

# Mathematical Proofs: A Transition to Advanced Mathematics Logic

Mathematical  
Proofs: A  
Transition to  
Advanced  
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Logic

# Example (2.7)

Consider the open sentences

$$P_1(x) : x = -3 \text{ and } P_2(x) : |x| = 3.$$

where  $x \in \mathbb{R}$ , that is, where the domain of  $x$  is  $\mathbb{R}$  in each case.

We can then form the following open sentences:

$$\sim P_1(x) : x \neq -3.$$

$$P_1(x) \vee P_2(x) : x = -3 \text{ or } |x| = 3.$$

$$P_1(x) \wedge P_2(x) : x = -3 \text{ and } |x| = 3.$$

$$P_1(x) \Rightarrow P_2(x) : \text{If } x = -3, \text{ then } |x| = 3.$$

For a specific real number  $x$ , the truth value of each resulting statement can be determined. For example,  $\sim P_1(-3) : -3 \neq -3$  is a false statement; while each of the remaining sentences above results in a true statement when  $x = -3$ . Both  $P_1(2) \vee P_2(2)$  and  $P_1(2) \wedge P_2(2)$  are false statements. On the other hand, both  $\sim P_1(2)$  and  $P_1(2) \Rightarrow P_2(2)$  are true statements. Infact, for each real number  $x \neq -3$ , the implication  $P_1(x) \Rightarrow P_2(x)$  is a true statement since  $P_1(x) : x = -3$  is a false statement. Thus,  $P_1(x) \Rightarrow P_2(x)$  is true for all  $x \in \mathbb{R}$ . We will see that open sentences that result in true statements for all values of the domain will be especially interesting to us.

Listed below are various ways of wording the implication

$P_1(x) \Rightarrow P_2(x)$ :

If  $x = -3$ , then  $|x| = 3$ .

$|x| = 3$  if  $x = -3$ .

$x = -3$  implies that  $|x| = 3$ .

$x = -3$  only if  $|x| = 3$ .

$x = -3$  is sufficient for  $|x| = 3$ .

$|x| = 3$  is necessary for  $x = -3$ .

# Example ( 2.9)

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Let  $S = \{2, 3, 5\}$  and let

$P(n) : n^2 - n + 1$  is prime      and       $Q(n) : n^3 - n + 1$  is prime.

be open sentences over the domain  $S$ . Determine the truth or false ness of the implication  $P(n) \Rightarrow Q(n)$  for each  $n \in S$ .

# Solution

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In this case, we have the following:

$P(2)$  : 3 is prime.  $P(3)$  : 7 is prime.  $P(5)$  : 21 is prime.;

$Q(2)$  : 7 is prime.  $Q(3)$  : 25 is prime.  $Q(5)$  : 121 is prime.

Consequently,  $P(2) \Rightarrow Q(2)$  and  $P(5) \Rightarrow Q(5)$  are true, while  $P(3) \Rightarrow Q(3)$  is false.

## Example (2.10)

Let  $S = \{1, 2\}$  and let  $T = \{-1, 4\}$ . Also, let

$$P(x, y) : | |x + y| - |x - y| | = 2 \quad \text{and} \quad Q(x, y) : xy + 1 = yx$$

be open sentences, where the domain of the variable  $x$  is  $S$  and the domain of  $y$  is  $T$ . Determine the truth or falseness of the implication  $P(x, y) \Rightarrow Q(x, y)$  for all  $(x, y) \in S \times T$ .

# Solution

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For  $(x, y) = (1, -1)$ , we have

$P(1, -1) \Rightarrow Q(1, -1)$  : If  $2 = 2$ , then  $1 = -1$ , which is false.

For  $(x, y) = (1, 4)$ , we have  $P(1, 4) \Rightarrow Q(1, 4)$ : If  $2 = 2$ , then  $1 = 4$ , which is also false.

For  $(x, y) = (2, -1)$ , we have  $P(2, -1) \Rightarrow Q(2, -1)$  : If  $2 = 2$ , then  $1 = 1$ . which is true;

while for  $(x, y) = (2, 4)$ , we have  $P(2, 4) \Rightarrow Q(2, 4)$ : If  $2 = 4$ , then  $32 = 16$ . which is true.

## Example ( 2.13)

We noted in Example 2.7 that for the open sentences  $P_1(x) : x = -3$ . and  $P_2(x) : |x| = 3$ . over the domain  $\mathbb{R}$ , the implication

$$P_1(x) \Rightarrow P_2(x) : \quad \text{If } x = -3, \text{ then } |x| = 3$$

is a true statement for each  $x \in \mathbb{R}$ . However, the converse  $P_2(x) \Rightarrow P_1(x)$ : If  $|x| = 3$ , then  $x = -3$  is a false statement when  $x = 3$  since  $P_2(3)$  is true and  $P_1(3)$  is false. For all other real numbers  $x$ , the implication  $P_2(x) \Rightarrow P_1(x)$  is true. Therefore, the biconditional  $P_1(x) \iff P_2(x) : x = -3$  if and only if  $|x| = 3$ . is false when  $x = 3$  and is true for all other real numbers  $x$

# Example (2.15)

Let  $S = \{0, 1, 4\}$ . Consider the following open sentences over the domain  $S$ :

$$P(n) : \frac{n(n+1)(2n+1)}{6} \text{ is odd} \quad Q(n) : (n+1)^3 = n^3 + 1.$$

Determine three distinct elements  $a, b, c$  in  $S$  such that  $P(a) \Rightarrow Q(a)$  is false,  $Q(b) \Rightarrow P(b)$  is false and  $P(c) \iff Q(c)$  is true.

# Solution

Observe that

$$P(0) : 0 \text{ is odd. } P(1) : 1 \text{ is odd. } P(4) : 30 \text{ is odd.}$$

$$Q(0) : 1 = 1. \quad Q(1) : 8 = 2. \quad Q(4) : 125 = 65.$$

Thus,  $P(0)$  and  $P(4)$  are false, while  $P(1)$  is true. Also,  $Q(1)$  and  $Q(4)$  are false, while  $Q(0)$  is true. Thus,  $P(1) \Rightarrow Q(1)$  and  $Q(0) \Rightarrow P(0)$  are false, while  $P(4) \iff Q(4)$  is true. Hence, we may take  $a = 1$ ,  $b = 0$  and  $c = 4$ .

## Example (2.18)

For statements  $P$  and  $Q$ , determine whether the compound statement  $(P \wedge Q) \Rightarrow (P \vee Q)$  is a tautology, a contradiction or neither.

**Solution** The truth table in Figure 2.14 shows that the compound statement  $(P \wedge Q) \Rightarrow (P \vee Q)$  is a tautology

| $P$ | $Q$ | $P \wedge Q$ | $P \vee Q$ | $(P \wedge Q) \Rightarrow (P \vee Q)$ |
|-----|-----|--------------|------------|---------------------------------------|
| T   | T   | T            | T          | T                                     |
| T   | F   | F            | T          | T                                     |
| F   | T   | F            | T          | T                                     |
| F   | F   | F            | F          | T                                     |

figure 1: Figure 2.14 Verifying that  $(P \wedge Q) \Rightarrow (P \vee Q)$  is a tautology

## Example (2.19)

For statements  $P$  and  $Q$ , determine whether the compound statement  $(P \wedge (\sim Q)) \iff (P \Rightarrow Q)$  is a tautology, a contradiction or neither.

### **Solution**

The truth table in Figure 2.15 shows that the compound statement  $(P \wedge (\sim Q)) \iff (P \Rightarrow Q)$  is a contradiction.

| $P$ | $Q$ | $\sim Q$ | $P \wedge (\sim Q)$ | $P \Rightarrow Q$ | $(P \wedge (\sim Q)) \iff (P \Rightarrow Q)$ |
|-----|-----|----------|---------------------|-------------------|----------------------------------------------|
| T   | T   | F        | F                   | T                 | F                                            |
| T   | F   | T        | T                   | F                 | F                                            |
| F   | T   | F        | F                   | T                 | F                                            |
| F   | F   | T        | F                   | T                 | F                                            |

figure 2: Figure 2.15 Verifying that  $(P \wedge (\sim Q)) \iff (P \Rightarrow Q)$  is a contradiction