

Math 481 – Midterm 2

Model Solution

Question 1

[5 marks]

For all $x \in [0, \pi]$,

$$\left| \frac{\sin(nx)}{n^3} \right| \leq \frac{1}{n^3}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges, the Weierstrass M-test implies that

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$$

converges uniformly on $[0, \pi]$.

Since each term is continuous and the series converges uniformly, we may integrate term by term:

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} dx = \sum_{n=1}^{\infty} \int_0^{\pi} \frac{\sin(nx)}{n^3} dx.$$

Now

$$\int_0^{\pi} \frac{\sin(nx)}{n^3} dx = \frac{1}{n^3} \left[-\frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{1 - \cos(n\pi)}{n^4}.$$

Thus the term is zero when n is even and equals $\frac{2}{n^4}$ when n is odd. Hence

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} dx = \sum_{m=1}^{\infty} \frac{2}{(2m-1)^4}.$$

Question 2

[5 marks]

Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

By the ratio test,

$$\lim_{n \rightarrow \infty} \left| \frac{x^{n+1}/(n+1)}{x^n/n} \right| = |x|.$$

Hence the radius of convergence is

$$R = 1.$$

If $x \in [-c, c]$, where $0 < c < 1$, then

$$\left| \frac{(-1)^{n-1}}{n} x^n \right| \leq \frac{c^n}{n} \leq c^n.$$

Since $\sum c^n$ converges, the given power series converges uniformly on $[-c, c]$.

The derivative series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1}.$$

This is a geometric series and converges uniformly on every $[-c, c]$, $0 < c < 1$. Therefore f is differentiable on $(-1, 1)$, and

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = 1 - x + x^2 - x^3 + \dots = \frac{1}{1+x}.$$

Since $f(0) = 0$,

$$f(x) = \int_0^x \frac{1}{1+t} dt = \ln(1+x).$$

Thus

$$f(x) = \ln(1+x), \quad -1 < x < 1.$$

At $x = 1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges by the alternating series test, and its value is $\ln 2$.

At $x = -1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-1)^n = - \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. Therefore the series diverges at $x = -1$.

Question 3

[5 marks]

Let

$$h_n(x) = e^{-x} \cos\left(\frac{x}{n}\right).$$

For every $x \geq 0$,

$$\lim_{n \rightarrow \infty} h_n(x) = e^{-x} \cos(0) = e^{-x}.$$

Also,

$$|h_n(x)| \leq e^{-x}.$$

The function e^{-x} is integrable on $[0, \infty)$, since

$$\int_0^{\infty} e^{-x} dx = 1.$$

By the Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^{\infty} e^{-x} \cos\left(\frac{x}{n}\right) dx = \int_0^{\infty} e^{-x} dx = 1.$$

Question 4

[5 marks]

We have

$$f_n(x) = xe^{-x^2} \mathbf{1}_{[0,n]}(x).$$

Since

$$[0, n] \subseteq [0, n+1],$$

we get

$$\mathbf{1}_{[0,n]}(x) \leq \mathbf{1}_{[0,n+1]}(x).$$

Because $xe^{-x^2} \geq 0$ for $x \geq 0$,

$$f_n(x) \leq f_{n+1}(x).$$

Thus (f_n) is increasing pointwise.

For every $x \geq 0$, eventually $x \in [0, n]$, so

$$\lim_{n \rightarrow \infty} f_n(x) = xe^{-x^2}.$$

By the Monotone Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx = \int_0^\infty xe^{-x^2} dx.$$

Using $u = x^2$, $du = 2x dx$,

$$\int_0^\infty xe^{-x^2} dx = \frac{1}{2} \int_0^\infty e^{-u} du = \frac{1}{2}.$$

Therefore,

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx = \frac{1}{2}.$$

Question 5

[2 marks]

The function f is equal to 1 only on the countable set

$$\left\{ \frac{1}{3^n} : n \in \mathbb{N} \right\},$$

and is zero elsewhere.

The possible discontinuities occur at the points

$$\left\{ \frac{1}{3^n} : n \in \mathbb{N} \right\} \cup \{0\}.$$

This set is countable, hence has measure zero.

Therefore, by the Lebesgue criterion for Riemann integrability, f is Riemann integrable on $[0, 1]$.

Since $f = 0$ except on a countable set,

$$\int_0^1 f(x) dx = 0.$$

Question 6**[3 marks]**

Let

$$g(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1], \\ 0, & x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

Every interval contains both rational and irrational numbers. Hence, near every point $x \in [0, 1]$, the function takes both values 0 and 1. Therefore g is discontinuous at every point of $[0, 1]$.

Since the set of discontinuities is all of $[0, 1]$, which has positive measure, the Lebesgue criterion implies that g is not Riemann integrable.

However, $g = \mathbf{1}_{\mathbb{Q} \cap [0, 1]}$. Since $\mathbb{Q} \cap [0, 1]$ is countable, it has Lebesgue measure zero. Therefore g is Lebesgue integrable, and

$$\int_{[0, 1]} g \, dm = m(\mathbb{Q} \cap [0, 1]) = 0.$$