Answer the following questions:

1 1. (1 Point) Find the real and imaginary parts, $u(x, y)$ and $v(x, y)$, of the following function:

$$f(z) = \frac{2z - i}{iz + 2}$$

$$f(z) = \frac{2(x+iy) - i}{i(x+iy) + 2} = \frac{2x + 2iy - i}{ix - y + 2}$$

$$= \frac{2x + i(2y-1)}{(2-y) + xi} \cdot \frac{(2-y) - xi}{(2-y) - xi} = \frac{[2x + i(2y-1)][(2-y) - xi]}{(2-y)^2 + x^2} = \frac{2x(2-y) - 2x^2 i + (2-y)(2y-1)i + x(2y-1)}{(2-y)^2 + x^2}$$

$$= \frac{2x(2-y) + x(2y-1) + i[(2-y)(2y-1) - (2x^2)]}{(2-y)^2 + x^2} = \frac{2x(2-y) + x(2y-1)}{(2-y)^2 + x^2} + \frac{(2-y)(2y-1) - (2x^2)}{(2-y)^2 + x^2} i$$

$$= \frac{4x - 2xy + 2xy - x}{(2-y)^2 + x^2} + \frac{4y - 2 - 2y^2 + y - 2x^2}{(2-y)^2 + x^2} i$$

$$= \underbrace{\frac{3x}{(2-y)^2 + x^2}}_{u(x,y)} + \underbrace{\frac{5y - 2y^2 - 2x^2 - 2}{(2-y)^2 + x^2}}_{v(x,y)} i$$

0.5 0.5

1 2. (1 Point) Find the real and imaginary parts, $u(x, y)$ and $v(x, y)$, of the following function:

$$e^{iz}$$

$$f(z) = e^{iz} = e^{i(x+iy)} = e^{ix - y} = e^{ix} \cdot e^{-y}$$

$$= e^{-y}(\cos x + i \sin x) = \underbrace{\frac{\cos x}{e^y}}_{u(x,y)} + \underbrace{\frac{\sin x}{e^y}}_{v(x,y)} i$$

0.5 0.5

3. (2 Points) Given the Cauchy-Riemann conditions: $u_x = v_y$ and $u_y = -v_x$, determine whether the previous functions are analytic or not.

1 : 0 0

$u(x,y) = \dfrac{3x}{(2-y)^2+x^2}$

$v(x,y) = \dfrac{5y-2y^2-2x^2-2}{(2-y)^2+x^2}$

$u_x = \dfrac{3[(2-y)^2+x^2] - 3x(2x)}{((2-y)^2+x^2)^2}$

$= \dfrac{3(2-y)^2 - 3x^2}{((2-y)^2+x^2)^2}$ 0.5

$v_y = \dfrac{(5-4y)((2-y)^2+x^2) + (5y-2y^2-2x^2-2)(2(2-y))}{((2-y)^2+x^2)^2} = \dfrac{-3x^2+3(2-y)}{(x^2+(2-y)^2)^2}$ 0.5

$= \dfrac{5(2-y^2)+5x^2-4y(2-y)^2-4x^2y+10y(2-y)-4y^2(2-y)-4x^2(2-y)-4(2-y)}{((2-y)^2+x^2)^2}$

$= \dfrac{20-\cancel{20y}+5y^2+5x^2-16y+16y^2-\cancel{4y^3}-4x^2y+\cancel{20y}-10y^2-8y^2+\cancel{4y^3}-8x^2+\cancel{4x^2y}-8+4y}{((2-y)^2+x^2)^2}$

$u_y = -\dfrac{3x(2(2-y)\cdot -1)}{((2-y)^2+x^2)^2}$

$= \dfrac{6x(2-y)}{((2-y)^2+x^2)^2}$ 0.5

$v_x = \dfrac{((2-y)^2+x^2)(-4x) - (5y-2y^2-2-2x^2)(2x)}{((2-y)^2+x^2)^2}$

$= \dfrac{(4-4y+y^2+x^2)(-4x) - 2x(5y-2y^2-2x^2-2)}{((2-y)^2+x^2)^2}$

$= \dfrac{-16x+16xy-4xy^2-4x^3-10yx+4xy^2+4x+4x^3}{((2-y)^2+x^2)^2}$

$= \dfrac{-12x+6xy}{((2-y)^2+x^2)^2} = -\dfrac{6x(2-y)}{((2-y)^2+x^2)^2}$ 0.5

$\therefore f(z)$ (1) is analytic

1

4. (1 Point) Given that $z_1 = 5 - i$ and $z_2 = 4i$, find

0.25 • $z_1 z_2 = (5-i)(4i)$

$= 20i + 4 = 4 + 20i$

0.25 • $z_1 - \overline{z_2} = (5-i) - (-4i)$

$= 5 - i + 4i = 5 + 3i$

0.5 • $\overline{z_1}/\overline{z_2} = \dfrac{5+i}{-4i} \cdot \dfrac{4i}{4i} = \dfrac{20i + 4(-1)}{16} = \dfrac{-1}{4} + \dfrac{5}{4}i$

② :

$u(x,y) = \dfrac{\cos x}{e^y}$	$v(x,y) = \dfrac{\sin x}{e^y}$
$u_x = \dfrac{-\sin x}{e^y}$ 0.5	$v_y(x,y) = \dfrac{-e^y \sin x}{e^{2y}} = \dfrac{-\sin x}{e^y}$ 0.5
$u_y = \dfrac{-\cos x}{e^y}$ 0.5	$v_x(x,y) = \dfrac{\cos x}{e^y}$ 0.5

$\therefore$ f(z) ② is analytic