$\Leftrightarrow$ (3) :

In order to see that (3)$\Rightarrow$ (2), apply (3) to $(z, \overline{z}) \in D \times D$ and obtain (2). To show that (2)$\Rightarrow$ (3), we consider the holomorphic function defined in the bi-disk by the formula

$$\begin{aligned} F(z,w) &= f_1(z)g_1(z) + \bar{f}_2(\overline{w})\bar{g}_2(\overline{w}) + f_1(z)\bar{g}_2(\overline{w}) \\ &\quad + (1-zw)^2\int_D \frac{\bar{f}_2(\zeta)g_1(\zeta) - h(\xi)}{(1-z\bar{\zeta})^2(1-wz)^2}dA(\zeta). \end{aligned}$$

In fact, F is holomorphic as it is holomorphic in each of its variables.

Assuming (2), F is identically zero on the set $\{(z,\overline{z}) : z \in D\}$ and by Proposition 1.1.17 we see that $F \equiv 0$ in D. Thus $F(z,\overline{w}) \equiv 0$, which is exactly the statement (3). This ends the proof. ■

Remark 3.1.7. *If f is any function from $L^\infty(\partial D)$, then we can write $f = f_1 + \overline{f_2}$ where f_1 and f_2 are in $BMOA$ as we noticed in Subsection 1.2.2. Recall that $S_z(e^{i\theta}) = \frac{1}{(1-\overline{z}e^{i\theta})}$ denotes the Szegö kernel and P_S denotes the Szegö projection from $L^2(\partial D)$ onto H^2. Then the application of the method of Proposition 3.1.6 leads to the fact that*

$$f_1(z)g_1(z) + \overline{f_2}(z)\overline{g_2}(z) + f_1(z)\overline{g_2}(z) + \frac{1}{S_z(z)}P_S(\overline{f_2}g_1S_z)(z) = h(z). \tag{3.1.1}$$

But for the last term of the L.H.S. of the latter, we have

$$\begin{aligned}\frac{1}{S_z(z)}P_S(uS_z)(z) &= \frac{1}{\frac{1}{1-\overline{z}z}}\frac{1}{2\pi}\int_0^{2\pi}\frac{uS_z(\theta)}{1-ze^{-i\theta}}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-|z|^2)}{1-ze^{-i\theta}}uS_z(\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|1-ze^{-i\theta}|^2}u(\theta)d\theta,\end{aligned}$$

which is the harmonic extension of any u that is integrable on ∂D. Hence every term in Equality (3.1.1) is obviously harmonic except $f_1\overline{g_2}$. It follows that $f_1\overline{g_2}$ is harmonic too which implies that

$$\frac{\partial f_1}{\partial z}\frac{\partial \overline{g_2}}{\partial \overline{z}} = 0.$$

Since $\frac{\partial f_1}{\partial z}$ is analytic, we see that either $\frac{\partial f_1}{\partial z} = 0$ or $\frac{\partial \overline{g_2}}{\partial \overline{z}} = 0$,"whence f_1 is constant or g_2 is constant. In other words either f is conjugate holomorphic or g is holomorphic.

In the Bergman space case the Berezin transform appears rather than the harmonic extension. Since the Berezin transform does not always yield harmonic functions, (according to part (3) of Remark 2.1.6), some more work should be done. In order to establish a Brown-Halmos type theorem, some preparatory assertions are needed. The first one gives us an important identity.

Lemma 3.1.8. *If f and g are holomorphic in D and $f\overline{g} = Bu$ where $u \in L^1 \cap C^2(D)$ and $\widetilde{\Delta} u \in L^1$, then we have*

$$f'(z)\overline{g'}(\overline{w}) = \int_D \frac{\widetilde{\Delta} u(\zeta)}{(1-\overline{\zeta}z)^2(1-\zeta w)^2} dA(\zeta). \tag{3.1.2}$$

Proof: Let us start with the equality $f\overline{g} = Bu$.

Taking the invariant Laplacian of both sides, by Lemma 2.2.3, we see that

$$(1-|z|^2)^2 \Delta(f\overline{g}) = B(\widetilde{\Delta} u),$$

i.e.

$$(1-|z|^2)^2 \Delta(f\overline{g}) = (1-|z|^2)^2 \int_D \frac{\widetilde{\Delta} u(\zeta)}{\left|1-z\overline{\zeta}\right|^4} dA(\zeta).$$

Dividing both sides of the latter by $(1-|z|^2)^2$, we arrive at

$$\Delta(f\overline{g}) = \frac{\partial}{\partial z}\left(f\frac{\partial \overline{g}}{\partial \overline{z}}\right) = \int_D \frac{\widetilde{\Delta} u(\zeta)}{\left|1-z\overline{\zeta}\right|^4} dA(\zeta).$$

Thus, we obtain

$$f'(z)\overline{g'}(z) = \int_D \frac{\widetilde{\Delta} u(\zeta)}{\left|1-z\overline{\zeta}\right|^4} dA(\zeta). \tag{3.1.3}$$

Next, we complexify this identity to obtain the identity (3.1.2). Note that the functions on both sides of Equation (3.1.2) are holomorphic in the bidisk

$$\{(z,w) : |z| < 1, |w| < 1\},$$

(as they are holomorphic in each of their variables), and from Equation (3.1.3) they are equal on the subset $\{(z,\overline{z}) : |z| < 1\}$. Hence, by Proposition 1.1.17 they are equal on the whole bidisk. This proves the lemma. ■

The second lemma shows that there is no rank one Toeplitz operators with a bounded symbol except the trivial one T_0. Here, we believe that much more is true, namely that even there is no finite rank Toeplitz operators.

Lemma 3.1.9. *If σ is a bounded (not necessarily harmonic) function in D and*

$$\dim T_\sigma L_a^2 \leq 1,$$

then $\sigma \equiv 0$.

Proof: As we have seen in the 7th step of the proof of Theorem 2.4.5, the Fourier transform in $\mathbb{R}^2$ is given by:

$$\mathcal{F}(f)(x,y) = \int_{\mathbb{R}^2} e^{-i(x,y)(z,w)} f(z,w) d\Omega.$$

With the inner product $(x,y)(z,w) = xz + yw$. So that from the fact that $\mathbb{C} \approx \mathbb{R}^2$, we might assume $w = u + iv \approx (u,v)$ and $\zeta = x + iy \approx (x,y)$. Thus we have that

$$w\zeta - \overline{w\zeta} = (u+iv)(x+iy) - (u-iv)(x-iy) = 2i(xv+yu) = 2i(v,u)(x,y).$$

In particular for the function σ, we have

$$\begin{aligned}
\mathcal{F}(\sigma)(v,u) &= \int_D e^{-i(v,u)(x,y)} \sigma(x,y) dA(\zeta) \\
&= \int_D e^{-\frac{1}{2}(w\zeta - \overline{w}\overline{\zeta})} \sigma(\zeta) dA(\zeta) \\
&= \int_D e^{-\frac{w}{2}\zeta} e^{\frac{\overline{w}}{2}\overline{\zeta}} \sigma(\zeta) dA(\zeta).
\end{aligned}$$

Put $e_w(\zeta) = e^{w\zeta}$, we get $e^{-\frac{w}{2}\zeta} = e_{-\frac{w}{2}}(\zeta)$ and $e^{\frac{\overline{w}}{2}\overline{\zeta}} = \overline{e_{\frac{w}{2}}(\zeta)}$. Hence, we obtain:

$$\begin{aligned}
\mathcal{F}(\sigma)(v,u) &= \int_D e_{-\frac{w}{2}} \overline{e_{\frac{w}{2}}} \sigma dA \\
&= \left\langle e_{-\frac{w}{2}}\sigma, e_{\frac{w}{2}} \right\rangle = \left\langle P(e_{-\frac{w}{2}}\sigma), e_{\frac{w}{2}} \right\rangle, \text{ as } e_{\frac{w}{2}} \in L_a^2 \text{ and } \sigma \text{ is bounded} \\
&= \left\langle T_\sigma e_{-\frac{w}{2}}, e_{\frac{w}{2}} \right\rangle.
\end{aligned}$$

Now as $\dim T_\sigma L_a^2 \leq 1$, by Definition 1.1.31 of rank one operator, we have $T_\sigma f = \langle f, \varphi \rangle F$ for some $\varphi, F \in L_a^2$. So, we see that

$$\begin{aligned}
\mathcal{F}(\sigma)(v,u) &= \left\langle \left\langle e_{-\frac{w}{2}}, \varphi \right\rangle F, e_{\frac{w}{2}} \right\rangle \\
&= \left\langle e_{-\frac{w}{2}}, \varphi \right\rangle \left\langle F, e_{\frac{w}{2}} \right\rangle \\
&= G(w)\overline{H}(w).
\end{aligned}$$

where G and H are entire functions. Indeed, since $G(w) = \langle e_{-\frac{w}{2}}, \varphi\rangle = \int_D e^{-\frac{w}{2}z}\overline{\varphi}(z)dA(z)$, we see that G is differentiable with respect to all $w \in \mathbb{C}$ and hence it is entire, and the same is true for H. Hence

$$|\mathcal{F}(\sigma)| = \left|G\overline{H}\right| = |GH|.$$

But $\mathcal{F}(\sigma)$ is continuous and goes to 0 at ∞, as $\mathcal{F}(\sigma)(w) = \langle T_\sigma e_{-\frac{w}{2}}, e_{\frac{w}{2}}\rangle$ is continuous and if $w \to \infty$ then $e_{-\frac{w}{2}} \to 0$ and GH is entire which, by Theorem 1.1.7, implies that GH is identically zero.

Thus, we have $\mathcal{F}(\sigma) \equiv 0$. Now, Theorem 1.1.39 tells us that σ must be identically zero, i.e. $\sigma \equiv 0$. This completes the proof of the lemma. ■

Now, we are in the position to prove the following strong result using Lemmas 3.1.8 and 3.1.9.

Proposition 3.1.10. *Suppose that f and g are holomorphic in D and $f\overline{g} = Bu$ where $u \in L^1 \cap C^2(D)$ and $\widetilde{\Delta}u \in L^1$. Then either f is constant or g is constant.*

Proof: Differentiate Identity (3.1.2) with respect to w and then let $w = 0$. Thus we obtain

$$\left(\frac{\partial}{\partial w}\overline{g'}(\overline{w})f'(z)\right)|_{w=0} = \left(\int_D \frac{\widetilde{\Delta}u(\zeta)}{(1-\overline{\zeta}z)^2}\frac{2\zeta(1-\zeta w)}{(1-\zeta w)^4}dA(\zeta)\right)|_{w=0}.$$

So we obtain

$$c_1 f'(z) = \int_D \frac{\zeta\widetilde{\Delta}u(\zeta)}{(1-\overline{\zeta}z)^2}dA(\zeta).$$

Again differentiate Identity (3.1.2) with respect to w and then let $w = 0$, we get

$$c_2 f'(z) = \left(\int_D \frac{\zeta\widetilde{\Delta}u(\zeta)}{(1-\overline{\zeta}z)^2}\frac{\zeta(1-\zeta w)^2}{(1-\zeta w)^6}dA(\zeta)\right)|_{w=0}.$$

That is to say

$$c_2 f'(z) = \int_D \frac{\zeta^2\widetilde{\Delta}u(\zeta)}{(1-\overline{\zeta}z)^2}dA(\zeta).$$

Continuing in this manner, we arrive at

$$c_k f'(z) = \int_D \frac{\zeta^k \sigma(\zeta)}{(1-\overline{\zeta}z)^2} dA(\zeta),$$

for some constants $c_k, k = 1, 2, \cdots$, where $\sigma(\zeta) = \widetilde{\Delta} u(\zeta)$.

Now, consider the Toeplitz operator T_σ with the possibly non-harmonic bounded symbol σ. The latter identity shows that

$$T_\sigma(\zeta^k) = \int_D \frac{\sigma(\zeta)\zeta^k}{(1-\overline{\zeta}z)^2} dA(\zeta) = c_k f'(z).$$

That is $T_\sigma(\zeta^k)$ is a multiple of f' for all non-negative integers k. There are two possibilities: either $T_\sigma(\zeta^k) = 0$ for all k or not. If the first holds then $T_\sigma p = 0$ for all polynomials and hence $T_\sigma = 0$ on L_a^2 which by (3) of Remark 1.3.3 implies that $\sigma = 0$. In the other case, there exists k such that $T_\sigma(\zeta^k) \neq 0$. That is, there is some $c_k \neq 0$. Now, since it is a multiple of $T_\sigma(\zeta^k)$ for some k, we see that $f' \in L_a^2$. Hence, if $\sigma \neq 0$ then T_σ is a rank one Toeplitz operator. But Lemma 3.1.9 says that there is no rank one Toeplitz operator with bounded symbol except T_0, whence $\sigma \equiv 0$ in all cases. Thus $\Delta u = 0$. Now, since

$$f'(z)\overline{g'}(z) = \int_D \frac{\widetilde{\Delta} u(\zeta)}{\left|1 - z\overline{\zeta}\right|^4} dA(\zeta),$$

we see that $f'(z)\overline{g'}(z) = 0$ for all $z \in D$. This implies that either f is constant or g is constant which finishes the proof of Proposition 3.1.10. ■

Now we are able to state and prove Ahern-Čučkovič's theorem, which is of Brown-Halmos type, for Bergman space Toeplitz operators with bounded harmonic symbols. Its proof relies heavily on Proposition 3.1.10.

Theorem 3.1.11. *Suppose that f and g are bounded harmonic functions and that h is a bounded C^2-function with the property that $\widetilde{\Delta} h$ is also bounded in D. Assume that $T_f T_g = T_h$, then one of the following holds:*

1. *f is conjugate holomorphic.*

2. *g is holomorphic.*

In either case $h = fg$.

Proof: From (2) of Proposition 3.1.6 we know that $f_1g_1+\overline{f_2}\overline{g_2}+f_1\overline{g_2} = B(h-\overline{f_2}g_1)$. Since f_1g_2 and $\overline{f_2}\overline{g_2}$ are harmonic and B reproduces harmonic functions, the latter can be rewritten as $f_1\overline{g_2} = B(u)$, where $u = h - \overline{f_2}g_1 - f_1g_1 - \overline{f_2}\overline{g_2}$.
Notice that $\widetilde{\Delta}u = \widetilde{\Delta}\left(h - \overline{f_2}g_1 - f_1g_1 - \overline{f_2}\overline{g_2}\right) = (1-|z|^2)^2\Delta(h-\overline{f_2}g_1) = \widetilde{\Delta}h - \widetilde{\Delta}\overline{f_2}g_1$ is bounded. This is so because $\widetilde{\Delta}h$ is bounded by assumption and

$$\begin{aligned}\widetilde{\Delta}\left(\overline{f_2}g_1\right) &= (1-|z|^2)^2\Delta(\overline{f_2}g_1)\\ &= (1-|z|^2)^2\frac{\partial}{\partial z}\left(\frac{\partial\overline{f_2}}{\partial\overline{z}}g_1+\overline{f_2}\frac{\partial g_1}{\partial\overline{z}}\right)\\ &= (1-|z|^2)^2\frac{\partial}{\partial z}\left(g_1\overline{f_2'}\right)\\ &= (1-|z|^2)^2g_1'\overline{f_2'},\end{aligned}$$

i.e. $\widetilde{\Delta}\left(\overline{f_2}g_1\right) = (1-|z|^2)^2\overline{f_2'}(z)g_1'(z)$ is bounded since f_2 and g_1 are Bloch functions. We note also that as h is a bounded $C^2(D)$ function then $h \in L^\infty \subset L^1$, i.e. $h \in L^1 \cap C^2(D)$. Similarly since $\overline{f_2}, g_1 \in \mathbf{B} \subset L^2_a \subset L^2$, we see that $\overline{f_2}, g_1 \in L^2$. So by Cauchy-Schwarz inequality, we get $\overline{f_2}g_1 \in L^1$. Also g_1 is analytic implies that g_1 is harmonic and hence $g_1 \in C^2(D)$. Similarly f_2 is analytic implies that $\overline{f_2}$ is harmonic and hence $\overline{f_2} \in C^2(D)$. Thus $h - \overline{f_2}g_1 \in C^2(D)$ and therefore $h - \overline{f_2}g_1 \in L^1 \cap C^2(D)$. Now, by Proposition 3.1.10, we conclude that either f_1 is constant or g_2 is constant. But f_1 is constant implies that f is conjugate holomorphic and g_2 is constant implies that g is holomorphic. Hence, we conclude that

$$T_fT_g = T_{fg} = T_h,$$

with $fg = h$, which completes the proof of the Theorem. ■

An important particular case of the above theorem is the case when h is also bounded and harmonic.

Corollary 3.1.12. *If f, g and h are bounded harmonic functions and $T_f T_g = T_h$ then one of the following holds:*

1. *f and g are holomorphic.*
2. *f and g are conjugate holomorphic.*
3. *f is constant.*
4. *g is constant.*

Proof: As h is bounded and harmonic, h is a bounded C^2- function with the property that $\widetilde{\Delta}h$ is also bounded in D.

Now, Theorem 3.1.11 asserts that either f is conjugate holomorphic or g is holomorphic and in either case $h = fg$ and hence $\Delta(fg) = \Delta h = 0$. Explicitly, we have

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \overline{z}}g + f\frac{\partial g}{\partial \overline{z}}\right) = 0.$$

If g is holomorphic then

$$\frac{\partial g}{\partial \overline{z}} = 0,$$

and hence

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \overline{z}}g\right) = 0.$$

In other words, we get

$$(\Delta f)g + \frac{\partial f}{\partial \overline{z}}\frac{\partial g}{\partial z} = 0,$$

which implies, since f is harmonic, that $\frac{\partial f}{\partial \overline{z}}\frac{\partial g}{\partial z} = 0$. Hence g is constant or f is holomorphic.

If $\overline{f}$ is holomorphic then $\frac{\partial \overline{f}}{\partial \overline{z}} = 0$ and, since $fg = h$, we have

$$\overline{f}\overline{g} = \overline{h}.$$

That is to say

$$\Delta\left(\overline{f}\overline{g}\right) = 0,$$

whence

$$\frac{\partial}{\partial z}\left(\overline{f}\frac{\partial \overline{g}}{\partial \overline{z}}\right) = 0.$$

So, we obtain

$$\frac{\partial \overline{f}}{\partial z}\frac{\partial \overline{g}}{\partial \overline{z}} = 0.$$

This implies that $\overline{f}$ is constant, (and hence f is constant), or $\overline{g}$ is holomorphic. This completes the proof. ■

Probably the most important consequence of Theorem 3.1.11 is the so called zero product problem, namely we have the following

Corollary 3.1.13. *There is no zero divisors among Toeplitz operators with bounded harmonic symbols.*

Proof: We want to show that if f and g are bounded harmonic functions and $T_f T_g = 0$, then either $f \equiv 0$ or $g \equiv 0$.

By Theorem 3.1.11, we see that $fg = 0$, (with the function $h = 0$ satisfies the assumptions of the theorem), and by Corollary 3.1.12 we have one of the following:

1. f and g are both holomorphic and hence $fg = 0$, i.e. $f \equiv 0$ or $g \equiv 0$.

2. f and g are both conjugate holomorphic and $fg = 0$, whence $f \equiv 0$ or $g \equiv 0$.

3. f is constant, say c, if $c \neq 0$ then $cg = 0$, i.e. $g \equiv 0$.

4. g is constant, say a, if $a \neq 0$ then $af = 0$, i.e. $f \equiv 0$.

Thus if f and g are bounded harmonic functions and $T_f T_g = 0$, then either $f \equiv 0$ or $g \equiv 0$ and the proof is complete. ■

A slight generalization of the above zero-product corollary is given in the following

Corollary 3.1.14. *If f, g and h are bounded harmonic symbols such that $T_f T_g = T_f T_h$ and f is not identically zero, then $g = h$.*

Proof: If $T_f T_g = T_f T_h$ then $T_f(T_g - T_h) = 0$. So, $T_f T_{g-h} = 0$, which implies, by Corollary 3.1.13, that $f \equiv 0$ or $g - h \equiv 0$. But as f is assumed not to be identically zero, we should have that $g = h$ and the proof is complete. ■

The next corollary says that if the inverse of a Toeplitz operator with bounded harmonic symbol is also a Toeplitz operator with bounded harmonic symbol then this can only happen in the most trivial way.

Corollary 3.1.15. *If f and g are bounded and harmonic and $T_f T_g = I$, then either f and g are both holomorphic or they are both conjugate holomorphic and in either case $f = \frac{1}{g}$.*

Proof: By Theorem 3.1.11, we have $T_f T_g = T_1$ and $fg = 1$. Since $fg = 1$, we see that f is not identically zero and g is not identically zero. If f is constant, say c, then $c \neq 0$ and $T_f T_g = T_{cg} = T_1$, i.e. $cg = 1$ and hence $g = \frac{1}{c}$. In other words, if f is constant then g is also constant (that is f and g are both holomorphic) and $f = \frac{1}{g}$. The case if g is constant is similar.

Hence, using Corollary 3.1.12, we see that if f and g are bounded and harmonic and $T_f T_g = I$, then either f and g are both holomorphic or they are both conjugate holomorphic and in either case $fg = 1$. ■

The next corollary says that there are no idempotent Toeplitz operators with bounded harmonic symbols other than 0 and I. Notice that this result is the Bergman space counterpart of Corollary 1.2.23.

Corollary 3.1.16. *If f is bounded and harmonic and $T_f^2 = T_f$, then $f \equiv 0$ or $f \equiv 1$.*

Proof: If $T_f^2 - T_f = 0$, then $T_f(T_f - T_1) = 0$, i.e. $T_f T_{f-1} = 0$. Hence, by Corollary 3.1.13, we infer that $f \equiv 0$ or $f - 1 \equiv 0$, i.e. $f \equiv 0$ or $f \equiv 1$. ■

The next result is the Bergman space analog of Corollary 1.2.22. Probably it is worth stressing that this result was proved earlier by Čučkovič in [21] using Banach algebra techniques and this original proof was long and rather complicated. Here, we give a shorter simple proof using Theorem 3.1.11 with no claim of originality.

Corollary 3.1.17. *Suppose that $f \in L^\infty$ is harmonic. Then T_f is an isometry if and only if f is a constant function of modulus 1.*

Proof: If T_f is an isometry, then by Remark 1.1.30 $T_f^* T_f = I$, i.e. $T_{\overline{f}} T_f = T_f T_{\overline{f}} = T_1$. This in its turn implies, by Theorem 3.1.11, that f is holomorphic and $\overline{f}$ is holomorphic. Hence f is constant and $\overline{f} f = 1$, i.e. $|f|^2 = 1$. Thus $|f| = 1$. Conversely, if f is a constant function of modulus 1, then it is clear that

$$T_f^* T_f = T_{\overline{f}} T_f = T_{\overline{f} f} = T_{|f|^2} = T_1 = I.$$

Hence T_f is an isometry, by Remark 1.1.30, which completes the proof. ■

Corollary 3.1.18. *If f and g are bounded harmonic functions and $T_f T_g = T_{fg}$, then either g is holomorphic or f is conjugate holomorphic.*

Proof: The fact that f and g are harmonic implies that they are of class C^2, and thus $fg \in C^2(D)$. In order to apply Theorem 3.1.11, we need only to check that

$\widetilde{\Delta} fg \in L^\infty$. As $f = f_1 + \overline{f_2}$ and $g = g_1 + \overline{g_2}$ with $f_i, g_i \in \mathbf{B}$ as we have seen in page 51, we see that

$$fg = f_1 g_1 + f_1 \overline{g_2} + \overline{f_2} g_1 + \overline{f_2}\,\overline{g_2}.$$

Introducing the $\overline{\partial}$ operator on both sides, we obtain

$$\begin{aligned}\frac{\partial}{\partial \overline{z}} fg &= f_1 \frac{\partial g_1}{\partial \overline{z}} + g_1 \frac{\partial f_1}{\partial \overline{z}} + \overline{g_2} \frac{\partial f_1}{\partial \overline{z}} + f_1 \frac{\partial \overline{g_2}}{\partial \overline{z}} + \overline{f_2} \frac{\partial g_1}{\partial \overline{z}} + g_1 \frac{\partial \overline{f_2}}{\partial \overline{z}} + \overline{f_2} \frac{\partial \overline{g_2}}{\partial \overline{z}} + \overline{g_2} \frac{\overline{f_2}}{\partial \overline{z}} \\ &= f_1 \frac{\partial \overline{g_2}}{\partial \overline{z}} + g_1 \frac{\partial \overline{f_2}}{\partial \overline{z}} + \overline{f_2} \frac{\partial \overline{g_2}}{\partial \overline{z}} + \overline{g_2} \frac{\partial \overline{f_2}}{\partial \overline{z}}.\end{aligned}$$

So, as f_i, g_i are harmonic and h is holomorphic, we obtain

$$\begin{aligned}\frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} fg &= \frac{\partial f_1}{\partial z} \frac{\partial \overline{g_2}}{\partial \overline{z}} + f_1 \frac{\partial^2 \overline{g2}}{\partial z \partial \overline{z}} + \frac{\partial g_1}{\partial z} \frac{\partial \overline{f_2}}{\partial \overline{z}} + g_1 \frac{\partial^2 \overline{f_2}}{\partial z \partial \overline{z}} \\ &\quad + \frac{\partial \overline{f_2}}{\partial z} \frac{\partial \overline{g_2}}{\partial \overline{z}} + \overline{f_2} \frac{\partial^2 \overline{g_2}}{\partial z \partial \overline{z}} + \frac{\partial \overline{g_2}}{\partial z} \frac{\partial \overline{f_2}}{\partial \overline{z}} + \overline{g_2} \frac{\partial^2 \overline{f_2}}{\partial z \partial \overline{z}} \\ &= f_1' \overline{g_2'} + g_1' \overline{f_2'} + \overline{g_2'} \frac{\partial \overline{f_2}}{\partial z} + \overline{f_2'} \frac{\partial \overline{g_2}}{\partial z}.\end{aligned}$$

Thus, we infer that

$$\begin{aligned}\left| (1-|z|^2)^2 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} fg \right| &\leq \left((1-|z|^2) |f_1'| \right) \left((1-|z|^2) |g_2'| \right) \\ &\quad + \left((1-|z|^2) |g_1'| \right) \left((1-|z|^2) |f_2'| \right) \\ &\quad + \left((1-|z|^2) |g_2'| \right) \left((1-|z|^2) \left| \frac{\partial \overline{f_2}}{\partial z} \right| \right) \\ &\quad + \left((1-|z|^2) |f_2'| \right) \left((1-|z|^2) \left| \frac{\partial \overline{g_2}}{\partial z} \right| \right). \qquad (3.1.1)\end{aligned}$$

Since f_i, g_i are Bloch functions, we obtain the boundedness of all terms except $(1 - |z|^2) \left| \frac{\partial \overline{f_2}}{\partial z} \right|$ and $(1-|z|^2) \left| \frac{\partial \overline{g_2}}{\partial z} \right|$, but they are bounded too because $f_2 \in \mathbf{B}$ implies that $\overline{f_2} \in \mathbf{B}_h$. Thus, using Remark 1.3.8 as well as the definition of $\mathbf{B}_h$, we obtain

$$\underset{D}{Sup} \left\{ (1-|z|^2) \left| \frac{\partial \overline{f_2}}{\partial z} \right| \right\} \leq \frac{1}{\sqrt{2}} \underset{D}{Sup} \left\{ (1-|z|^2) \left| \nabla \overline{f_2} \right| \right\} < \infty,$$

and

$$\underset{D}{Sup}\left\{(1-|z|^2)\left|\frac{\partial \overline{g_2}}{\partial z}\right|\right\} \leq \frac{1}{\sqrt{2}}\underset{D}{Sup}\left\{(1-|z|^2)\left|\nabla \overline{g_2}\right|\right\} < \infty.$$

So that, all terms of the right hand side of Equality (3.1.4) are bounded. Thus, we obtain $\left|\widetilde{\Delta} fg\right| = \left|(1-|z|^2)^2 \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} fg\right| < \infty$. This means that $\widetilde{\Delta} fg \in L^\infty$. Now, Theorem 3.1.11 tells us that, in this case, either g is holomorphic or f is conjugate holomorphic, which completes the proof. ■

Remark 3.1.19. *Even though we are interested primarily in Toeplitz operators with bounded symbols, operators with unbounded symbols arise naturally. In contrast to the Hardy space case, unbounded symbols can give rise to bounded Toeplitz operators on the Bergman space.*
For example if $F \in L^1$ and has compact support K in D, then we can define

$$T_F f(z) = \int_D \frac{F(\zeta) f(\zeta)}{(1-\overline{\zeta} z)^2} dA(\zeta).$$

Then, using the fact that $\frac{1}{\left|1-\overline{\zeta}z\right|^2}$ is bounded on K, we see that

$$\begin{aligned} |T_F f(z)| &\leq \int_D \frac{|F(\zeta) f(\zeta)|}{\left|1-\overline{\zeta} z\right|^2} dA(\zeta) \\ &\leq c\underset{K}{Sup}|f| \int_K |F|\, dA \\ &\leq c\,\|f\|_2 \int_K |F|\, dA. \end{aligned}$$

The last inequality is obtained from the fact that $f \in L^2_a$ and the L^2-norm dominates the sup norm over any compact set K because $f(w) = \langle f, K_w \rangle$. Indeed, on any compact $K \Subset D$, we have

$$f(w) = \int_K f(z)\overline{K_w}(z) dA(z).$$

Thus, we get

$$\begin{aligned} |f(w)| &\leq \int_K |f(z)|\left|\overline{K_w}(z)\right| dA(z) \\ &\leq \|f\|_2 \left(\int_K \left|\overline{K_w}(z)\right|^2 dA(z)\right)^{\frac{1}{2}}. \end{aligned}$$

Hence, since K_w *is bounded on* K, *we obtain*

$$\underset{K}{Sup}\,|f(w)| \leq c\,\|f\|_2\,.$$

So, we have that the sup norm of $T_F f$ *is dominated by a constant times* $\|f\|_2$, *and hence* $\|T_F f\|_2 \leq c\,\|f\|_2$ *for some constant* c. *More generally, if* F *is nearly bounded, in the sense of Definition 3.1.4, then* T_F *is bounded on* L_a^2 *because* F *can be written as an* L^1- *function with compact support plus a bounded function.*

3.2 Examples

In this section we provide some examples that show that the Brown-Halmos type theorem 3.1.11 fails for general symbols, even for symbols continuous up to the boundary.

Example 3.2.1. *Recall the equivalence of (1) and (2) of Proposition 3.1.6 and take* $f(z) = z = z + 0$, $g(z) = \overline{z} = 0 + \overline{z}$ *and* $h(z) = u(z) = 1 - \log\frac{1}{|\zeta|^2}$. *Since we showed in Example 2.4.3 that* $z\overline{z} = B(u)$, *we have*

$$T_z T_{\overline{z}} = T_{u(z)}.$$

Remark that u *is nearly bounded which can be seen as follows: for any* s, *with* $0 < s < 1$, $|z| \leq s$ *we have that* $|z|^2 \leq s^2$. *Then* $\frac{1}{|z|^2} \geq \frac{1}{s^2}$; *so that* $1 - \log\frac{1}{|z|^2} \leq 1 - \log\frac{1}{s^2}$, *whence* $u_1 = 1 - \log\frac{1}{|z|^2}$ *is integrable and with compact support namely* $\overline{D}(0,s)$. *On the other hand on* $D_s = \{z : s < |z| < 1\}$, u *is bounded, as* $\lim\limits_{|z|\to 1}(1 - \log\frac{1}{|z|^2}) = 1$. *We conclude that* u *can be written as* $u_1 + u_2$ *where* u_1 *is an* L^1-*function with compact support* $\overline{D}(0,s)$ *and* u_2 *is a bounded function on* D_s.

The following example tells us that the Brown-Halmos theorem 3.1.11 is not true for the Bergman space Toeplitz operators unless we make restrictions on the symbols.

Example 3.2.2. *Let us start with* $T_z T_{\overline{z}} = T_{u(z)}$, *with* $u(\zeta) = 1 - \log\frac{1}{|\zeta|^2}$. *If we compose both sides of the above display on the right by* T_z, *we get* $(T_z T_{\overline{z}})T_z = T_{u(z)} T_z$. *Since the symbol* z *is analytic, by part (2) of Proposition 3.1.6, we see that* $T_z T_{\overline{z}z} = T_{zu(z)}$, *i.e.*

$$T_z T_{|z|^2} = T_{zu(z)}.$$

The last equation is of the form $T_fT_g = T_h$, where f, g and h are continuous on the closed unit disk, ($\lim_{\zeta\to 0}\zeta \log \frac{1}{|\zeta|^2} = 0$), but neither $\overline{f} = \overline{z}$ nor $g = |z|^2 = z\overline{z}$ is holomorphic because Cauchy-Riemann equations are not satisfied.

Remark 3.2.3. *Example **3.2.2** shows that the condition $\widetilde{\Delta}h \in L^\infty$ in Theorem 3.1.11 can not be dropped as $h(z) = zu(z)$ has unbounded invariant Laplacian. Indeed, we have*

$$\begin{aligned}\widetilde{\Delta}h &= \widetilde{\Delta}(z - z\log\frac{1}{|z|^2})\\ &= (1-|z|^2)^2\Delta(z + z\log z\overline{z}) = (1-|z|^2)^2\frac{\partial}{\partial z}(\frac{zz}{z\overline{z}})\\ &= (1-|z|^2)^2\frac{\partial}{\partial z}(\frac{z}{\overline{z}}) = (1-|z|^2)^2\frac{1}{\overline{z}},\end{aligned}$$

which is not bounded in D.

The following example tells us that Theorem 3.1.11 does not remain true if we just require that the functions f, g and h have their invariant Laplacian bounded in D.

Example 3.2.4. *If we start with $T_zT_{z\overline{z}} = T_{zu(z)}$ and we compose on the right again by T_{z^2}, we get $T_zT_{\overline{z}z^3} = T_{z^3u(z)}$, (as z^2 is holomorphic).*
In the latter equation, which is of the form $T_fT_g = T_h$, all three symbols have bounded invariant Laplacian which can be seen as follows:

1. *Since $f(z) = z$, we have $\Delta z = 0$ and $\widetilde{\Delta}z = 0$ which is bounded.*
2. *For $g(z) = \overline{z}z^3$, we have $\Delta\overline{z}z^3 = \frac{\partial}{\partial z}\frac{\partial}{\partial\overline{z}}(\overline{z}z^3) = \frac{\partial}{\partial z}(z^3) = 3z^2$. So that*

   $$\widetilde{\Delta}g = 3(1-|z|^2)^2z^2,$$

   which is clearly bounded.
3. *With regard to $h(z) = z^3u(z)$, we have $\widetilde{\Delta}h = (1-|z|^2)^2\Delta(z^3u(z))$. But*

   $$\Delta z^3u = \frac{\partial}{\partial z}\frac{\partial}{\partial\overline{z}}(z^3 + z^3\log z\overline{z}) = \frac{\partial}{\partial z}(\frac{z^3z}{z\overline{z}}) = \frac{\partial}{\partial z}(\frac{z^3}{\overline{z}}) = 3\frac{z^2}{\overline{z}}.$$

   Thus we get

   $$\widetilde{\Delta}h = 3(1-|z|^2)^2\frac{z^2}{\overline{z}},$$

whence

$$\left|\widetilde{\Delta}h\right| = 3(1-|z|^2)^2\frac{|z|^2}{|\overline{z}|} = 3(1-|z|^2)^2\,|z| < \infty.$$

But neither $\overline{f}$ nor g is holomorphic.

Example 3.2.5. *$T_uT_v = T_w$, where $u(\zeta) = \zeta$, $v(\zeta) = \overline{\zeta}^2\zeta$ and $w(\zeta) = 2\zeta\overline{\zeta} - 1$. In fact, by Lemma 2.4.4, We have that $z\overline{z}^2 = B(f)(z)$ where $f(\zeta) = 2\overline{\zeta} - \frac{1}{\zeta}$. Since $z = z + 0$ and $\overline{z}^2 = 0 + \overline{z^2}$, then the equivalence of (1) and (2) in Proposition 3.1.6 implies that*

$$T_zT_{\overline{z}^2} = T_{2\overline{z}-\frac{1}{z}}.$$

Now if we compose both sides of the last equation on the right by T_z we see that, since z is holomorphic, $T_{\overline{z}^2}T_z = T_{\overline{z}^2z}$ and $T_{2\overline{z}-\frac{1}{z}}T_z = T_{2\overline{z}z-1}$, whence we obtain:

$$T_zT_{\overline{z}^2z} = T_{2\overline{z}z-1}.$$

3.3 A stronger Brown-Halmos type theorem

In this section we use the characterization of all triples (f, g, u) where f and g are non-constant holomorphic functions on the unit disk D and u is integrable on D such that $f\overline{g} = Bu$, namely Theorem 2.4.5, to give an improvement of Theorem 3.1.11. The next corollary can be viewed as a fairly considerable improvement of Theorem 3.1.11. It asserts that, under some reasonable conditions, a necessary and sufficient condition for $T_fT_g = T_h$ to hold is that the analytic and coanalytic components of the symbols f and g respectively have certain specific form, namely they are compositions of analytic polynomials with disk automorphisms and that h has certain form given in the assertion.

Corollary 3.3.1. *Suppose that f_1, f_2, g_1 and g_2 are holomorphic in D and that $f = f_1 + \overline{f_2}$ as well as $g = g_1 + \overline{g_2}$ are bounded in D and that $h \in L^1$ and neither $\overline{f}$ nor g is holomorphic. Then $T_fT_g = T_h$ if and only if the following holds: there are non-constant holomorphic polynomials p and q with $\deg(pq) \leq 3$ and an $a \in D$ such that $f_1 = p\circ\phi_a$ and $g_2 = q\circ\phi_a$. Here, f_2 and g_1 can be arbitrary bounded holomorphic functions in D and h must be of the form $h = u\circ\phi_a + \overline{f_2}g_1 + f_1g_1 + \overline{f_2}\overline{g_2}$, where $p\overline{q} = B(u)$.*

Proof: It was shown in Proposition 3.1.6 that $T_f T_g = T_h$ if and only if

$$f_1\overline{g_2} = B(h - \overline{f_2}g_1 - f_1 g_1 - \overline{f_2}\overline{g_2}).$$

Notice that the statement that neither $\overline{f}$ nor g is holomorphic is equivalent to the statement that neither f_1 nor g_2 is constant. Indeed, if f_1 is constant then $f = c + \overline{f_2}$ and $\overline{f}$ is holomorphic and if g_2 is constant then $g = g_1 + c$ which is holomorphic. Now, suppose that neither $\overline{f}$ nor g is holomorphic and $T_f T_g = T_h$. Then it follows from Theorem 2.4.5 that $f_1 = p \circ \phi_a$ and $g_2 = q \circ \phi_a$ for non-constant polynomials p and q with $\deg(pq) \leq 3$. This implies that

$$p \circ \phi_a . \overline{q} \circ \phi_a = B(h - \overline{f_2}g_1 - f_1 g_1 - \overline{f_2}\overline{g_2}).$$

But since

$$B(u \circ \phi_a) = (B(u)) \circ \phi_a = (p\overline{q}) \circ \phi_a = p \circ \phi_a . \overline{q} \circ \phi_a,$$

we see that

$$B(u \circ \phi_a) = B(h - \overline{f_2}g_1 - f_1 g_1 - \overline{f_2}\overline{g_2}).$$

Hence, by the injectivity of the Berezin transform, we obtain

$$u \circ \phi_a = h - \overline{f_2}g_1 - f_1 g_1 - \overline{f_2}\overline{g_2},$$

and

$$h = u \circ \phi_a + \overline{f_2}g_1 + f_1 g_1 + \overline{f_2}\overline{g_2},$$

Next suppose that p and q are non-constant polynomials with $\deg(pq) \leq 3$, $f_1 = p \circ \phi_a$, $g_2 = q \circ \phi_a$ and f_2 as well as g_1 are bounded holomorphic functions and

$$h = u \circ \phi_a + \overline{f_2}g_1 + f_1 g_1 + \overline{f_2}\overline{g_2}.$$

Then, we see that

$$B(h - \overline{f_2}g_1 - f_1g_1 - \overline{f_2}\overline{g_2}) = B(u \circ \phi_a) = p \circ \phi_a . \overline{q} \circ \phi_a = f_1\overline{g_2}.$$

Therefore, by Proposition 3.1.6, we might have $T_fT_g = T_h$. This concludes the proof of this corollary. ■

The next Corollary is a theorem of Brown-Halmos type in the sense of Definition 1.2.19. It states that if f and g are bounded harmonic functions and $h \in L^1$ is locally bounded, then $T_fT_g = T_h$ implies that $\overline{f}$ or g is holomorphic. Which gives an improvement of Theorem 3.1.11, where it is only assumed that h is a bounded C^2-function with the property that $\widetilde{\Delta}h$ is also bounded in D. The fact that this result is a generalization of Theorem 3.1.11 is more transparent than Corollary 3.3.1 does.

Corollary 3.3.2. *Suppose that f_1, f_2, g_1 and g_2 are holomorphic in D and that $f = f_1 + \overline{f_2}$ and $g = g_1 + \overline{g_2}$ are bounded in D and that $h \in L^1$ is bounded on D, or even bounded on compact subsets of D. Then $T_fT_g = T_h$ implies that $\overline{f}$ or g is holomorphic.*

Proof: Let us suppose that $T_fT_g = T_h$ with neither $\overline{f}$ nor g is holomorphic. By Corollary 3.3.1, we should have $p\overline{q} = B(u)$ where p and q are bounded holomorphic polynomials with $\deg(pq) \le 3$ and

$$h = u \circ \phi_a + \overline{f_2}g_1 + f_1g_1 + \overline{f_2}\overline{g_2}.$$

If $p(z) = Az + B$ and $\overline{q}(z) = a\overline{z}^2 + b\overline{z} + c$, then

$$(p\overline{q})(z) = c_1z\overline{z}^2 + c_2z\overline{z} + c_3z + c_4\overline{z}^2 + c_5\overline{z} + c_6,$$

where c_i are constants. Taking the Berezin transform of both sides and using the fact that the Berezin transform reproduces harmonic functions and that, by Lemma 2.4.4, we have $B\left(2\overline{\zeta}-\frac{1}{\zeta}\right)(z)=z\overline{z}^2$ and that $B\left(1+\log|\zeta|^2\right)(z)=z\overline{z}$ as we have seen in Example 2.4.3, we arrive at

$$u(\zeta)=c_1\left(2\overline{\zeta}-\frac{1}{\zeta}\right)+c_2(1+\log|\zeta|^2)+c_3\zeta+c_4\overline{\zeta}^2+c_5\overline{\zeta}+c_6.$$

This means that u differs by a bounded function, namely $c_2+2c_1\overline{\zeta}+c_3\zeta+c_4\overline{\zeta}^2+c_5\overline{\zeta}+c_6$, from $c_1\log|\zeta|^2+\frac{c_2}{\zeta}$.

Similarly, if $p(z)=Az^2+Bz+c$ and $\overline{q}(z)=a\overline{z}+b$, then we get

$$(p\overline{q})(z)=c_1\overline{z}z^2+c_2z^2+c_3z\overline{z}+c_4z+c_5\overline{z}+c_6.$$

Arguing in the same manner as above and using part (4) of Remark 2.1.6, we obtain

$$u(\zeta)=c_1\left(2\zeta-\frac{1}{\overline{\zeta}}\right)+c_2\zeta^2+c_3(1+\log|\zeta|^2)+c_4\zeta+c_5\overline{\zeta}+c_6.$$

We conclude that u differs by a bounded function, namely $2c_1\zeta+c_2\zeta^2+c_3+c_4\zeta+c_5\overline{\zeta}+c_6$, from $c_1\log|\zeta|^2+\frac{c_3}{\overline{\zeta}}$. Hence we infer that u differs by a bounded function from

$$c_1\log|\zeta|^2+\frac{c_2}{\zeta}+\frac{c_3}{\overline{\zeta}},$$

where at least one of the c_j must differ from zero. But the latter expression can be bounded near zero only if all the c_j are zero. Thus u is not bounded near zero and hence $u\circ\phi_a$ is not bounded near a. This means that h is not bounded and there exists a compact subset Ω_0 of D containing a on which h is not bounded . Thus, if h is bounded or even bounded on compact subsets of D, then $T_fT_g=T_h$ implies that $\overline{f}$ or g is holomorphic. ■

3.4 The case of weighted Bergman space

In this section, we are interested in the analog of Theorem 3.1.11 in the case of weighted Bergman space Toeplitz operators. In fact, we are trying here to follow the steps that have been introduced in Section 3.1. Unfortunately we could not reach the final step, but we prove some powerful results which can be considered as a good contribution in this direction.

Proposition 3.4.1. *The weighted Bergman space Toeplitz operators have the following properties:*

1. *If* $T_u = 0$ *then* $u = 0$.
2. $T_u^* = T_{\overline{u}}$.
3. *If* f *is holomorphic, then* $T_u T_f = T_{uf}$ *and* $T_{\overline{f}} T_u = T_{\overline{f}u}$ *for any* u.
4. *If* f *is holomorphic and not identically zero, then* T_f *is one-to-one.*
5. *If* $g \in A_\alpha^2$ *and* $w \in D$ *then* $P_\alpha(\overline{g} K_w) = \overline{g}(w) K_w^{(\alpha)}$.

Proof: (1) If $T_u = 0$ then $T_u K_w^{(\alpha)} = 0$ which implies that

$$\left\langle T_u K_w^{(\alpha)}, K_w^{(\alpha)} \right\rangle_\alpha = B_\alpha[T_u](w) = B_\alpha[u](w) = 0, \forall w \in D.$$

Thus $u = 0$, as B_α is one-to-one.

(2) For any $h, f \in A_\alpha^2$,

$$\begin{aligned} \langle T_u^* h, f \rangle_\alpha &= \langle h, T_u f \rangle_\alpha = \langle h, P_\alpha(uf) \rangle_\alpha = \langle h, uf \rangle_\alpha \\ &= \langle \overline{u} h, f \rangle_\alpha = \langle P_\alpha(\overline{u} h), f \rangle_\alpha = \langle T_{\overline{u}} h, f \ \rangle_\alpha. \end{aligned}$$

Hence $T_u^* = T_{\overline{u}}$.

(3) If f is holomorphic, then for any $g \in A_\alpha^2$ we have:

(i) $T_u T_f(g) = T_u(P_\alpha(fg)) = T_u(fg) = P_\alpha(ufg) = T_{uf}(g)$, whence $T_u T f = T_{uf.}$

(ii) $T_{\overline{f}} T_u = (T_{\overline{u}} T_f)^* = (T_{\overline{u}f})^* = T_{\overline{f}u}$.

(4) Let $u \in \ker T_f$, then $T_f(u) = P_\alpha(fu) = 0$. Since f is holomorphic, we obtain $fu = 0$. But the product of two analytic functions is zero only in the case when one of them is zero. Since f is not identically zero then $u = 0$. Hence $\ker T_f = \{0\}$.

(5) If $g \in A^2_\alpha$, $w \in D$, then

$$\begin{aligned} P_\alpha\left(\overline{g}K_w^{(\alpha)}\right)(z) &= \left\langle P_\alpha(\overline{g}K_w^{(\alpha)}), K_z^{(\alpha)}\right\rangle_\alpha \ , z \in D \\ &= \left\langle \overline{g}K_w^{(\alpha)}, K_z^{(\alpha)}\right\rangle_\alpha = \left\langle K_w^{(\alpha)}, gK_z^{(\alpha)}\right\rangle_\alpha \\ &= \overline{\left\langle gK_z^{(\alpha)}, K_w^{(\alpha)}\right\rangle} = \overline{gK_z^{(\alpha)}(w)} \\ &= \overline{g}(w)K_w^{(\alpha)}(z). \end{aligned}$$

Thus $P_\alpha\left(\overline{g}K_w^{(\alpha)}\right) = \overline{g}(w)K_w^{(\alpha)}$ and the proof is complete. ■

The following result is just the analog of Lemma 3.1.5.

Lemma 3.4.2. *If f,g and h are in $L^\infty(D)$. Then we have the following easy and useful fact:*

$$T_fT_g = T_h \text{ if and only if } T_fT_gK_w^{(\alpha)} = T_hK_w^{(\alpha)}, \forall w \in D.$$

Proof: It is clear that $T_fT_g = T_h$ implies that $T_fT_gK_w^{(\alpha)} = T_hK_w^{(\alpha)}, \forall w \in D$. For the other implication, if

$$T_fT_gK_w^{(\alpha)} = T_hK_w^{(\alpha)}, \forall w \in D,$$

then

$$\left\langle T_fT_gK_w^{(\alpha)}, K_w^{(\alpha)}\right\rangle_\alpha = \left\langle T_hK_w^{(\alpha)}, K_w^{(\alpha)}\right\rangle_\alpha,$$

In other words, we have

$$(1-|w|^2)^{\alpha+2}\left\langle T_fT_gK_w^{(\alpha)}, K_w^{(\alpha)}\right\rangle_\alpha = (1-|w|^2)^{\alpha+2}\left\langle T_hK_w^{(\alpha)}, K_w^{(\alpha)}\right\rangle_\alpha.$$

So, we obtain

$$B_\alpha[T_fT_g - T_h](w) = 0, \forall w \in D.$$

Thus, by the injectivity of the Berezin transform, we get $T_fT_g = T_h$. ■

Proposition 3.4.3. *Suppose that $f = f_1 + \overline{f_2}$ and $g = g_1 + \overline{g_2}$ are bounded harmonic functions with f_i, g_i holomorphic in D and $\alpha > -1$. Then the following are equivalent:*

1. $T_f T_g = T_g T_f$.
2. $f_1(z)\overline{g_2}(z) - g_1(z)\overline{f_2}(z) = B_\alpha(f_1\overline{g_2} - g_1\overline{f_2})(z)$, *for all* $z \in D$.
3. $f_1(z)\overline{g_2}(\overline{w}) - g_1(z)\overline{f_2}(\overline{w}) = (1 - zw)^{\alpha+2} \int_D \frac{f_1(\zeta)\overline{g_2}(\zeta) - g_{1(\zeta)}\overline{f_2}(\zeta)}{(1-z\overline{\zeta})^2(1-w\zeta)^2} dA_\alpha(\zeta)$,
   for all $(z, w) \in D \times D$

Proof: (1)⇔(3): By Lemma 3.4.2, we have

$$T_f T_g = T_g T_f \Leftrightarrow T_f T_g K_w^{(\alpha)} = T_g T_f K_w^{(\alpha)}.$$

This is equivalent to

$$T_f \left(P_\alpha \left(g_1 K_w^{(\alpha)} + \overline{g_2} K_w^{(\alpha)}\right)\right) = T_g \left(P_\alpha \left(f_1 K_w^{(\alpha)} + \overline{f_2} K_w^{(\alpha)}\right)\right),$$

which in its turn equivalent to

$$T_f \left(g_1 K_w^{(\alpha)} + \overline{g_2}(w) K_w^{(\alpha)}\right) = T_g \left(f_1 K_w^{(\alpha)} + \overline{f_2}(w) K_w^{(\alpha)}\right).$$

The last equality is exactly the following

$$P_\alpha \left((f_1 + \overline{f_2}) \left(g_1 K_w^{(\alpha)} + \overline{g_2}(w) K_w^{(\alpha)}\right)\right) = P_\alpha \left((g_1 + \overline{g_2}) \left(f_1 K_w^{(\alpha)} + \overline{f_2}(w) K_w^{(\alpha)}\right)\right).$$

The last is also equivalent to

$$\begin{aligned} f_1 g_1 K_w^{(\alpha)} + \overline{g_2}(w) f_1 K_w^{(\alpha)} + P_\alpha \left(\overline{f_2} g_1 K_w^{(\alpha)}\right) + \overline{g_2}(w) P_\alpha \left(\overline{f_2} K_w^{(\alpha)}\right) \\ = g_1 f_1 K_w^{(\alpha)} + \overline{f_2}(w) g_1 K_w^{(\alpha)} + P_\alpha \left(\overline{g_2} f_1 K_w^{(\alpha)}\right) + \overline{f_2}(w) P_\alpha \left(\overline{g_2} K_w^{(\alpha)}\right), \end{aligned}$$

which can be rewritten as

$$f_1(z)\overline{g_2}(w) - g_1(z)\overline{f_2}(w) = \frac{1}{K_w^{(\alpha)}(z)} \left[P_\alpha(\overline{g_2} f_1 K_w^{(\alpha)}) - P_\alpha(\overline{f_2} g_1 K_w^{(\alpha)})\right](z), \text{ for all } z \in D$$

The latter is equivalent to

$$f_1(z)\overline{g_2}(w) - g_1(z)\overline{f_2}(w) = (1 - z\overline{w})^{\alpha+2} \int_D \frac{f_1(\zeta)\overline{g_2}(\zeta) - g_{1(\zeta)}\overline{f_2}(\zeta)}{(1 - z\overline{\zeta})^{\alpha+2}(1 - \overline{w}\zeta)^{\alpha+2}} dA_\alpha(\zeta),$$

which is just equation (3) with w replaced by $\overline{w}$.

(3)⇒(2): Applying (3) to $(z, \overline{z}) \in D \times D$, we see that (3)⇒(2). To show that (2) implies (3), consider the holomorphic function defined in the bi-disk by the formula:

$$F(z, w) = f_1(z)\overline{g_2}(\overline{w}) - g_1(z)\overline{f_2}(\overline{w}) + (1 - zw)^{\alpha+2} \int_D \frac{g_{1(\zeta)}\overline{f_2}(\zeta) - f_1(\zeta)\overline{g_2}(\zeta)}{(1 - z\overline{\zeta})^{\alpha+2}(1 - w\zeta)^{\alpha+2}} dA_\alpha(\zeta).$$

Assuming (2), F is identically zero on the set $\{(z, \overline{z}) : z \in D\}$ and Proposition 1.1.17 implies that $F \equiv 0$ in $D \times D$. So $F(z, \overline{w}) = 0$, which is the statement (3) and the proof is complete. ■

Chapter 4

Some Examples on Products of Bergman Space Toeplitz Operators

In this chapter we introduce several examples related to products of the Bergman space Toeplitz operators with radial symbols using the Mellin transform and the concept of radialization. In fact, these examples inspire a lot of ideas about the restrictions on the symbols of Toeplitz operators in order to have a theorem of Brown-Halmos type, in the sense of Definition 1.2.19, for Bergman space Toeplitz operators. Despite that the main purpose of this chapter is to give a method that enables us to do the calculus of Toeplitz operators, we give an important result concerning the zero product problem among Bergman space Toeplitz operators. Unless mentioned, all the results in this chapter are due to Ahern and Čučkovič [4].

4.1 Mellin transform and Toeplitz operators

First, let us do some computations:

Lemma 4.1.1. *If $f \in L^1(0,1)$ then $T_f z^k = c_k z^k$ where $c_k = (2k+2)\widetilde{f}(2k+2)$, where $\widetilde{f}$ stands for the Mellin transform of f introduced in Definition 1.1.40.*

Proof: If $f \in L^1(0,1)$, then f is clearly radial.

$$\begin{aligned} T_f z^k(w) &= \int_D \frac{f(w)w^k}{(1-z\overline{w})^2} dA(w) \\ &= \sum_{j=0}^{\infty} (j+1) z^j \int_D w^k \overline{w}^j f(w) dA(w) \\ &= \sum_{j=0}^{\infty} (j+1) z^j \int_0^1 \int_0^{2\pi} r^{k+j+1} e^{i(k-j)\theta} f(r) \frac{d\theta dr}{\pi}. \end{aligned}$$

In the inside integral, observe that

$$\int_0^{2\pi} e^{i(k-j)\theta} d\theta = \begin{cases} 2\pi, & \text{if } j = k. \\ 0, & \text{if } j \neq k. \end{cases}$$

Hence, we obtain

$$T_f z^k(w) = 2(k+1) z^k \int_0^1 r^{2k+1} f(r) dr.$$

On the other hand, we have

$$\int_0^1 r^{2k+1} f(r) dr = \int_0^1 f(r) r^{(2k+2)-1} dr = \widetilde{f}(2k+2).$$

Whence, we conclude that $T_f z^k = (2k+2)\widetilde{f}(2k+2) z^k$. ■

Recall Definitions 1.1.4, 1.1.13, 1.1.26 and remark the following:

Remark 4.1.2. *The previous lemma shows, in particular, that T_f is a diagonal operator when f is radial. This observation yields the following facts:*

1. *T_f is bounded if and only if $\{c_k\}$ is bounded. This is due to the fact that*

$$\|T_f\| = \sup_k |c_k|.$$

2. *T_f is invertible if and only if $\{c_k\}$ is bounded and also bounded away from 0 as the spectrum of T_f is $\sigma_{T_f} = \overline{\{c_k\}}$. So that, if $c_k \to 0$ then $0 \in \sigma_{T_f}$ and T_f is not invertible and if $\{c_k\}$ is bounded away from 0 then T_f is invertible.*

3. *T_f is compact if and only if $c_k \to 0$ when $k \to \infty$. Indeed, if $c_k \nrightarrow 0$ as $k \to \infty$ then 0 is not the only limit point of σ_{T_f}, thus σ_{T_f} can not be the spectrum of a compact operator. Conversely, if $c_k \to 0$ as $k \to \infty$; consider the finite rank diagonal operators T_N defined on L^2_a by*

$$T_N z^k = \begin{cases} c_k z^k & \text{if } k \leq N, \\ 0 & \text{if } k > N. \end{cases}$$

Then $\|T_f - T_N\| \leq \sup_{k>N} |c_k| \to 0$ as $N \to \infty$; i.e. T_f is a norm limit of a sequence of finite rank operators. Hence T_f is compact.

Lemma 4.1.3. *If f, g and h are in $L^1(0,1)$, then $T_f T_g(z^k) = T_h(z^k)$ for all $k = 0, 1, \ldots,$ if and only if*

$$(f * g)(t) = \int_t^1 h(s) \frac{ds}{s} = (h * 1)(t).$$

Proof: First, we have

$$\begin{aligned} T_f T_g(z^k) &= T_f(c_k z^k) = T_f((2k+2)\widetilde{g}(2k+2)z^k) \\ &= (2k+2)\widetilde{g}(2k+2)T_f(z^k) \\ &= (2k+2)^2 \widetilde{f}(2k+2)\widetilde{g}(2k+2)z^k, \end{aligned}$$

and

$$T_h(z^k) = (2k+2)\widetilde{h}(2k+2)z^k.$$

So, we see that

$$T_f T_g(z^k) = T_h(z^k),$$

is equivalent to the fact that

$$(2k+2)\widetilde{f}(2k+2)\widetilde{g}(2k+2) = \widetilde{h}(2k+2), \text{ for all } k.$$

The latter holds if and only if the bounded holomorphic functions $z\widetilde{f}(z)\widetilde{g}(z)$ and $\widetilde{h}(z)$ agree on the non-Blaschke sequence $\{2k+2\}$ in $\mathbf{\Pi}$, which means that the holomorphic function $z\widetilde{f}(z)\widetilde{g}(z) - \widetilde{h}(z) = 0$ on the non-Blaschke sequence $\{2k+2\}$ in $\mathbf{\Pi}$, (see Remark 1.1.15 and Example 1.1.14). Thus,

$$z\widetilde{f}(z)\widetilde{g}(z) = \widetilde{h}(z),$$

by the principle of analytic continuation. So, we have that $T_f T_g(z^k) = T_h(z^k)$ if and only if $\widetilde{f}(z)\widetilde{g}(z) = \frac{1}{z}\widetilde{h}(z)$.

But since the Mellin transform $\widetilde{1}$ of the constant function 1 reads as

$$\widetilde{1}(z) = \int_0^1 1(r) r^{z-1} dr = \left[\frac{r^z}{z}\right]_0^1 = \frac{1}{z},$$

we see that $\frac{1}{z} = \widetilde{1}$. Hence, by the injectivity of the Mellin transform, we infer that:

$$T_f T_g(z^k) = T_h(z^k) \Leftrightarrow \widetilde{f}\widetilde{g} = \widetilde{h}\widetilde{1} \Leftrightarrow \widetilde{(f * g)} = \widetilde{(h * 1)} \Leftrightarrow f * g = h * 1,$$

which is equivalent to

$$(f * g)(t) = \int_t^1 h(s)\frac{ds}{s}.$$

This completes the proof of the lemma. ■

4.2 Radialization of functions and operators

Definition 4.2.1. *[68] Let f be a function in L^1 and let $rad(f)$ be the function defined by*

$$rad(f)(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}z)dt.$$

We say that $rad(f)$ is the radialization of f.

Remark 4.2.2. *Let f be a function in L^1. From Definition 1.1.4 and Definition 4.2.1, we see that f is radial if and only if it is equal to its radialization.*

Generalizing the idea of radialization to operators, we obtain the following definition

Definition 4.2.3. *[68] For a bounded operator A on L_a^2, we define the radialization $Rad(A)$ of A to be the operator*

$$Rad(A) = \frac{1}{2\pi}\int_0^{2\pi} U_t^* A U_t dt,$$

where U_t is the unitary operator $(U_t f)(z) = f(e^{-it}z)$ for f in L_a^2 and z in D. Accordingly, we will say that the operator A is radial whenever $A = Rad(A)$.

Remark 4.2.4. *[68] The definition of $Rad(A)$ means that for f and g in L_a^2, we have*

$$\langle Rad(A)f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle U_t^* A U_t f, g\rangle dt.$$

The following result tells us that the radialization of functions commutes with the Berezin transform.

Lemma 4.2.5. *[68] Let $f \in L^1$. Then we have*

$$B(rad(f))(z) = rad(B(f))(z).$$

Proof: Using the substitution $u = we^{it}$ and the fact that $k_z(w) = k_{e^{it}z}(e^{it}w)$ *as well as* Lemma 2.4.2, we obtain for $z \in D$:

$$\begin{aligned}
B(rad(f))(z) &= \int_D rad(f)(w)\,|k_z(w)|^2\,dA(w)\\
&= \frac{1}{2\pi}\int_D \left(\int_0^{2\pi} f(we^{it})dt\right)|k_z(w)|^2\,dA(w)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\int_D f(we^{it})\,|k_z(w)|^2\,dA(w)\right)dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\int_D f(we^{it})\left|k_{e^{it}z}(e^{it}w)\right|^2 dA(w)\right)dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\int_D f(u)\,|k_{e^{it}z}(u)|^2\,dA(u)\right)dt\\
&= \frac{1}{2\pi}\int_0^{2\pi} B(f)(e^{it}z)dt = rad(B(f))(z).
\end{aligned}$$

This completes the proof. ■

The following result is a slight generalization of the latter.

Proposition 4.2.6. *[68]*

1. *For all z in D, we have*

$$B(Rad(A))(z) = rad(B(A))(z).$$

2. *An operator A is radial if and only if its Berezin transform function is radial.*

Proof: 1. Since $U_t k_z(w) = k_z(e^{-it}w) = k_{e^{it}z}(w)$, we have that

$$\begin{aligned} B\left(Rad(A)\right)(z) &= \langle Rad(A)k_z, k_z\rangle \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle AU_t k_z, U_t k_z\rangle dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle Ak_{e^{it}z}, k_{e^{it}z}\rangle dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} B(A)(e^{it}z)dt = rad(B(A))(z). \end{aligned}$$

Thus $B(Rad(A))(z) = rad(B(A))(z)$ for all $z \in D$.

2. From (1) we see that if A is a radial operator, then its Berezin transform must be a radial function. Because $rad(B(A)) = B\left(Rad(A)\right) = B(A)$, as A is radial. Conversely, suppose that $B(A)$ is radial. Then $B(RadA) = rad(B(A)) = B(A)$. By the injectivity of the Berezin transform we infer that $RadA = A$. Thus, if the Berezin transform of an operator A is radial then A is radial and the proof is complete. ■

Remark 4.2.7. *From Lemma 4.2.5 we have that f is radial if and only if $B(f)$ is radial and so, by Proposition 4.2.6, the Toeplitz operator T_f is radial if and only if f is radial, (as $B(T_f) = B(f)$ by part (5) of Remark 2.1.6).*

Lemma 4.2.8. *[68] Let A be a radial bounded operator on L^2_a. Then A is a diagonal operator with respect to the standard basis $\{e_n\}$ of L^2_a.*

Proof: *It is easy to see that*

$$\begin{aligned}
\langle Ae_n, e_m\rangle &= \langle RadA\ e_n, e_m\rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle U_t^* A U_t e_n, e_m\rangle\, dt = \frac{1}{2\pi}\int_0^{2\pi} \langle AU_t e_n, U_t e_m\rangle\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left\langle A\left(e^{-int}e_n\right), e^{-imt}e_m\right\rangle dt = \frac{1}{2\pi}\int_0^{2\pi} \left\langle e^{-int}A\left(e_n\right), e^{-imt}e_m\right\rangle dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n-m)t}\langle Ae_{n,}e_m\rangle\, dt = 0,\ \ whenever\ n \neq m.
\end{aligned}$$

Thus A is a diagonal operator. ■

Remark 4.2.9. *From Lemma 4.2.8, we see that if T_f is a bounded radial Toeplitz operator on L_a^2 then T_f is diagonal. Indeed, we are certain that the converse is also true in this case as we are going to see in the next lemma which is needed in the sequel.*

Note that this lemma is among our contributions, so it is completely new.

Lemma 4.2.10. *If T_f is diagonal, then T_f is radial and hence f is also radial.*

Proof: If T_f is diagonal, then $T_f z^k = c_k z^k$, c_k is constant.

For any $h, g \in L_a^2$, we have

$$\langle (Rad(T_f))\, h, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle U_t^* T_f U_t h, g\rangle dt.$$

Since $h \in L_a^2$ it can be rewritten as $h(z) = \sum_{k=0}^{\infty} a_k z^k$. Then, we obtain

$$U_t h(z) = h(e^{-it}z) = \sum_{k=0}^{\infty} a_k e^{-ikt} z^k.$$

In order to find U_t^* we should write

$$\langle U_t h, g\rangle = \langle h, U_t^* g\rangle.$$

Thus, we see that

$$\langle h, U_t^* g\rangle = \int_D U_t h(z)\overline{g}(z)dA(z) = \int_D h(e^{-it}z)\overline{g}(z)dA(z).$$

Let $w = e^{-it}z$, then $z = we^{it}$ and, by Lemma 2.4.2, $dA(w) = dA(z)$. Thus, we obtain

$$\langle h, U_t^* g\rangle = \int_D h(w)\overline{g}(e^{it}w)dA(w).$$

Hence $U_t^* g(w) = g(e^{it}w)$. Now, since $T_f z^k = c_k z^k$, we obtain

$$\begin{aligned}
\langle (Rad(T_f))h, g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle U_t^* T_f\left(\sum a_k e^{-ikt} z^k\right), g\right\rangle dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left\langle U_t^* \left(\sum_{k=0}^{\infty} c_k a_k e^{-ikt} z^k\right), g\right\rangle dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left\langle \sum_{k=0}^{\infty} c_k a_k z^k, g\right\rangle dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle T_f h, g\rangle dt = \langle T_f h, g\rangle.
\end{aligned}$$

Hence we see that $Rad(T_f) = T_f$. In other words, T_f is radial and hence, by Remark 4.2.7, we infer that f is radial ■

4.3 Non-trivial examples of radial f and g ensuring $T_f T_g = T_h$

Lemma 4.1.3 gives a general method of constructing non-trivial examples, in the sense of Definition 3.1.1, of radial f and g such that $T_f T_g = T_h$ for some h. In this section we introduce some of them.

The following result is a corollary of Lemma 4.1.3.

Corollary 4.3.1. *Suppose that f and g are bounded radial functions on $(0,1)$ and that $f * g \in C^1[0,1]$, if we just define $h(t) = -t(f*g)'(t)$ then $T_f T_g = T_h$.*

Proof: If we put

$$h(t) = -t(f * g)'(t),$$

then we obtain

$$\frac{h(t)}{t} = -(f * g)'(t).$$

So by integration, we see that

$$\int_t^1 \frac{h(s)}{s} ds = -\int_t^1 (f * g)'(s) ds = -(f * g)(1) + (f * g)(t).$$

Now if f and g are bounded, we see that $(f * g)(1) = \int_1^1 f(s) g(\frac{1}{s}) \frac{ds}{s} = 0$, whence

$$\int_t^1 \frac{h(s)}{s} ds = (f * g)(t).$$

Now, by Lemma 4.1.3, we have that $T_f T_g = T_h$. ■

A concrete example can be given as follows:

Example 4.3.2. *If $f(t) = t^\alpha$ and $g(t) = t^\beta$ with $\alpha, \beta > 0$ and $\alpha \neq \beta$ then we have $T_f T_g = T_h$ where*

$$h(t) = \frac{\alpha t^\alpha - \beta t^\beta}{\alpha - \beta},$$

Indeed, from one hand, we have

$$\begin{aligned} (f * g)(t) &= \int_t^1 s^\alpha \frac{t^\beta}{s^\beta} \frac{ds}{s} = t^\beta \int_t^1 s^{\alpha-\beta-1} ds \\ &= \left[t^\beta \frac{s^{\alpha-\beta}}{\alpha - \beta} \right]_t^1 = \frac{t^\beta - t^\alpha}{\alpha - \beta}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_t^1 \frac{h(s)}{s} ds &= \int_t^1 \frac{\alpha s^\alpha - \beta s^\beta}{\alpha - \beta} \frac{ds}{s} \\ &= \frac{1}{\alpha - \beta} \left(\alpha \int_t^1 \frac{s^\alpha}{s} ds - \beta \int_t^1 \frac{s^\beta}{s} ds \right) \\ &= \frac{1}{\alpha - \beta} \left(\left[\alpha \frac{s^\alpha}{\alpha} \right]_t^1 - \left[\beta \frac{s^\beta}{\beta} \right]_t^1 \right) \\ &= \frac{1}{\alpha - \beta} \left(1 - t^\alpha - 1 + t^\beta \right) = \frac{t^\beta - t^\alpha}{\alpha - \beta}. \end{aligned}$$

So that the condition of Lemma 4.1.3 is satisfied, whence $T_fT_g = T_h$. *A simpler way to see this is to just apply Corollary 4.3.1. The functions* $f(t) = t^\alpha$ *and* $g(t) = t^\beta$ *are bounded on* $(0,1)$ *as radial functions and* $h(t) = \frac{\alpha t^\alpha - \beta t^\beta}{\alpha-\beta} = -t\left(\frac{t^\beta - t^\alpha}{\alpha-\beta}\right)' = -t\,(f*g)'\,(t)$. *So that by Corollary 4.3.1, we infer that* $T_fT_g = T_h$. ■

Remark 4.3.3. *It is not true that for every radial* $f,\ g \in L^\infty$ *there is* $h \in L^1(0,1)$ *such that* $T_fT_g = T_h$. *The reason is that in case* $T_fT_g = T_h$ *we have by Lemma 4.1.3*

$$\widetilde{f}(z)\widetilde{g}(z) = \frac{\widetilde{h}(z)}{z}.$$

In particular

$$\left|\widetilde{f}(z)\widetilde{g}(z)\right| \le \frac{c}{|z|}.$$

In other words $\left|z\widetilde{f}(z)\widetilde{g}(z)\right| \le c$ *which is not true in general because* $|z|$ *is not bounded in* $\mathbf{\Pi}$.

Now, we give an example of a continuous function f with compact support in $(0,1)$ such that $f*f$ is not $O\left(\frac{1}{|z|}\right)$ and hence $T_fT_f \ne T_h$, for any $h \in L^1(0,1)$.

Example 4.3.4. *Let us calculate*

$$\widetilde{f}(2+i2^n) = \int_0^1 f(r) r^{1+i2^n} dr.$$

In the right hand side of the latter we make the variable change $r = e^{-t}$, *(and so* $dr = -rdt$*), to obtain*

$$\begin{aligned} \int_0^1 f(r) r^{1+i2^n} dr &= -\int_\infty^0 f(e^{-t}) e^{-2t-it2^n} dt \\ &= \int_0^\infty f(e^{-t}) e^{-2t-it2^n} dt. \end{aligned}$$

Now, we choose the function f *in such a way that*

$$f(e^{-t})e^{-2t} = \begin{cases} a_0 + \sum_{k=1}^\infty \frac{1}{k^2} e^{i2^k t}, & \text{if } 2\pi \le t \le 4\pi \\ 0, & \text{otherwise.} \end{cases}$$

Here a_0 is chosen so that the series vanishes at 2π and at 4π, and consequently f is continuous. Interchanging the integral and the sum in latter on $[0,\infty)$, we obtain

$$\begin{aligned}
\widetilde{f}(2+i2^n) &= \int_{2\pi}^{4\pi}\left(a_0 e^{-i2^n t} + \sum_{k=1}^{\infty}\frac{1}{k^2}e^{i2^k t}e^{-i2^n t}\right)dt \\
&= \left[a_0\frac{e^{-i2^n t}}{-i2^n}\right]_{2\pi}^{4\pi} + \sum_{k=1}^{\infty}\frac{1}{k^2}\int_{2\pi}^{4\pi} e^{it(2^k-2^n)}dt \\
&= \sum_{k=1}^{\infty}\frac{1}{k^2}\int_{2\pi}^{4\pi} e^{it(2^k-2^n)}dt = \begin{cases} 0, \textit{ if } n \neq k, \\ \frac{2\pi}{n^2}, \textit{ if } n = k. \end{cases}
\end{aligned}$$

Thus, we conclude that

$$\widetilde{f * f}(2+i2^n) = \widetilde{f}(2+i2^n)\widetilde{f}(2+i2^n) = \frac{4\pi^2}{n^4},$$

which is not $O\left(\frac{1}{2^n}\right)$.

4.4 Non-trivial examples of bounded f and g with $T_fT_g = T_{fg}$

In this section we give examples of bounded radial functions f and g such that $T_fT_g = T_{fg}$ with neither $\overline{f}$ nor g is holomorphic. This means that the Bergman space analog of the original Brown-Halmos theorem [16] does not hold in the case of radial symbols. To proceed, fix g as follows:

$$g(t) = \begin{cases} 1, \text{ if } 0 < t < \frac{1}{2}, \\ 0, \text{ if } \frac{1}{2} < t < 1. \end{cases}$$

our main task is to find infinitely many bounded f such that $T_fT_g = T_{fg}$ in a non-trivial way.

As we have seen in Lemma 4.1.3, the fact that $T_fT_g = T_{fg}$ is equivalent to the fact that

$$f * g = fg * 1.$$

So, let us show that $f * g = fg * 1$. From one hand, we have

$$(f * g)(t) = \int_t^1 f(s) g\left(\frac{t}{s}\right) \frac{ds}{s},$$

with

$$\begin{aligned} g\left(\frac{t}{s}\right) &= \begin{cases} 1, \text{ if } 0 < \frac{t}{s} < \frac{1}{2}, \\ 0, \text{ if } \frac{1}{2} < \frac{t}{s} < 1 \end{cases} \\ &= \begin{cases} 1, \text{ if } \quad \frac{s}{t} > 2, \\ 0, \text{ if } 2 > \frac{s}{t} > 1 \end{cases} \\ &= \begin{cases} 1, \text{ if } \quad s > 2t, \\ 0, \text{ if } 2t > s > t. \end{cases} \end{aligned}$$

In other words,

$$(f * g)(t) = \begin{cases} \int_{2t}^1 \frac{f(s)}{s} ds, \text{ if } 0 < t < \frac{1}{2}, \\ 0, \qquad\qquad \text{ if } \frac{1}{2} < t < 1. \end{cases}$$

On the other hand, we have

$$(fg * 1)(t) = \int_t^1 f(s) g(s) \frac{ds}{s} = \begin{cases} \int_t^{\frac{1}{2}} \frac{f(s)}{s} ds, \text{ if } 0 < t < \frac{1}{2}, \\ 0, \qquad\qquad \text{ if } \frac{1}{2} < t < 1. \end{cases}$$

Since $T_f T_g = T_{fg}$ is equivalent to $f * g = fg * 1$, we infer that

$$T_f T_g = T_{fg},$$

holds if and only if

$$\int_{2t}^1 \frac{f(s)}{s} ds = \int_t^{\frac{1}{2}} \frac{f(s)}{s} ds, \quad \text{for all } t \text{ such that } 0 < t < \frac{1}{2}.$$

Now, put $s = 2u$ in the right hand side of the latter then $ds = 2du$ and

$$2t < s < 1 \Rightarrow t < u < \frac{1}{2}.$$

Hence, we obtain

$$\int_{2t}^1 \frac{f(s)}{s} ds = \int_t^{\frac{1}{2}} \frac{f(2u)}{u} du.$$

So that, we get

$$\int_t^{\frac{1}{2}} \frac{f(s)}{s} ds = \int_t^{\frac{1}{2}} \frac{f(2u)}{u} du,$$

i.e.

$$\int_t^{\frac{1}{2}} \frac{f(s)}{s} ds = \int_t^{\frac{1}{2}} \frac{f(2s)}{s} ds,$$

which is equivalent to

$$f(s) = f(2s), \text{ for all } s \text{ such that } 0 < s < \frac{1}{2}.$$

So if we have a bounded function f defined originally only on $\left(\frac{1}{2}, 1\right)$ and we define $f(s) = f(2^k s)$ for $\frac{1}{2^{k+1}} < s < \frac{1}{2^k}$, then we have a function satisfying $f(s) = f(2s)$ for all $s < \frac{1}{2}$, (an explicit example on such function will be given later on, namely Example 4.4.1). In other words $\forall f \in L^\infty\left(\frac{1}{2}, 1\right)$, $\exists \mathfrak{f}$ defined on $\left(0, \frac{1}{2}\right)$ such that $\mathfrak{f}(s) = f(2^k s)$. In fact, this construction gives a one-to-one linear mapping from $L^\infty\left(\frac{1}{2}, 1\right)$ onto the set of all radial functions f such that $T_f T_g = T_{fg}$, namely the mapping

$$\begin{array}{rcl} L^\infty\left(\dfrac{1}{2}, 1\right) & \longrightarrow & \{f \in L^\infty(0,1) : f \text{ is radial and } T_f T_g = T_{fg}\} \\ f & \longrightarrow & f^*, \end{array}$$

where

$$f^*(s) = \begin{cases} f(s), \text{ if } \frac{1}{2} < s < 1, \\ f(2^k s), \text{ if } \frac{1}{2^{k+1}} < s < \frac{1}{2^k}. \end{cases}$$

Example 4.4.1. *If $f(r) = r$ for $\frac{1}{2} < r < 1$, then*

$$\mathfrak{f}(r) = \begin{cases} 2r, \ \textit{if } \frac{1}{2^2} < r < \frac{1}{2}, \\ 2^2 r, \ \textit{if } \frac{1}{2^3} < r < \frac{1}{2^2}, \\ \cdot \\ \cdot \\ 2^k r, \ \textit{if } \frac{1}{2^{k+1}} < r < \frac{1}{2^k}, \end{cases}$$

and

$$f^*(r) = \begin{cases} r, \ if \ \frac{1}{2} < r < 1, \\ 2r, \ if \ \frac{1}{2^2} < r < \frac{1}{2}, \\ \cdot \\ \cdot \\ 2^k r, \ if \ \frac{1}{2^{k+1}} < r < \frac{1}{2^k}. \end{cases}$$

It is clear that $f^*(r) = f^*(2r)$ *for all* $r < \frac{1}{2}$ *and* $T_{f^*}T_g = T_{f^*g}$.■

4.5 Non-trivial examples of $T_fT_g = I$ without restrictions

It was shown in Corollary 3.1.15 that if f and g are bounded and harmonic and $T_fT_g = I$, then either f and g are both holomorphic or they are both conjugate holomorphic and in either case $f = \frac{1}{g}$.

In this section we show that there are non-trivial examples of $T_fT_g = I$ without restrictions on f and g.

From Lemma 4.1.3, we see that the condition $T_fT_g = I = T_1$ holds if and only if $f * g = 1 * 1$ which is equivalent to the fact that

$$\widetilde{(f * g)}(z) = \widetilde{(1 * 1)}(z).$$

The latter, in its turn, is equivalent to

$$\widetilde{f}(z)\widetilde{g}(z) = \widetilde{1}(z)\widetilde{1}(z) = \frac{1}{z^2}.$$

To find examples, we take an f and then look for g.

Example 4.5.1. *If* $f(r) = r^{\alpha}$ *with* $0 < \alpha$, *then*

$$\widetilde{f}(z) = \int_0^1 f(r)r^{z-1}dr = \int_0^1 r^{\alpha+z-1}dr = \left[\frac{r^{z+\alpha}}{z+\alpha}\right]_0^1 = \frac{1}{z+\alpha}.$$

In order to obtain $T_fT_g = I$, *we assume that*

$$\frac{1}{z+\alpha}\widetilde{g}(z) = \frac{1}{z^2}.$$

Then, we get

$$\widetilde{g}(z) = \frac{z+\alpha}{z^2} = \frac{1}{z} + \frac{\alpha}{z^2},$$

taking the inverse Mellin transform, we obtain

$$g(r) = 1 - \alpha \log r.$$

Indeed, if $g(r) = 1 - \alpha \log r$*, then*

$$\begin{aligned} \widetilde{g}(z) &= \int_0^1 (r^{z-1} - \alpha r^{z-1} \log r) dr \\ &= \frac{r^z}{z}\Big|_0^1 - \alpha \left(\lim_{a \to 0^+} \left[\frac{r^z}{z} \log r \right]_a^1 - \int_0^1 \frac{r^{z-1}}{z} dr \right) \\ &= \left[\frac{1}{z} + \alpha \frac{r^z}{z^2} \right]_0^1 = \frac{1}{z} + \frac{\alpha}{z^2}. \end{aligned}$$

This g *is not bounded but it does give rise to a bounded* T_g*, (as* g *is nearly bounded).*■

To get an example with both f and g are bounded, it seems to work a little more. The idea is to replace the unbounded part $\log r$ with the bounded function $r \log r$.

Example 4.5.2. *If* $f(r) = 1 + 2r \log r$ *and* $g(r) = 1 + 2\sin(\log r)$*, then* $T_f T_g = I$ *as* $\widetilde{f}(z)\widetilde{g}(z) = \frac{1}{z^2}$*. To see this, compute*

$$\begin{aligned} \widetilde{f}(z) &= \int_0^1 (1 + 2r \log r) r^{z-1} dr \\ &= \int_0^1 r^{z-1} dr + 2 \int_0^1 r^z \log r \, dr \\ &= \frac{1}{z} + 2 \left(\lim_{a \to 0^+} \left[\frac{r^{z+1}}{z+1} \log r \right]_a^1 - \frac{1}{z+1} \int_0^1 r^z dr \right) \\ &= \frac{1}{z} - \frac{2}{(z+1)^2}, \end{aligned}$$

as well as

$$\begin{aligned} \widetilde{g}(z) &= \int_0^1 (1 - 2\sin(\log r)) r^{z-1} dr \\ &= \int_0^1 r^{z-1} dr - 2 \int_0^1 r^{z-1} \sin(\log r) dr. \end{aligned}$$

For the second integral in the R.H.S of the latter, we have

$$\int_0^1 r^{z-1}\sin(\log r)dr = \lim_{a\to 0^+}\left[\frac{r^z}{z}\sin(\log r)\right]_a^1 - \frac{1}{z}\int_0^1 r^{z-1}\cos(\log r)dr$$
$$= -\frac{1}{z}\left(\lim_{a\to 0^+}\left[\frac{r^z}{z}\cos(\log r)\right]_a^1 + \frac{1}{z}\int_0^1 r^{z-1}\sin(\log r)dr\right)$$
$$= -\frac{1}{z^2} - \frac{1}{z^2}\int_0^1 r^{z-1}\sin(\log r)dr.$$

This implies that

$$(1+\frac{1}{z^2})\int_0^1 r^{z-1}\sin(\log r)dr = \frac{z^2+1}{z^2}\int_0^1 r^{z-1}\sin(\log r)dr = \frac{-1}{z^2}.$$

The latter yields

$$\int_0^1 r^{z-1}\sin(\log r)dr = -\frac{1}{z^2+1}.$$

Hence, we obtain

$$\widetilde{g}(z) = \frac{1}{z} + \frac{2}{z^2+1},$$

and therefore

$$\widetilde{f}(z)\widetilde{g}(z) = \left(\frac{1}{z} - \frac{2}{(z+1)^2}\right)\left(\frac{1}{z} + \frac{2}{z^2+1}\right)$$
$$= \frac{1}{z^2} + \frac{2}{z(z^2+1)} - \frac{2}{z(z+1)^2} - \frac{4}{(z+1)^2(z^2+1)}$$
$$= \frac{1}{z^2} + \frac{2(z+1)^2 - 2(z^2+1) - 4z}{z(z^2+1)(z+1)^2}$$
$$= \frac{1}{z^2} + \frac{2z^2+4z+2-2z^2-2-4z}{z(z^2+1)(z+1)^2} = \frac{1}{z^2}.\blacksquare$$

Remark 4.5.3. *If $f \in L^1(0,1)$ and T_f is bounded and invertible, then it is rarely true that the inverse of T_f is again a Toeplitz operator. As we have seen in Remark 4.1.2 that T_f is both bounded and invertible is equivalent to the condition that $\{(2k+2)\widetilde{f}(2k+2)\}$ should be both bounded and bounded away from zero. On the other hand suppose that $T_f^{-1} = T_g$ for some $g \in L^1$. Since $T_g(z^k) \in L_a^2$, we see that $T_fT_g(z^k) = T_f(T_g(z^k)) = T_f\left(\sum d_j z^j\right) = \sum d_j T_f(z^j) = \sum d_j c_j z^j$.*
Hence, we get $d_kc_k = 1$ and $d_jc_j = 0$ for $j \neq k$, i.e. $T_g z^k = d_k z^k = \frac{1}{c_k}z^k$. Then $T_fT_g(z^k) = z^k$ means that $T_g(z^k) = d_k z^k$ for some constant d_k. By Lemma 4.2.10, it

follows that g is radial. But, from Lemma 4.1.3, for radial g we know that $T_f T_g = I$ *is equivalent to* $\widetilde{f}(z)\widetilde{g}(z) = \frac{1}{z^2}$ *for all* $z \in \mathbf{\Pi}$*. In particular for such g to exist it is necessary that* $\widetilde{f}$ *has no zeros in* $\mathbf{\Pi}$*; otherwise* $\widetilde{f}(z)\widetilde{g}(z) \neq \frac{1}{z^2}$ *in* $\mathbf{\Pi}$, *which is not true in general as the next example shows.*

Example 4.5.4. *Let* $f(r) = 2r^5 - 1$*. Then*

$$\begin{aligned} \widetilde{f}(z) &= \int_0^1 (2r^5 - 1)r^{z-1}dr = \left[2\frac{r^{z+5}}{z+5} - \frac{r^z}{z}\right]_0^1 \\ &= \frac{2}{z+5} - \frac{1}{z} = \frac{z-5}{z(z+5)}. \end{aligned}$$

Thus, we infer that

$$(2k+2)\widetilde{f}(2k+2) = \frac{(2k+2)(2k+2-5)}{(2k+2)(2k+2+5)} = \frac{2k-3}{2k+7} \to 1 \quad as \quad k \to \infty.$$

Which is both bounded and bounded away from zero but $\widetilde{f}(5) = 0$*. Hence* T_f^{-1} *cannot be a Toeplitz operator.* ■

4.6 The zero product problem for radial Toeplitz operators

We have seen in Corollary 3.1.13 that if f and g are bounded harmonic functions and $T_f T_g = 0$ then either $f \equiv 0$ or $g \equiv 0$. It is still not known if this zero product theorem holds without restrictions on the symbols. Actually the only known case is the just mentioned one. Ahern and Čučkovič have proved even more, namely: if one of the symbols is radial and the other is arbitrary then there is no zero divisors for such Toeplitz operators.

Theorem 4.6.1. *[4] If* $T_f T_g = 0$ *and one of the functions f or g is radial then one of them must be identically zero.*

Proof: First, we may assume that g is radial and not identically zero and we show that $f \equiv 0$.

According to Remark 1.3.7, write $f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta}$. Thus, the fact that $T_fT_g = 0$ implies

$$\langle T_fT_gz^k, z^l\rangle = 0 \text{ for all } k, l \geq 0.$$

Hence, we see that

$$\left\langle T_f((2k+2)\widetilde{g}(2k+2)z^k), z^l\right\rangle = 0,$$

which can be written as

$$\langle P(f(2k+2)\widetilde{g}(2k+2)z^k), z^l\rangle = 0.$$

This means that

$$\langle f(2k+2)\widetilde{g}(2k+2)z^k, z^l\rangle = 0.$$

Consequently, we get

$$\sum_{n=-\infty}^{\infty} \left\langle f_n(r)e^{in\theta}(2k+2)\widetilde{g}(2k+2)z^k, z^l\right\rangle = 0,$$

which yields

$$\sum_{n=-\infty}^{\infty} (2k+2)\widetilde{g}(2k+2)\left\langle f_n(r)e^{in\theta}z^k, z^l\right\rangle = 0.$$

For fixed n and any k, choose l such that $n+k = l$. Then we have

$$(2k+2)\widetilde{g}(2k+2)\langle f_n(r)e^{in\theta}z^k, z^l\rangle = 0.$$

The latter can be written as

$$(2k+2)\widetilde{g}(2k+2)\int_0^1\int_0^{2\pi} f_n(r)e^{in\theta}r^ke^{ik\theta}r^le^{-il\theta}\frac{rdrd\theta}{\pi} = 0,$$

which implies that

$$2(2k+2)\widetilde{g}(2k+2)\int_0^1 f_n(r)r^{k+l+1}drd\theta = 0.$$

Hence, we obtain

$$(2k+2)\widetilde{g}(2k+2)\widetilde{f_n}(k+l+2)=0, \text{ with } l=n+k.$$

If $n \geq 0$, the latter equality reads as

$$(2k+2)\widetilde{g}(2k+2)\widetilde{f_n}(n+2k+2)=0. \tag{4.6.1}$$

But, by definition of the Mellin transform, we see that

$$\begin{aligned} \widetilde{f_n}(n+2k+2) &= \int_0^1 f_n(r)r^{n+2k+2-1}dr \\ &= \int_0^1 f_n(r)r^n r^{2k+1}dr = \widetilde{f_n r^n}(2k+2). \end{aligned}$$

Combining the two latter identities, we obtain

$$(2k+2)\widetilde{g}(2k+2)\widetilde{f_n r^n}(2k+2)=0.$$

Arguing in the same manner as in the proof of Lemma 4.1.3, since the analytic function $z\widetilde{g}(z)\widetilde{f_n r^n}(z)$ vanishes on the non-Blaschke sequence $\{2k+2\}$ in $\mathbf{\Pi}$, we see that

$$z\widetilde{g}(z)\widetilde{f_n r^n}(z)=0 \text{ on } \mathbf{\Pi}.$$

Now, as g is not identically zero and the Mellin transform is one-to-one, we infer that $\widetilde{g}$ is not identically zero. This implies that $\widetilde{f_n r^n} \equiv 0$, whence $f_n r^n \equiv 0$. So, $f_n \equiv 0$ for $n \geq 0$. Now, if $n<0$, from Equation (4.6.1) and the fact that $k=l-n$ we see that

$$(2l-2n+2)\widetilde{g}(2l-2n+2)\widetilde{f_n}(2l+2-n)=0.$$

Since $2l-2n+2 \neq 0$, $l \geq 0$ and $n<0$, we obtain

$$\widetilde{g}(2l-2n+2)\widetilde{f_n}(2l+2-n)=0.$$

Hence, we arrive to

$$\left(\int_0^1 g(r)r^{-2n}r^{2l+1}dr\right)\left(\int_0^1 f_n(r)r^{-n}r^{2l+1}dr\right) = 0.$$

From which we conclude that

$$\widetilde{gr^{-2n}}(2l+2)\widetilde{f_n r^{-n}}(2l+2) = 0.$$

Again, since the analytic function $\widetilde{gr^{-2n}}\ \widetilde{f_n r^{-n}}$ vanishes on the non-Blaschke sequence $\{2l+2\}$ in Π, we see that

$$\widetilde{gr^{-2n}}(z)\widetilde{f_n r^{-n}}(z) = 0 \text{ on } \mathbf{\Pi}.$$

Now, since g is not identically zero, also gr^{-2n} is not identically zero. So by the injectivity of the Mellin transform, we see that $\widetilde{gr^{-2n}}$ is not identically zero. Hence we conclude that $\widetilde{f_n r^{-n}} = 0$, which implies that $f_n \equiv 0$ if $n < 0$. So far, we have proved that if g is radial and not identically zero and $T_f T_g = 0$ then $f \equiv 0$.
Now, if the other case occurs, namely if f is radial not identically zero and $T_f T_g = 0$ then we see that $(T_f T_g)^* = T_{\overline{g}} T_{\overline{f}} = 0$ and $\overline{f}$ is also radial and not identically zero. Arguing in the same manner as above, we see that the latter implies that $\overline{g} \equiv 0$, whence $g \equiv 0$, and the theorem is proved. ■

Chapter 5

Commutants of Bergman Space Toeplitz Operators

In this chapter we present one of the most important results about Toeplitz operators on the Bergman space. This result turns around the problem of commutativity of those operators. Its Hardy space analog, namely Theorem 1.2.18, was proved by Brown and Halmos [16]; while the Bergman space analog which will be given in Section 5.1, is due to Axler and Čučkovič [13]. It is remarkable that the relevant necessary and sufficient conditions are the same for all cases; unless that for the Bergman space case the harmonicity of symbols is imposed in addition. In Section 5.2, we describe the commutants of certain analytic Toeplitz operators. In the last section we study the commutants of analytic Toeplitz operators and we conclude with a theorem of Brown-Halmos type.

5.1 Commuting Toeplitz operators with harmonic symbols

In their nice paper, Axler and Čučkovič [13] have proved an invariant mean value property and have deduced the forthcoming theorem. At that time it was not known that the only invariant functions under the Berezin transform are the harmonic ones. As we have seen in Chapter 2, namely Theorem 2.1.5, this latter fact becomes a powerful tool in the theory of Bergman space Toeplitz operators. We use this deep result to provide a different proof of Axler-Čučkovič's theorem but first we need the following proposition which is just Proposition 3.1.6 with a slight modification.

Proposition 5.1.1. *Suppose that $f = f_1 + \overline{f_2}$, $g = g_1 + \overline{g_2}$ are bounded harmonic functions with holomorphic components f_i and g_i, $i = 1, 2$. Then the following are equivalent:*

1. $T_f T_g = T_g T_f$.
2. $f_1(z)\overline{g_2}(z) - g_1(z)\overline{f_2}(z) = B(f_1\overline{g_2} - g_1\overline{f_2})(z)$, *for all* $z \in D$.
3. *For all* $(z, w) \in D \times D$,

$$f_1(z)\overline{g_2}(\overline{w}) - g_1(z)\overline{f_2}(\overline{w}) = (1 - zw)^2 \int_D \frac{f_1(\zeta)\overline{g_2}(\zeta) - g_{1(\zeta)}\overline{f_2}(\zeta)}{(1 - z\overline{\zeta})^2(1 - w\zeta)^2} dA(\zeta).$$

Proof: (1)⇔(3): By Lemma 3.1.5, we have

$$T_f T_g = T_g T_f \text{ if and only if } T_f T_g K_w = T_g T_f K_w.$$

Thus we see that

$$T_f(P(g_1 K_w + \overline{g_2} K_w)) = T_g(P(f_1 K_w + \overline{f_2} K_w)),$$

that is

$$T_f(g_1 K_w + \overline{g_2}(w) K_w) = T_g(f_1 K_w + \overline{f_2}(w) K_w),$$

which is equivalent to

$$P\left(\left(f_1 + \overline{f_2}\right)\left(g_1 K_w + \overline{g_2}(w) K_w\right)\right) = P\left((g_1 + \overline{g_2})\left(f_1 K_w + \overline{f_2}(w) K_w\right)\right).$$

The latter can be rewritten as

$$f_1g_1k_w + \overline{g_2}(w)f_1k_w + P(\overline{f_2}g_1k_w) + \overline{g_2}(w)P(\overline{f_2}k_w)$$
$$= g_1f_1k_w + \overline{f_2}(w)g_1k_w + P(\overline{g_2}f_1k_w) + \overline{f_2}(w)P(\overline{g_2}k_w),$$

that is

$$f_1(z)\overline{g_2}(w) - g_1(z)\overline{f_2}(w) = \frac{1}{k_w(z)}\left[P(\overline{g_2}f_1k_w) - P(\overline{f_2}g_1k_w)\right](z), \text{ for all } z \in D.$$

Which is equivalent to

$$f_1(z)\overline{g_2}(w) - g_1(z)\overline{f_2}(w) = (1 - z\overline{w})^2 \int_D \frac{f_1(\zeta)\overline{g_2}(\zeta) - g_{1(\zeta)}\overline{f_2}(\zeta)}{(1 - z\overline{\zeta})^2(1 - \overline{w}\zeta)^2} dA(\zeta).$$

This is just the identity (3) above with w replaced by $\overline{w}$, whence we are done.

(3)⇒(2): Applying (3) to $(z, \overline{z}) \in D \times D$, we see that (3)⇒(2).

(2)⇒(3): To show that (2) implies (3), consider the holomorphic function defined in the bi-disk by the formula:

$$F(z, w) = f_1(z)\overline{g_2}(\overline{w}) - g_1(z)\overline{f_2}(\overline{w}) + (1 - zw)^2 \int_D \frac{f_1(\zeta)\overline{g_2}(\zeta) - g_{1(\zeta)}\overline{f_2}(\zeta)}{(1 - z\overline{\zeta})^2(1 - w\zeta)^2} dA(\zeta).$$

Assuming (2), F is identically zero on the set $\{(z, \overline{z}) : z \in D\}$ and Proposition 1.1.17 then implies that $F \equiv 0$ in $D \times D$. Hence $F(z, \overline{w}) = 0$; which is the statement (3). This completes the proof. ■

Now, we are in the position to provide a different proof to Axler-Čučkovič's theorem

Theorem 5.1.2. *Suppose that f and g are bounded harmonic functions on D. Then*

$$T_fT_g = T_gT_f,$$

if and only if one of the following conditions holds

1. *f and g are both holomorphic in D.*

2. $\overline{f}$ *and* $\overline{g}$ *are both holomorphic in* D.

3. *there exist constants* $a, b \in \mathbb{C}$, *not both* 0, *such that* $af + bg$ *is constant on* D.

Proof: If (1), (2) or (3) holds then it is easy to see that $T_f T_g = T_g T_f$. Now if $T_f T_g = T_g T_f$ then the equivalence of (1) and (2) in the previous proposition yields

$$f_1\overline{g_2} - g_1\overline{f_2} = B(f_1\overline{g_2} - g_1\overline{f_2}),$$

which means that $f_1\overline{g_2} - g_1\overline{f_2}$ is harmonic, (by Theorem 2.1.5 as the harmonic functions are the only functions which are invariant under the Berezin transform). Hence, we see that

$$\Delta(f_1\overline{g_2} - g_1\overline{f_2}) = 0,$$

i.e.

$$\Delta f_1\overline{g_2} = \Delta g_1\overline{f_2}.$$

In other words

$$\frac{\partial}{\partial z}\left(f_1\frac{\partial\overline{g_2}}{\partial\overline{z}}\right) = \frac{\partial}{\partial z}\left(g_1\frac{\partial\overline{f_2}}{\partial\overline{z}}\right).$$

This reduces to

$$f_1'\overline{g_2}' = g_1'\overline{f_2}' \text{ on } D. \tag{5.1.1}$$

We finish the proof by showing that Equation (5.1.1) implies that (1), (2) or (3) holds. If g_1' is identically 0 on D, then Equation (5.1.1) shows that either g_2' is identically 0 on D, which implies that g is constant on D and (3) would hold with $a = 0, b \neq 0$, or f_1' is identically 0 on D so both $\overline{f}$ and $\overline{g}$ would be holomorphic on D and (2) holds. Similarly, if g_2' is identically 0 on D, then Equation (5.1.1) shows that either (3) or (1) would hold. Thus we may assume that neither g_1' nor g_2' is identically 0 on D, and so Equation (5.1.1) shows that

$$\frac{f_1'}{g_1'} = \overline{\left(\frac{f_2'}{g_2'}\right)}, \tag{5.1.2}$$

at all points of D except the countable set consisting of the zeros of $g_1'g_2'$. The left hand side of Equation (5.1.2) is a holomorphic function, (on D with the zeros of $g_1'g_2'$ deleted), and the right hand side is the complex conjugate of a holomorphic function on the same domain. So, both sides must be equal to a constant $c \in \mathbb{C}$. Thus $f_1' = cg_1'$ and $f_2' = \bar{c}g_2'$ on D. Hence $f_1 - cg_1$ and $\overline{f_2} - c\overline{g_2}$ are constant on D, and so $f_1 - cg_1 + \overline{f_2} - c\overline{g_2} = f - cg$ is constant on D. i.e. (3) holds and the proof is complete. ■

5.2 Commutants of T_{z^n}

Writing $\mathbf{L}(L_a^2)$ for the algebra of all bounded linear operators on L_a^2. Let $\mathcal{T}$ denote the norm closed subalgebra of $\mathbf{L}(L_a^2)$ generated by Toeplitz operators. In this section we show that, for each positive integer n, the intersection of the commutants of T_{z^n} and $\mathcal{T}$ is the set of all analytic Toeplitz operators. In addition, we prove the analogous result for T_{u^n}, where u is a disk automorphism.

Definition 5.2.1. *For a Hilbert space $\mathcal{H}$, if $S \subset \mathbf{L}(\mathcal{H})$, then*

$$\{S\}' = \{B \in \mathbf{L}(\mathcal{H}) : AB = BA, \text{ for all } A \in S\},$$

is the commutant of S.

Denote by Π the natural projection from $\mathbf{L}(L_a^2)$ onto $\mathbf{L}(L_a^2)/_{\mathbf{K}}$, where $\mathbf{K}$ is the ideal of all compact operators. That is

$$\begin{aligned} \Pi : \mathbf{L}(L_a^2) &\longrightarrow \mathbf{L}(L_a^2)/_{\mathbf{K}}, \\ A &\longrightarrow A + \mathbf{K}. \end{aligned}$$

Which is linear and multiplicative. In the proof of the main result of this section, we need the following lemma, which is interesting in its own right.

Lemma 5.2.2. *If $S \in \mathcal{T}$, then $ST_z - T_zS \in \mathbf{K}$ where $\mathbf{K}$ denotes the ideal of all compact operators.*

Proof: First let us assume that $S = T_\varphi$, $\varphi \in L^\infty$. From Lemma 1.3.16 we have that

$$T_\varphi T_z - T_z T_\varphi = T_{z\varphi} - T_z T_\varphi = H^*_{\overline{z}} H_\varphi.$$

Since $z \in \mathbf{B}_0$, the operator $H^*_{\overline{z}}$ is compact by Lemma 1.3.15. So that, $T_\varphi T_z - T_z T_\varphi$ is compact. Now, since $\Pi : \mathbf{L}(L^2_a) \to \mathbf{L}(L^2_a)/_{\mathbf{K}}$ denotes the natural projection, then

$$\Pi(T_\varphi T_z - T_z T_\varphi) = \Pi(H^*_{\overline{z}} H_\varphi) = 0,$$

which implies that

$$\Pi(T_\varphi)\Pi(T_z) = \Pi(T_z)\Pi(T_\varphi), \text{ for every } \varphi \in L^\infty. \tag{5.2.1}$$

In fact, an arbitrary operator S in $\mathcal{T}$ is the limit of sums of operators of the form $T_{\varphi_1} \ldots T_{\varphi_l}$, i.e. $S = \lim S_l$ where each S_l is of the form $\sum T_{\varphi_1} \ldots T_{\varphi_l}$. So, we have

$$ST_z - T_z S = \lim(S_l T_z - T_z S_l). \tag{5.2.2}$$

But if $S = T_{\varphi_1} \ldots T_{\varphi_l}$, then Equation (5.2.1) leads to

$$\Pi(ST_z - T_z S) = 0.$$

Hence $ST_z - T_zS \in \mathbf{K}$, which means that the right hand side of Equation (5.2.2) belongs to $\mathbf{K}$. This completes the proof. ■

Now, we are able to introduce the main result of this Section, namely Theorem 5.2.3. It asserts that $\{T_{z^n}\}' \cap \mathcal{T}$ is the set of all analytic Toeplitz operators.

Theorem 5.2.3. *Suppose that $S \in \mathcal{T}$ commutes with T_{z^n}, $n \in \mathbb{N}$. Then $S = T_\psi$ for some $\psi \in H^\infty$.*

Proof: The equation $ST_{z^n} = T_{z^n}S$ gives us the following:

Let $g_i = Sz^i$, for $i = 0, 1, \ldots, n-1$. Then, for any such i, we have that

$$\begin{aligned} Sz^{n+i} &= SP(z^{n+i}) = ST_{z^n}z^i = T_{z^n}Sz^i \\ &= P(z^n g_i) = z^n g_i. \end{aligned}$$

Continuing in this manner, we obtain

$$Sz^{kn+i} = z^{kn}g_i, \text{ for } k = 0, 1, 2, \ldots$$

Let

$$\begin{aligned} X_0 &= \overline{span\{e_{kn},\ k = 0, 1, 2, \ldots\}}, \\ X_1 &= \overline{span\{e_{kn+1},\ k = 0, 1, 2, \ldots\}}, \\ &. \\ &. \\ &. \\ X_{n-1} &= \overline{span\{e_{kn+(n-1)},\ k = 0, 1, 2, \ldots\}}. \end{aligned}$$

Then, we see that the Bergman space can be written as

$$L_a^2 = X_0 \oplus X_1 \oplus \ldots \oplus X_{n-1}$$

i.e. each $f \in L_a^2$ can be written as

$$f = f_0 + f_1 + \ldots + f_{n-1},$$

where $f_i \in X_i$, $i = 0, 1, 2, \ldots, n-1$.

We know that each $f_0 \in X_0$ has Fourier series expansion given by

$$f_0 = \sum_{k=0}^{\infty} \langle f_0, e_{kn} \rangle e_{kn},$$

putting $\sigma_m = \sum_{k=0}^{m} \langle f_0, e_{kn}\rangle e_{kn}$, the latter can be seen as follows

$$f_0 = \lim_{m\to\infty} \sigma_m.$$

Since the point evaluation functional is bounded on L_a^2, (as we have seen in Subsection 1.3.1), we have

$$f_0(z) = \lim_{m\to\infty} \sigma_m(z), \text{ for each } z \in D.$$

So that, we get

$$(f_0 g_0)(z) = \lim_{m\to\infty} (\sigma_m g_0)(z), \text{ for each } z \in D. \tag{5.2.3}$$

Since $Sz^{kn} = z^{kn} g_0$, $k = 0, 1, 2, \ldots$, it follows that

$$S(\sigma_m) = \sigma_m g_0, \text{ for every } m \in \mathbb{N}.$$

By continuity of S, we have

$$(Sf_0) = S\left(\lim_{m\to\infty} \sigma_m\right) = \lim_{m\to\infty} S(\sigma_m) = \lim_{m\to\infty} (\sigma_m g_0).$$

So, we obtain

$$(Sf_0)(z) = \lim_{m\to\infty} (\sigma_m g_0)(z), \text{ for each } z \in D. \tag{5.2.4}$$

Comparing Equations (5.2.3) and (5.2.4) we conclude that

$$Sf_0 = g_0 f_0, \text{ for each } f_0 \in X_0.$$

Since $f_1 \in X_1$ has Fourier series expansion given by

$$f_1 = \sum_{k=0}^{\infty} \langle f_1, e_{kn+1}\rangle e_{kn+1} = \lim_{m\to\infty} \sigma_m,$$

where $\sigma_m = \sum_{k=0}^{m} \langle f_1, e_{kn+1}\rangle e_{kn+1}$, repeating the above reasoning we obtain

$$Sf_1 = \frac{g_1 f_1}{z}, \text{ for } f_1 \in X_1.$$

Since the point evaluation functional is bounded on L_a^2, (as we have seen in Subsection 1.3.1), we have

$$f_1(z) = \lim_{m\to\infty} \sigma_m(z), \text{ for each } z \in D.$$

So, we get

$$(f_1 g_1)(z) = \lim_{m\to\infty} (\sigma_m g_1)(z), \text{ for each } z \in D. \tag{5.2.5}$$

Since $Sz^{kn+1} = z^{kn} g_1$, $k = 0, 1, 2, \dots$, it follows that

$$S(\sigma_m) = \frac{\sigma_m g_1}{z}, \text{ for every } m \in \mathbb{N}.$$

By continuity of S, we have

$$(Sf_1) = S\left(\lim_{m\to\infty} \sigma_m\right) = \lim_{m\to\infty} S(\sigma_m) = \lim_{m\to\infty} \left(\frac{\sigma_m g_1}{z}\right).$$

So that

$$(Sf_1)(z) = \lim_{m\to\infty} \left(\frac{\sigma_m g_1}{z}\right)(z), \text{ for each } z \in D. \tag{5.2.6}$$

Comparing Equations (5.2.5) and (5.2.6), we conclude that

$$Sf_1 = \frac{1}{z}(g_1 f_1), \text{ for each } f_1 \in X_1;$$

and so on. Thus the operator S can be described as follows:

$$Sf = g_0 f_0 + \frac{g_1}{z} f_1 + \frac{g_2}{z^2} f_2 + \cdots + \frac{g_{n-1}}{z^{n-1}} f_{n-1}. \tag{5.2.7}$$

Now, express $ST_z - T_z S$ in terms of Equation (5.2.7). It is easy to see that

$$ST_z f = g_1 f_0 + \frac{g_2}{z} f_1 + \frac{g_3}{z^2} f_2 + \cdots + z g_0 f_{n-1}.$$

From this and Equation (5.2.7), we have that

$$(ST_z - T_z S)f = g_1 f_0 + \frac{g_2}{z} f_1 + \frac{g_3}{z^2} f_2 + \cdots + z g_0 f_{n-1} - z g_0 f_0 - g_1 f_1 - \frac{g_2}{z} f_2 - \cdots - \frac{g_{n-1}}{z^{n-2}} f_{n-1}.$$

It follows that

$$(ST_z - T_zS)f = f_0(g_1 - zg_0) + f_1\left(\frac{g_2}{z} - g_1\right) + f_2\left(\frac{g_3}{z^2} - \frac{g_2}{z}\right) + \cdots + f_{n-1}\left(zg_0 - \frac{g_{n-1}}{z^{n-2}}\right).$$

Now, Lemma 5.2.2 tells us that $(ST_z - T_zS)\mid_{X_0} = M_{g_1 - zg_0}$ is a compact map from X_0 into L^2_a. Putting $\varphi = g_1 - zg_0$, we obtain

$$M_\varphi \mid_{X_j} = M_{z^j}(M_\varphi \mid_{X_0})(M_{z^{-j}} \mid_{X_j}).$$

The latter hinges on the fact that the operators on both sides are defined on X_j and for each $f_j \in X_j$ we have that

$$\begin{aligned} M_{z^j}\left(M_\varphi \mid_{X_0}\right)\left(M_{z^{-j}} \mid_{X_j}\right) f_j &= M_{z^j}(M_\varphi \mid_{X_0})(z^{-j} f_j) \\ &= M_{z^j}((g_1 - zg_0)z^{-j} f_j) = (g_1 - zg_0) f_j \\ &= (M_\varphi \mid_{X_j}) f_j. \end{aligned}$$

It immediately, follows that $M_\varphi \mid_{X_j}$is compact for all $j - 1, \ldots, n-1$, whence M_φ is compact on $X_0 \oplus X_1 \oplus \ldots \oplus X_{n-1} = L^2_a$. By Theorem 1.3.14 we conclude that $\varphi = 0$. Therefore $g_0 = \frac{g_1}{z}$. Similarly, $(ST_z - T_zS)\mid_{X_1} = M_{\frac{g_2}{z} - g_1} : X_1 \to L^2_a$ is a compact operator. Thus $M_{\frac{g_2}{z} - g_1} M_z \mid_{X_0} = M_{g_2 - zg_1} : X_0 \to L^2_a$ is compact. Arguing in the same manner as above, we see that $g_2 - zg_1 = 0$. So $g_1 = \frac{g_2}{z}$, and therefore $g_0 = \frac{g_2}{z^2}$. If we continue in this manner, Equation(5.2.7) shows that

$$Sf = g_0 f_0 + g_0 f_1 + g_0 f_2 + \cdots + g_0 f_{n-1} = g_0 f = T_{g_0} f, \text{ for every } f \in L^2_a.$$

But as $g_0 f = Sf \in L^2_a$ for each $f \in L^2_a$, the function g_0 must be in H^∞. To conclude the proof, just take $\psi = g_0$. ■

Theorem 5.2.3 is also valid for the Hardy space. In fact, from its proof we notice the following

Remark 5.2.4.

1. *We can replace the assumption that* $S \in \mathcal{T}$ *by the weaker assumption*

$$ST_z - T_zS \in \mathbf{K}.$$

2. *Slightly modified arguments give a proof for the Hardy space case.*

The next result is the analog of Theorem 5.2.3 for T_{u^n} where u is a disk automorphism.

Corollary 5.2.5. *Let* u *be a disk automorphism and let* $S \in \mathcal{T}$ *commute with* T_{u^n}, *for some* $n \in \mathbb{N}$. *Then,* S *is an analytic Toeplitz operator.*

Proof: Let u be a disk automorphism, i.e. $u(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$ where $a \in D$. It is easy to see that $u^{-1}(z) = e^{-i\theta}\frac{z+a}{1+\overline{a}z}$. Define an operator $V : L_a^2 \to L_a^2$ by

$$Vf = f \circ u^{-1}.$$

It is clear that V is a bounded linear operator with inverse operator given by

$$V^{-1}f = f \circ u.$$

We are going to show that $T_zV = VT_u$:

From one hand, we have

$$T_zVf = T_z(f \circ u^{-1}) = P(z(f \circ u^{-1})) = z(f \circ u^{-1}).$$

From the other hand, we see that

$$VT_uf = V(P(uf)) = V(uf) = uf \circ u^{-1}, \text{ for all } f \in L_a^2.$$

But

$$z(f \circ u^{-1})(z) = z\left(f\left(e^{-i\theta}\frac{z+a}{1+\overline{a}z}\right)\right),$$

and

$$\begin{aligned}
((uf)\circ u^{-1})(z) &= uf\left(e^{-i\theta}\frac{z+a}{1+\overline{a}z}\right)\\
&= u\left(e^{-i\theta}\frac{z+a}{1+\overline{a}z}\right) f\left(e^{-i\theta}\frac{z+a}{1+\overline{a}z}\right)\\
&= \left(\frac{\frac{z+a}{1+\overline{a}z}-a}{1-\overline{a}\frac{z+a}{1+\overline{a}z}}\right) f\left(e^{-i\theta}\frac{z+a}{1+\overline{a}z}\right)\\
&= \left(\frac{z+a-a-|a|^2 z}{1+\overline{a}z-\overline{a}z-|a|^2}\right) f\left(e^{-i\theta}\frac{z+a}{1+\overline{a}z}\right)\\
&= \left(\frac{z(1-|a|^2)}{1-|a|^2}\right) f\left(e^{-i\theta}\frac{z+a}{1+\overline{a}z}\right)\\
&= zf\left(e^{-i\theta}\frac{z+a}{1+\overline{a}z}\right).
\end{aligned}$$

Hence, we obtain

$$T_z V = VT_u. \tag{5.2.8}$$

Therefore, by mathematical induction, we get

$$T_{z^n} V = VT_{u^n}, \text{ for every } n \in \mathbb{N}. \tag{5.2.9}$$

Indeed, for $n = 2$, compose both sides of Equation (5.2.8) on the left by T_z, we get $T_z(T_z V) = T_z(VT_u) = (T_z V)T_u = (VT_u)T_u$. Since both z and u are analytic functions, part(2) of Proposition 3.1.3 leads to the fact that

$$T_{z^2} V = T_z(T_z V) = (VT_u)T_u = VT_{u^2}.$$

Then we assume that $T_{z^k} V = VT_{u^k}$ for some integer $k > 2$. Now, we can show that $T_{z^{k+1}} V = VT_{u^{k+1}}$ by the same argument of the case $n = 2$.

Now, suppose that $S \in \mathcal{T}$ and $ST_{u^n} = T_{u^n}S$ for some $n \in \mathbb{N}$. Formula (5.2.9) implies that

$$SV^{-1}T_{z^n}V = SV^{-1}VT_{u^n} = ST_{u^n} = T_{u^n}S = V^{-1}T_{z^n}VS.$$

So, we obtain

$$VSV^{-1}T_{z^n}V = T_{z^n}VS.$$

Thus, we get

$$VSV^{-1}T_{z^n} = T_{z^n}VSV^{-1},$$

i.e. $VSV^{-1} \in \{T_{z^n}\}'$. Since u is analytic and

$$\begin{aligned}(1-|z|^2)u'(z) &= (1-|z|^2)e^{i\theta}\frac{(1-\overline{a}z+\overline{a}z-|a|^2)}{(1-\overline{a}z)^2} \\ &= (1-|z|^2)e^{i\theta}\frac{(1-|a|^2)}{(1-\overline{a}z)^2},\end{aligned}$$

which clearly has the limit 0 as $|z| \to 1$, we infer that $u \in \mathbf{B}_0$. whence, using the same argument as in the proof of Lemma 5.2.2, we have that $T_{\varphi_i}T_u - T_uT_{\varphi_i} \in \mathbf{K}$ for all $i = 1, \cdots n$. Thus we conclude that $ST_u - T_uS \in \mathbf{K}$.

Now, let us observe that the operator $B = VSV^{-1}$ has the property that $BT_z - T_zB$ is in $\mathbf{K}$. Indeed, we have

$$BT_z - T_zB = VSV^{-1}T_z - T_zVSV^{-1}.$$

So making use of Equation (5.2.8), we obtain

$$BT_z - T_zB = VSV^{-1}T_z - VT_uSV^{-1} = V(SV^{-1}T_z - T_uSV^{-1}).$$

Again from Equation (5.2.8), we have that $V^{-1}T_z = T_uV^{-1}$. Hence, we obtain

$$BT_z - T_zB = V(ST_uV^{-1} - T_uSV^{-1}) = V(ST_u - T_uS)V^{-1},$$

which belongs to **K.** Hence, by Remark 5.2.4, we should have $B = T_\varphi$ for some $\varphi \in H^\infty$. Thus $S = V^{-1}T_\varphi V$. If we let $\psi = \varphi \circ u$, then we obtain

$$\begin{aligned} Sf &= V^{-1}T_\varphi(f \circ u^{-1}) \\ &= (\varphi(f \circ u^{-1})) \circ u \\ &= (\varphi \circ u)f = T_\psi f, \text{ for every } f \in L_a^2. \end{aligned}$$

This means that $S = T_\psi$ for some $\psi \in H^\infty$, and the proof is complete. ■

5.3 Commutants of analytic Toeplitz operators

Recall that the general problem we are interested in during this chapter is: *If two Toeplitz operators commute, what is the relationship between their symbols?* If we were working on the Hardy space of ∂D, then Theorem 1.2.18 answers such question. On the Bergman space, the situation is more complicated and Theorem 1.2.18 fails. For example, any two Toeplitz operators with radial symbols commute. Indeed, Lemma 4.1.1 shows that every Toeplitz operator with radial symbol has a diagonal matrix with respect to the usual orthonormal basis; and it is well-known that two diagonal matrices commute.

Despite the difficulty of the general problem, we have some nice results in some particular cases, namely:

1. Axler-Čučkovič's theorem on the commutativity of Toeplitz operators with bounded harmonic symbols, namely Theorem 5.1.2.

2. Čučkovič's theorem on characterization of the commutants of T_{z^n}, namely Theorem 5.2.3.

3. Čučkovič-Rao's results [23], on the radial case, namely: if T_φ and T_ψ commute and φ is radial, then ψ is radial.

4. Čučkovič-Rao's results [23], on the monomial case, namely: if T_φ and T_ψ commute and $\varphi = z^m\overline{z}^n$, then $\psi(re^{i\theta}) = \sum_{j=-\infty}^{\infty}\psi_j(r)e^{ij\theta}$, where $\{\psi_j\}$ are some functions depending upon m, n.

In the sequel we show that if two Toeplitz operators on the Bergman space commute and one of them is analytic and non constant, then the other one is also analytic. The proof of this result hinges on the following approximation theorem due to C. J. Bishop [20]:

Theorem 5.3.1. *Let φ be a non constant bounded analytic function on a bounded open domain Ω. Then, the norm closed subalgebra of $L^\infty(\Omega, dA)$ generated by $\overline{\varphi}$ and the bounded analytic functions on Ω contains $C(\overline{\Omega})$.*

The following result is due to Axler, Čučkovič and Rao:

Theorem 5.3.2. *If φ is a non constant bounded analytic function on D and ψ is a bounded measurable function on D such that T_φ and T_ψ commute, then ψ is analytic.*

Proof: Suppose that φ is a nonconstant bounded analytic function on the open unit disk D and ψ is a bounded measurable function on D such that $T_\varphi T_\psi = T_\psi T_\varphi$. Write $\psi = f + u$ with $f \in L_a^2$ and $u \in L^2 \ominus L_a^2$. If n is a nonnegative integer, then

$$T_{\varphi^n}T_\psi(1) = T_{\varphi^n}P(f+u) = T_{\varphi^n}f = \varphi^n f, \tag{5.3.1}$$

and

$$T_\psi T_{\varphi^n}(1) = P(\psi\varphi^n) = P(f\varphi^n + u\varphi^n) = f\varphi^n + P(u\varphi^n). \tag{5.3.2}$$

But since we have supposed that $T_\varphi T_\psi = T_\psi T_\varphi$, we can show by induction that $T_{\varphi^n}T_\psi = T_\psi T_{\varphi^n}$ since φ is analytic. Thus Equations (5.3.1) and (5.3.2) imply that

$$T_\psi T_{\varphi^n}(1) - T_{\varphi^n}T_\psi(1) = 0,$$

whence, we get

$$f\varphi^n + P(u\varphi^n) - \varphi^n f = 0.$$

That is to say $P(u\varphi^n) = 0$. Hence if $h \in L^2_a$, we have

$$0 = \langle P(h), P(u\varphi^n)\rangle = \langle h, u\varphi^n\rangle = \int_D \overline{u} h \overline{\varphi^n} dA.$$

Since the latter holds for every bounded analytic function h on D and every nonnegative integer n, Theorem 5.3.1 implies that

$$\int_D \overline{u} w dA = 0, \text{ for every } w \in C(\overline{D}).$$

But $C(\overline{D})$ is dense in L^2; it follows that $\langle w, u\rangle = 0$ for all $w \in C(\overline{D})$. Hence $u = 0$. Thus $\psi = f$; which means that ψ is analytic and the proof is complete. ■

Remark 5.3.3. *Theorem 5.3.2 is still valid if we replace D by a bounded open domain* Ω.

We conclude this section by a theorem of Brown-Halmos type in the sense of Definition 1.2.19. It can be viewed as a corollary of Theorem 5.3.2. It says that if $T_fT_g = T_{fg}$ with either f or g is harmonic then $T_fT_g = T_{fg}$ holds only in the trivial way. Theorem 5.3.4 is still valid if we assume that $f \in L^\infty$ and $g \in L^\infty_h$.

Theorem 5.3.4. *If* $f = f_1 + \overline{f_2}$ *where* f_1 *and* f_2 *are in* H^∞*and* $g \in L^\infty$ *and* $T_fT_g = T_{fg}$*, then either* $\overline{f}$ *or* g *is in* H^∞.

Proof: It is clear that

$$T_fT_g = (T_{f_1+\overline{f_2}})T_g = T_{f_1}T_g + T_{\overline{f_2}}T_g,$$

and

$$T_{fg} = T_{f_1g+\overline{f_2}g} = T_{f_1g} + T_{\overline{f_2}g}.$$

Hence $T_f T_g = T_{fg}$ implies that

$$T_{f_1} T_g + T_{\overline{f_2}} T_g = T_{f_1 g} + T_{\overline{f_2} g}.$$

But since f_1 and f_2 are analytic, part (2) of Proposition 3.1.3 tells us that

$$T_{f_1 g} = T_g T_{f_1} \text{ and } T_{\overline{f_2} g} = T_{\overline{f_2}} T_g.$$

Thus we obtain

$$T_{f_1} T_g = T_g T_{f_1}.$$

Now, if f_1 is constant then $\overline{f}$ is holomorphic. Also, if f_1 is non-constant, then Theorem 5.3.2 implies that g is holomorphic, which completes the proof. ■

Concluding Remarks and Perspectives

Having spent a relatively short time in the framework of this master thesis, we realized how fertile this area is. Accordingly, hereafter we exhibit some perspectives in the form of remarks and questions for further research. The investigation of some of them is in progress.

Question 1: Regarding the weighted analog of the above results, an attempt has been done in the framework of this thesis, namely we tempt to establish a Brown-Halmos type theorem for Toeplitz operators on the weighted Bergman space with radial weight. Serious difficulties occur at once. A lot has been already done namely Theorem 2.3.3, Proposition 3.4.1, Lemma 3.4.2, Proposition 3.4.3 and we are still working on. Note that the weighted Bergman space Toeplitz operators were studied by several authors, we mention for example [51, 60]

Question 2: The zero product problem seems to be curiously resistant; it occurs for the case of two operators as a corollary to Brown-Halmos theorem [16] in the Hardy space setting. We learned that S. Axler has proved it for three operators by the same method of [16], but unfortunately we ignore the exact reference. Always, in the Hardy space case, some progress has been done by C. Gu in [34] for five operators. K. Y. Guo in [35] has generalized the results of the latter to the product of six operators. For arbitrary products, the conjecture is believed to be true but without any convincing proof till now. The Bergman space case is certainly terribly complicated. The case of two Toeplitz operators with harmonic symbols is a simple corollary of the celebrated Brown-Halmos type theorem of Ahern and Čučković [2]. While for general symbols, the matter is very delicate; even the case of two operators is still open.

Question 3: An immediate corollary of Brown-Halmos theorem in the Hardy space setting asserts that: If T_f is invertible, then T_f^{-1} is again a Toeplitz operator if and only if f is analytic or f is co-analytic. The proof uses the fact that T is a Toeplitz operator $\Leftrightarrow$ $\mathbf{S}^*T\mathbf{S} = T$, $\mathbf{S}$ is the unilateral shift. This fact for Bergman space Toeplitz operators is known to be false [27]. Then, a corresponding proof needs another approach. For instance, if one can prove that T_f is invertible $\Rightarrow$ f is invertible, the result follows immediately.

Question4: P. Halmos in [36] posed ten problems; the fifth one reads as: is every subnormal

Toeplitz operator either analytic or normal? The question in its high generality seems to be delicate, but some work have been done on a weaker version of it, namely subnormality is replaced by normality. The Hardy space case was discussed again in [37] and the references are therein; and completely solved by Amemiya, Ito and Wong in [7]. While the Bergman space case is still pending; nevertheless Faour in [30] has discussed this problem and made some progress. Can we go further?

Question 5: The second research direction that the celebrated paper by P. Ahern and Ž. Čučkovič [2] concerns the range of the Berezin transform. More precisely Example 2.4.3 solves the equation $Bu(z) = z\overline{z}$ and gives $u(\zeta) = 1 - \log \frac{1}{|\zeta|^2}$; whereas Lemma 2.4.4 characterizes the solution of the equation $Bu(z) = z\overline{z}^2$ and shows that u must be of the form $u(\zeta) = 2\overline{\zeta} - \frac{1}{\zeta}$. P. Ahern proves rather more, namely he characterizes all solutions of $Bu = f\overline{g}$ with f and g are holomorphic in D and neither is constant and $u \in L^1$. It should be interesting to seek for solutions of such a problem for other classes of symbols and in more general situations.

Question 6: Lemma 3.1.9 asserts that there is no rank one Toeplitz operator with bounded symbol except T_0. It is, then, fairly believed that there are no finite rank Toeplitz operators. Such conjecture seems to be hard and equally interesting.

Question 7: What about the kernel of a Bergman space Toeplitz operator T_f? The Hardy space case is more transparent. Indeed, Coburn's Lemma, see [24], states that a Hardy space Toeplitz operator is either injective or has dense range. This result is no longer true for Bergman space Toeplitz operators. On the other hand, different aspects related to kernels of Hardy space Toeplitz operators were considered by Dyakonov [26], Gu [34], Hayashi [39] and Sarason [56]. It will be interesting, however, to think about such research direction for the Bergman space Toeplitz operators in the light of the latter references.

Question 8: Fredholm theory related to Hardy space Toeplitz operators is by now well understood [18, 24]. The Bergman space case is less understood. A few related results can be found in [46]. It sounds interesting to investigate the Fredholm properties to more extent.

Question 9: Spectral properties of Hardy space Toeplitz operators is nowadays more transparent. Very deep results in this direction can be found in Douglas' book [24]. However the spectral study of Bergman space Toeplitz operators has not attracted the attention it deserves. Again in [47] and [41] some light has been shed on such direction. Though, this situation is still less understood. It might be interesting to investigate more spectral properties of Bergman space Toeplitz operators.

Question 10: Theorem 5.2.3 asserts that: Let $S \in \mathbf{L}(L_a^2)$ with the property that $ST_z - T_zS \in \mathbf{K}$. Then if $ST_{z^n} - T_{z^n}S = 0$ for some positive integer n, S must be an analytic Toeplitz operator. Accordingly, Ž. Čučkovič [21] asks what is the set of all functions f such that $ST_f - T_f S = 0$ implies that S is an analytic Toeplitz operator? So his question is reported here again.

Question 11: S. Axler, Ž. Čučkovič and N. V. Rao in [12] posed three problems. For the sake of completeness, we report them here again

1. If an operator S is in the algebra generated by the Toeplitz operators commutes with a nonconstant analytic Toeplitz operator,then is S itself Toeplitz and hence (by Theorem 5.3.2) analytic?

2. Suppose φ is a bounded harmonic function on the unit disk D that is neither analytic nor co-analytic. If ψ is a bounded measurable function on the disk such that T_φ and T_ψ commute, must ψ be of the form $a\varphi + b$ for some constants a, b? This question would have a negative answer if the disk were replaced by an annulus centered at the origin because $T_{\log|z|}$ commutes with every Toeplitz operator with radial symbol.

3. What is the situation on Bergman spaces in higher dimensions?

Question 12: It is very well-known that Hardy space Toeplitz operators are characterized by the interwining relation $S^*TS = T$ as Theorem 1.2.15 asserts. However, Bergman space Toeplitz operators fail to satisfy such kind of characterizations as asserted by Engliš [27]. In this context, a related problem is worth stressing, namely: R. Martínez-Avendãno in [43, 44, 45] has generalized the concept of Hardy space Hankel operators through the interwining relations they satisfy. N. Faour in [31] established an interwining relation for Bergman space little Hankel operators. Accordingly, it might be interesting to think about further generalizations of Bergman space Hankel operators.

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