King Saud University
Department of Mathematics
M-203 (Differential & Integral Calculus)
Final Examination
Summer-Semester 1437-1438
Mark:40 Time: 3 hrs

Q1:(4+4+4) **(a)** Use the integral test to show that series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n(\ln n)^2}$ is absolutely convergent.

(b) Determine whether the following series converges or diverges:

1- $\sum_{n=1}^{\infty} \frac{100^n}{n!}$. 2- $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$.

(c) Find the Taylor series for $f(x) = 10^x$ about $c = 1$.

Q2:(5+5+4) **(a)** Find the area A of the region in the xy-plane bounded by the graphs of $y = 4x - x^2$ and $y = -x$.

(b) Find the volume of the tetrahedron bounded by the coordinates planes $x = 0$, $y = 0$, $z = 0$ and the plane $x + y + z = 1$.

(c) Use spherical coordinates to find the centroid of hemispherical solid $z = \sqrt{a^2 - x^2 - y^2}$ of radius a where the mass density $\delta(x, y, z) = 1$.

Q3:(7+7) **(a)** Show that the line integral $\int_{(0,0,0)}^{(3,2,1)} y^2z^4dx + 2xyz^4dy + 4xy^2z^3dz$ is independent of path, and find its value.

(b) Use Green theorem to evaluate the line integral $\oint_C y^2dx + 3xydy$, where C is the boundary of upper half region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$.

Model Answer of Final Exam

Math 203 — 55/37-38

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Q1 a) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)^2}$

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let $\displaystyle\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

- By using Integral Test, let $f(x) = \dfrac{1}{x(\ln x)^2}$, $x \geq 2$

Clearly, f is a positive valued fn, Continuous and decreasing on $[2, \infty)$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{(\ln x)^2} \cdot \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{(\ln x)^{-1}}{-1} \right]_2^t$$

$$= \lim_{t\to\infty} \left[\frac{-1}{\underbrace{\ln t}_{0}} + \frac{1}{\ln 2} \right]$$

$$= \frac{1}{\ln 2} \Rightarrow c'gt$$

$\Rightarrow \displaystyle\sum_{n=2}^{\infty} |a_n|$ is c'gt

$\therefore \displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln n)^2}$ is absolutely c'gt.

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b)(i) $\sum_{n=1}^{\infty} \frac{100^n}{n!}$

- By using Ratio Test

(2)

Let $a_n = \frac{100^n}{n!}$

$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{(100)^n}$

$= \frac{100}{n+1}$

$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{100}{n+1} = 0 < 1$

$\therefore \sum_{n=1}^{\infty} \frac{100^n}{n!}$ is c'gt. #

(ii) • By using AST

let $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$

(2)

$\Rightarrow a_n = \frac{n}{n^2+1} = f(n)$

, let $f(x) = \frac{x}{x^2+1}$, $x \geqslant 1$

$\Rightarrow f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$

$\Rightarrow f$ is decreasing $\underline{fn}$ on $[1,\infty)$

(1)

$\Rightarrow a_n \geqslant a_{n+1} > 0$

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0 \quad (2)$

$(1),(2) \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ is c'gt.

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c) $f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(c)}{n!} (x-c)^n \quad , c \in \mathbb{R}$

Taylor Series

$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c) + \cdots$

$+ \frac{f^{n}(c)}{n!}(x-c)^n + \cdots$

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$c=1 \ , \ f(x) = 10^x$

$, f'(x) = \ln 10 . 10^x \ , \ f''(x) = (\ln 10)^2 10^x, \ldots, f^{n}(x) = (\ln 10)^n 10^x$

$\therefore f(x) = 10 + 10 \ln 10 . (x-1) + \frac{10(\ln 10)^2}{2!}(x-1)^2 + \cdots + \frac{10(\ln 10)^n}{n!}(x-1)^n + \cdots$

$\Rightarrow 10^x = 10 \sum_{n=0}^{\infty} \frac{(\ln 10)^n (x-1)^n}{n!}$

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Q2 a) Find the area A of the region in the xy-plane bounded by the graphs of $y = 4x - x^2$ and $y = -x$

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Ans:

Area, $A = \int_0^5 \int_{-x}^{4x-x^2} dy\, dx$

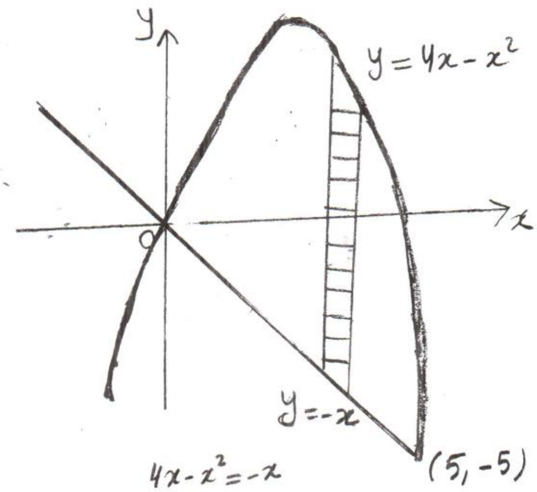
$$A = \int_0^5 \left[y \right]_{-x}^{4x-x^2} dx$$

$$A = \int_0^5 (5x - x^2)\, dx$$

$$A = \left[\frac{5x^2}{2} - \frac{x^3}{3} \right]_0^5$$

$$= \frac{5^3}{2} - \frac{5^3}{3}$$

$$\therefore A = \frac{125}{6} \simeq 20.8 \quad \#$$



$4x - x^2 = -x$

$\Rightarrow x^2 - 5x = 0$

$\Rightarrow x(x-5) = 0 \Rightarrow x = 0, x = 5$

the intersection points are $(0,0)$, $(5,-5)$

b) Find the volume of the tetrahedron bounded by the Coordinate planes $x=0, y=0, z=0$ and the plane $x+y+z=1$

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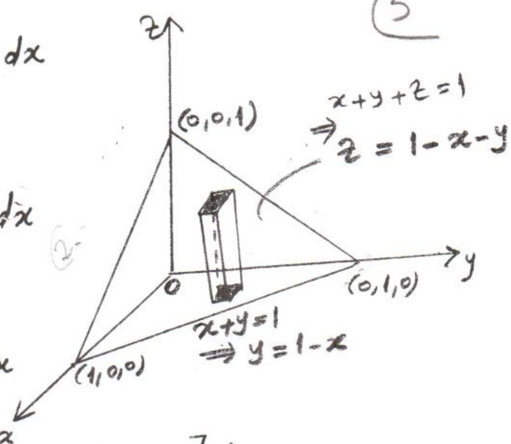
Ans:

$$\text{Volume} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$



$$V = \int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx$$

$$V = \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$V = \int_0^1 \left[(1-x) - x(1-x) - \frac{1}{2}(1-x)^2 \right] dx$$

$$V = \int_0^1 (1-x) \left[1-x - \frac{1}{2}(1-x) \right] dx$$

$$V = \int_0^1 (1-x) \left(\frac{1}{2} - \frac{1}{2}x \right) dx$$

$$V = \frac{1}{2} \int_0^1 (1-x)^2\, dx$$

$$V = \frac{-1}{2} \int_0^1 (1-x)^2 \cdot -dx$$

$$V = \frac{-1}{2} \left[\frac{(1-x)^3}{3} \right]_0^1 = \frac{1}{6}$$

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c) Find the centroid of hemispherical solid of radius a where the mass density $\delta(x,y,z)=1$, by using Spherical Coordinates.

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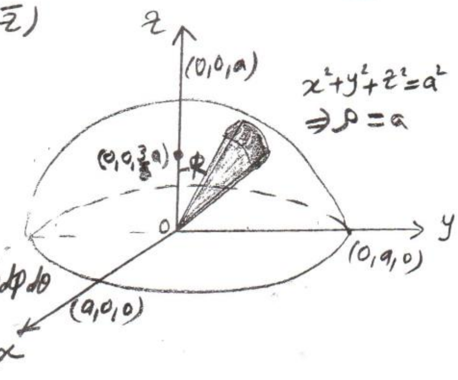

Ans: the center of mass at $(0,0,\bar{z})$

$$\bar{z} = \frac{M_{xy}}{V} \quad ①$$

$$M_{xy} = \iiint_Q z \, dV$$

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho\cos\varphi \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho^3 \cdot \sin\varphi\cos\varphi \, d\varphi \, d\theta$$

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2} \left[\frac{\rho^4}{4}\right]_0^a \underline{\sin\varphi}\,\underline{\cos\varphi} \, d\varphi \, d\theta$$

$$M_{xy} = \frac{1}{4}a^4 \int_0^{2\pi} \left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2} d\theta$$

$$M_{xy} = \frac{1}{8}a^4 \int_0^{2\pi} d\theta = \frac{1}{8}a^4(2\pi) = \frac{1}{4}\pi a^4 \quad ②$$

$\because$ V. of hemisphere $= \frac{2}{3}\pi a^3$ ③

Subs. ②, ③ in ①

$$\Rightarrow \bar{z} = \frac{1/4\,\pi a^4}{2/3\,\pi a^3} = \frac{3}{8}a$$

$\therefore$ the centroid at $(0,0,\frac{3}{8}a)$. #

Q3 (a) $\int_C \vec{F}\cdot d\vec{r}$ is independent of path if $\exists$ a fn f s.t $\vec{F} = \nabla f$

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$\Rightarrow \frac{\partial f}{\partial x} = y^2 z^4$

$\Rightarrow f(x,y,z) = \int y^2 z^4 \, dx$

$= y^2 z^4 x + g_1(y,z)$

, $\frac{\partial f}{\partial y} = 2xyz^4$

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$\Rightarrow f(x,y,z) = 2\int xyz^4 \, dy$

$= xy^2 z^4 + g_2(x,z)$

, $\frac{\partial f}{\partial z} = 4xy^2 z^3$

$\Rightarrow f(x,y,z) = 4\int xy^2 z^3 \, dz$

$= xy^2 z^4 + g_3(x,y)$

$\therefore f(x,y,z) = xy^2 z^4 + C$

$$\Rightarrow \int_{(0,0,0)}^{(3,2,1)} y^2 z^4 dx + 2xyz^4 dy + 4xy^2 z^3 dz = \left[xy^2 z^4\right]_{(0,0,0)}^{(3,2,1)} = 12 \quad \#$$

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(b) $\oint_C y^2 dx + 3xy \, dy$

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$= \iint_R (3y - 2y) \, dA = \iint_R y \, dA$ Green th.

$= 2\int_0^{\pi/2}\int_1^2 r\sin\theta \; r \, dr \, d\theta$

$= 2\int_0^{\pi/2} \left[\frac{r^3}{3}\right]_1^2 \sin\theta \, d\theta$

$= \frac{14}{3}\int_0^{\pi/2} \sin\theta \, d\theta$

$= \frac{14}{3}\left[-\cos\theta\right]_0^{\pi/2} = \frac{14}{3}$ #