King Saud University, Mathematics Department
Math 204. Time: 3H, Full Marks: 40, 23/08/2017
Final Exam

Question 1. [4,4] **a)** Solve the initial value problem

$$\begin{cases} xy' + (3x+1)y = e^{-3x}, & x > 0 \\ y(1) = 1 \end{cases}$$

b) Find the general solution of the differential equation

$$(4y + yx^2)dy - (2x + xy^2)dx = 0$$

Question 2. [4,5] **a)** Solve the differential equation

$$(\tan x - \sin x \sin y)dx + (\cos x \cos y)dy = 0, \quad 0 < x < \pi/2.$$

b) Find and sketch the largest region of the xy-plane for which the following IVP has a unique solution

$$\sqrt{x^2 - 4}.\frac{dy}{dx} = 1 + e^x \ln y, \quad y(-3) = 4.$$

Question 3. [4,4] **a)** Solve the differential equation

$$2y'' + y' - y = 4 - 2e^{-x}.$$

b) Find the general solution of the differential equation

$$y'' - \frac{3}{x}y' - \frac{5}{x^2}y = x^4, \quad x > 0.$$

Question 4. [3] Determine whether the functions: $f_1(x) = \ln(4 - x^2)$, $f_2(x) = \ln(4 + 4x + x^2)$, $f_3(x) = \ln(4 - 4x + x^2)$ are linearly dependent or linearly independent on the interval $(-1, 1)$.

Question 5. [6,6] **a)** Obtain the Fourier series expansion of the function

$$f(x) = \begin{cases} -\sin x, & -\frac{\pi}{2} < x < 0 \\ \sin x, & 0 < x < \frac{\pi}{2} \end{cases}$$

of period π and deduce that $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$.

b) Find the Fourier integral of the function

$$g(x) = \begin{cases} 0, & x < -\frac{\pi}{2} \\ \cos x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0, & x > \frac{\pi}{2} \end{cases}$$

and deduce that $\int_0^{\infty} \frac{1}{1-\alpha^2} \cos \frac{\pi\alpha}{2} d\alpha = \frac{\pi}{2}$.

• Model answer of Math 204 Final Exam SS | 1437-38

8

Q1 a) Solve the initial value problem

$$\begin{cases} x\, y' + (3x+1)y = e^{-3x} & , x > 0 \\ y(1) = 1 \end{cases}$$

Ans: $\frac{dy}{dx} + \left(3 + \frac{1}{x}\right)y = \frac{1}{x} e^{-3x}$

where $P(x) = 3 + \frac{1}{x}$, $Q(x) = \frac{1}{x} e^{-3x}$

, the Integ. factor $\mu(x) = e^{\int P(x)dx}$

$= e^{\int (3+\frac{1}{x})dx}$

4

$= e^{3x + \ln x}$

$\therefore \mu(x) = x e^{3x}$

$\Rightarrow$ the general soln is given by

$\mu y = \int \mu(x)\, Q(x)\, dx$

$x e^{3x} y = \int x e^{3x} \cdot \frac{1}{x} e^{-3x} dx$

$x e^{3x} y = \int dx$

$x e^{3x} y = x + c$

$\Rightarrow \boxed{y = e^{-3x} + \frac{c}{x} e^{-3x}}$, $x > 0$

$\because y(1) = e^{-3} + c e^{-3} = 1$

$\Rightarrow c = \frac{1 - e^{-3}}{e^{-3}} = e^{3} - 1$

$\therefore \boxed{y = e^{-3x} + \left(\frac{e^{3}-1}{x}\right) e^{-3x}}$ #

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b) Solve the given DE by separation of variables

$(4y + yx^2)\,dy - (2x + xy^2)\,dx = 0$

<u>Ans:</u> $\dfrac{dy}{dx} = \dfrac{2x + xy^2}{4y + yx^2}$

4

$\dfrac{dy}{dx} = \dfrac{x(2+y^2)}{y(4+x^2)}$ ---- $\times \dfrac{y}{2+y^2}$

$\Rightarrow \displaystyle\int \frac{y\,dy}{2+y^2} = \int \frac{x}{4+x^2}\,dx$

$\Rightarrow \frac{1}{2}\ln(2+y^2) = \frac{1}{2}\ln(4+x^2) + \ln c_1$ ---- $\times 2$

$\ln(2+y^2) = \ln(4+x^2) + \ln C$

$\Rightarrow \quad 2+y^2 = C(4+x^2)$

Q2 a) Solve the differential equation

$(\tan x - \sin x \sin y)\, dx + \cos x \cos y \, dy = 0$

9

4

Ans: $\frac{\partial M}{\partial y} = -\sin x \cos y = \frac{\partial N}{\partial x}$

$\Rightarrow$ Exact DE

$\frac{\partial f}{\partial x} = \tan x - \sin x \sin y \qquad (1)$

$\frac{\partial f}{\partial y} = \cos x \cos y \qquad (2)$

$(2) \Rightarrow f(x,y) = \int \cos x \cos y \, dy + h(x)$

$f(x,y) = \cos x \sin y + h(x) \qquad (*)$

$\Rightarrow \frac{\partial f}{\partial x} = -\sin x \sin y + h'(x) \qquad (3)$

$(1), (3) \Rightarrow h'(x) = \tan x$

$\Rightarrow h(x) = \int \tan x \, dx$

$h(x) = \ln|\sec x|$

$(*) \Rightarrow f(x,y) = \cos x \sin y + \ln|\sec x| = c$

or

$f(x,y) = \cos x \sin y - \ln|\cos x| = c$

b) Find and sketch the largest region of the xy-plane for which the initial value problem

$$\begin{cases} \sqrt{x^2-4}\,\dfrac{dy}{dx} = 1 + e^x \ln y \\ y(-3) = 4 \end{cases}$$

5

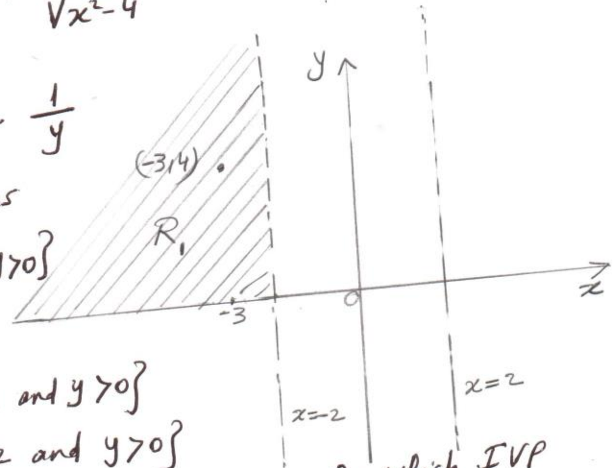
has a unique solution

Ans:

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-4}} + \frac{e^x}{\sqrt{x^2-4}} \cdot \ln y \;, \quad y > 0 \text{ and } |x| > 2$$

$$= f(x,y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{e^x}{\sqrt{x^2-4}} \cdot \frac{1}{y}$$



f and $\frac{\partial f}{\partial y}$ are continuous on $R = \{(x,y) : |x| > 2 \text{ and } y > 0\}$

$= R_1 \cup R_2$

where $R_1 = \{(x,y) : x < -2 \text{ and } y > 0\}$

and $R_2 = \{(x,y) : x > 2 \text{ and } y > 0\}$

$\because (-3,4) \in R_1$ $\therefore$ R_1 is the largest region for which IVP has a unique solution.

Q3 a) Solve the differential eqn

$$2y'' + y' - y = 4 - 2e^{-x}$$

8

Ans: For $2y''+y'-y=0$

$\Rightarrow 2m^2+m-1=0$

$\Rightarrow (2m-1)(m+1)=0$

$\Rightarrow m=\frac{1}{2},\ m=-1$

4

$\therefore \boxed{y_c = c_1 e^{x/2} + c_2 e^{-x}}$

the DE is $2y''+y'-y = 4-2e^{-x}$ (1)

For $4 \Rightarrow y_{1p} = A$

, $2e^{-x} \Rightarrow y_{2p} = Bxe^{-x}$

$\therefore \boxed{y_p = A + Bxe^{-x}}$

$\Rightarrow y_p' = -Bxe^{-x} + Be^{-x}$, $y_p'' = Bxe^{-x} - 2Be^{-x}$ (2)

Subs. (2) in (1)

$\Rightarrow \cancel{2Bxe^{-x}} - 4Be^{-x} - \cancel{Bxe^{-x}} + Be^{-x} - A - \cancel{Bxe^{-x}} = 4-2e^{-x}$

$\Rightarrow -A - 3Be^{-x} = 4 - 2e^{-x}$

$\Rightarrow A=-4,\ B=\frac{2}{3}$

$\therefore \boxed{y_p = -4 + \frac{2}{3}xe^{-x}}$

$\therefore$ The general soln of the give DE is

$$\boxed{y = c_1e^{x/2} + c_2e^{-x} - 4 + \frac{2}{3}xe^{-x}}$$ #

b) Find the general soln of the DE

$$y'' - \frac{3}{x}y' - \frac{5}{x^2}y = x^4, \quad x > 0$$

4

Ans:

$$y'' - \frac{3}{x}y' - \frac{5}{x^2}y = x^4$$ non-homo. DE , $f(x) = x^4$

Multiply by x^2 $\Rightarrow$ $\boxed{x^2y'' - 3xy' - 5y = x^6}$

let $y = x^m \Rightarrow y' = m x^{m-1}$, $y'' = m(m-1)x^{m-2}$

For $x^2y'' - 3xy' - 5y = 0$

$\Rightarrow m(m-1) - 3m - 5 = 0$

$\Rightarrow m^2 - 4m - 5 = 0$ ch. eqn $\Rightarrow m_1 = 5, \; m_2 = -1$

$\therefore \boxed{y_c = c_1 x^5 + c_2 x^{-1}}$ $\to y_1 = x^5$, $y_2 = x^{-1}$, $f(x) = x^4$

$$W(y_1, y_2) = \begin{vmatrix} x^5 & x^{-1} \\ 5x^4 & -x^{-2} \end{vmatrix} = -6x^3$$

$\Rightarrow y_p = u_1 x^5 + u_2 x^{-1}$ (*)

To get y_p Solve the system of eqns (by using Cramer's rule)

$$\begin{cases} x^5 u_1' + x^{-1}u_2' = 0 \\ 5x^4 u_1' - x^{-2}u_2' = x^4 \end{cases}$$

$$W_1 = \begin{vmatrix} 0 & x^{-1} \\ x^4 & -x^{-2} \end{vmatrix} = -x^3, \quad W_2 = \begin{vmatrix} x^5 & 0 \\ 5x^4 & x^4 \end{vmatrix} = x^9$$

$$u_1' = \frac{W_1}{W} = \frac{-x^3}{-6x^3} = \frac{1}{6} \Rightarrow u_1 = \int \frac{1}{6}\,dx = \boxed{\frac{1}{6}x}$$

$$, \; u_2' = \frac{W_2}{W} = \frac{x^9}{-6x^3} = -\frac{1}{6}x^6 \Rightarrow u_2 = \int -\frac{1}{6}x^6\,dx = \boxed{\frac{-1}{42}x^7}$$

(*) $\Rightarrow y_p = \frac{1}{6}x(x^5) - \frac{1}{42}x^7(x^{-1}) = \frac{1}{7}x^6$

$\therefore$ the general soln of the give DE is $y = y_c + y_p$

$$\Rightarrow \boxed{y = c_1 x^5 + c_2 x^{-1} + \frac{1}{7}x^6}$$

Q4 Determine whether the functions:

$f_1(x) = \ln(4-x^2)$, $f_2(x) = \ln(4+4x+x^2)$, $f_3(x) = \ln(4-4x+x^2)$ are linearly dependent or linearly independent on the interval $(-1,1)$

3

Ans: $\because$ we have, $-2f_1 + f_2 + f_3 = 0$

$\therefore f_1, f_2$ and f_3 are linearly dependent.

$\ln(4+4x+x^2) = \ln(2+x)^2$, $\ln(4-4x+x^2) = \ln(2-x)^2$

$\Rightarrow \ln(4+4x+x^2) + \ln(4-4x+x^2) = 2\ln(4-x^2)$

Q5 a) obtain the Fourier Series expansion of the function

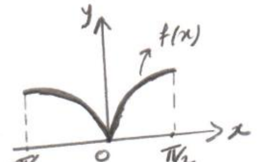
$$f(x) = \begin{cases} -\sin x, & -\frac{\pi}{2} < x < 0 \\ \sin x, & 0 < x < \frac{\pi}{2} \end{cases}$$

12

of period π and deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$

Ans: The given fn is an even function and its Fourier Cosine Series expansion is $\quad , p = \pi/2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2nx$$



$$a_0 = \frac{2}{\pi/2} \int_0^{\pi/2} f(x)\,dx$$

$$a_0 = \frac{4}{\pi} \int_0^{\pi/2} \sin x\,dx = \frac{4}{\pi} \left[-\cos x\right]_0^{\pi/2} = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi/2} \int f(x) \cos 2nx\,dx = \frac{4}{\pi} \int_0^{\pi/2} \sin x \cos 2nx\,dx$$

6

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} \left[\sin(2n+1)x - \sin(2n-1)x\right] dx$$

$$a_n = -\frac{4}{\pi}\left(\frac{1}{4n^2-1}\right)$$

This gives

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{4n^2-1}\right)\cos 2nx$$

- Taking $x=0$, We obtain $\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$ #

b) Find the Fourier integral of the function

$$g(x) = \begin{cases} 0 & x < -\pi/2 \\ \cos x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & x > \pi/2 \end{cases}$$

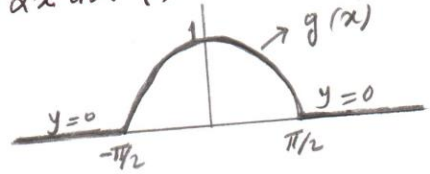
and deduce that $\int_0^{\infty}\frac{1}{1-\alpha^2}\cos\frac{\pi\alpha}{2}\,d\alpha = \frac{\pi}{2}$

6

Ans: The given fn is an even function

$\Rightarrow$ It has Fourier cosine integral representation

$$f(x) = \frac{2}{\pi}\int_0^{\infty} A(\alpha)\cos\alpha x\,d\alpha \quad (1)$$



where

$$A(\alpha) = \int_0^{\infty} f(x)\cos\alpha x\,dx$$

$$A(\alpha) = \int_0^{\pi/2}\cos x\cos\alpha x\,dx$$

$$A(\alpha) = \frac{1}{2}\int_0^{\pi/2}\left[\cos(\alpha+1)x + \cos(\alpha-1)x\right]dx$$

$$A(\alpha) = \frac{1}{2}\left[\frac{1}{\alpha+1}\sin(\alpha+1)x + \frac{1}{\alpha-1}\sin(\alpha-1)x\right]_0^{\pi/2}$$

$$A(\alpha) = \frac{1}{2}\left[\frac{1}{\alpha+1}\sin(\alpha+1)\frac{\pi}{2} + \frac{1}{\alpha-1}\sin(\alpha-1)\frac{\pi}{2}\right]$$

$$A(\alpha) = \frac{1}{2(\alpha^2-1)}\left[(\alpha-1)\sin(\alpha+1)\frac{\pi}{2} + (\alpha+1)\,\Sigma\,(\alpha-1)\frac{\pi}{2}\right]$$

$$A(\alpha) = \frac{1}{2(\alpha^2-1)}\left[\cancel{\alpha\sin(\alpha+1)\frac{\pi}{2}} + \cancel{\alpha\sin(\alpha-1)\frac{\pi}{2}}\right.$$

$$\left. - \sin(\alpha+1)\pi/2 + \sin(\alpha-1)\pi/2\right]$$

$$A(\alpha) = \frac{1}{2(\alpha^2-1)}\left(-2\cos\frac{\alpha\pi}{2}\sin\frac{\pi}{2}\right)$$

$$A(\alpha) = \frac{1}{1-\alpha^2}\cos\frac{\alpha\pi}{2} \qquad (2)$$

Subs. (2) in (1)

$$\Rightarrow\ f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1-\alpha^2}\cos\frac{\alpha\pi}{2}\cos\alpha x\, d\alpha$$

- Taking $x=0$, We obtain

$$\int_0^\infty \frac{1}{1-\alpha^2}\cos\frac{\pi\alpha}{2}\, d\alpha = \frac{\pi}{2}$$

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