KING SAUD UNIVERSITY COLLEGE OF SCIENCE

M203 DEPARTMENT OF MATHEMATICS TIME: 90 Minutes

(SEMESTER 2, 1439-1440)

Second Mid-term Exam

Note: All questions carry equal Marks.

Q1. Evaluate the iterated integral $\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{\sqrt{x^3+1}} dxdy$.

Q2. Evaluate the integral $\iint_R \frac{x^2}{x^2+y^2} dA$, where the region R is bounded by the graphs of the equations: $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Q3. Find the surface area of the portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line $y = x$ and the parabola $y = x^2$.

Q4. Use cylindrical coordinates to evaluate the integral:

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x^2 dzdydx$$

Q5. A solid Q is bounded by the sphere $z = \sqrt{4 - x^2 - y^2}$ and the xy-plane and the mass density of Q is given by the function: $\delta(x, y, z) = 2 + x^2 + y^2 + z^2$. Use spherical coordinates to find the moment of inertia of Q with respect to the z-axis.

M- 203

II Mid-term Exam. (II semester 1439/1440)

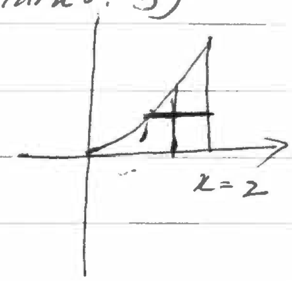
Time: 90 Minutes Max Marks: 35

Solutions

Q#1) Evaluate the iterated integral $\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{\sqrt{x^3+1}}\,dx\,dy$

[Marks: 5]

Solution: Given. $\left.\begin{array}{l}\sqrt{y} \le x \le 2 \\ 0 \le y \le 4\end{array}\right\}$



Changing to: $\left.\begin{array}{l}0 \le y \le x^2 \\ 0 \le x \le 2\end{array}\right\}$

$$\therefore \int_0^2 \int_0^{x^2} \frac{1}{\sqrt{x^3+1}}\,dy\,dx$$

3 $= \int_0^2 x^2 \cdot \frac{1}{\sqrt{1+x^3}}\,dx$

Put $x^3+1 = u \ni$:

$3x^2 dx = du$ $= \frac{4}{3}$ 2

$= \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du$

$= \frac{1}{3}\times 2 u^{\frac{1}{2}} + c$

$= \frac{2}{3}\left[(x^3+1)^{\frac{1}{2}}\right]_0^2 = \frac{4}{3}$

Q#2) Evaluate the Integral $\iint_R \frac{x^2}{x^2+y^2}\,dA$, where the region R is bounded by the graphs of the equations:

$x^2+y^2=1$ and $x^2+y^2=4$ [Marks: 5]

Soln. we use Polar Coordinates: $\int_0^{2\pi}\int_1^2 \frac{\cancel{r^2}\cos^2\theta}{\cancel{r^2}}\,r\,dr\,d\theta$ 3

$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_1^2 \cos^2\theta\,d\theta = \frac{3}{2}\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$

$= \frac{3}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{3}{\cancel{4}\,2}\times \cancel{2}\pi = \frac{3}{2}\pi$

Q#3) Find the surface area of the portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line $y = x$ and the parabola $y = x^2$. [Marks: 5]

Soln. We have $z = \sqrt{4x^2+4y^2}$ ~~$= f(x,y)$~~

$$f_x(x,y) = \frac{1}{\cancel{2}\sqrt{4x^2+4y^2}} \times \cancel{8}^{4}x \quad \text{and} \quad f_y(x,y) = \frac{1}{\cancel{2}\sqrt{4x^2+4y^2}} \times \cancel{8}y$$

$$\therefore S.A = \iint_R \sqrt{1 + \frac{16x^2}{4x^2+4y^2} + \frac{16y^2}{4x^2+4y^2}}\, dA$$

$$= \iint_R \sqrt{\frac{x^2+y^2+4x^2+4y^2}{x^2+y^2}}\, dA = \iint_R \underbrace{\sqrt{\frac{5x^2+5y^2}{x^2+y^2}}}_{=\sqrt{5}} \quad (2)$$

$$= \sqrt{5}\int_0^1 \int_{x^2}^{x} dy\, dx = \sqrt{5}\int_0^1 (x - x^2)\, dx \quad (2)$$

$$= \sqrt{5}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \sqrt{5}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\sqrt{5}}{6} \quad (1)$$

Q#4) Use cylindrical coordinates to evaluate the integral.

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x^2\, dz\, dy\, dx.$$

[Marks: 5]

Soln.

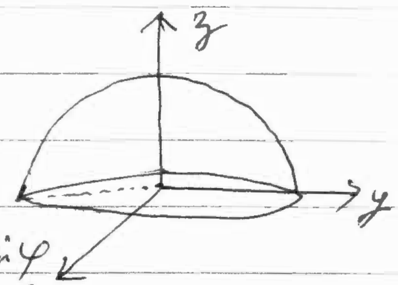
$$\int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2\cos^2\theta\; r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^3 r \cdot r^2 \cos^2\theta\,(9 - r^2)\, dr\, d\theta \quad (3)$$

$$= \int_0^{2\pi} \cos^2\theta \left[9\frac{r^4}{4} - \frac{r^6}{6}\right]_0^3 d\theta = \int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right) d\theta \left(\frac{729}{4} - \frac{2\ldots}{2\ldots}\right)$$

$$= \frac{1}{24}\, 2\pi\, (729 - 486) = 94.?$$

Q#5) A solid Q is bounded by the sphere $z=\sqrt{4-x^2-y^2}$ and the xy-plane and the mass density of Q is given by the function: $\delta(x,y,z) = 2+x^2+y^2+z^2$. Use Spherical Coordinates to find the moment of Inertia of Q with respect to the z-axis. [Marks: 5]

Soln. $I_z = \iiint_Q (x^2+y^2)\,\delta(x,y,z)\,dV$ 1



$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^2\sin^2\varphi\,(2+\rho^2)\,\rho^2\sin\varphi\; d\rho\, d\varphi\, d\theta$$ 3

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (2\rho^4\sin^3\varphi+\rho^6\sin^3\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left[2\frac{\rho^5}{5}+\frac{\rho^7}{7}\right]_0^2 \sin^3\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left(\frac{64}{5}+\frac{128}{7}\right)\sin^3\varphi\, d\theta$$

$$= 2\pi\left(\frac{1084}{35}\right)(1-\cos^2\varphi)\sin\varphi\, d\varphi$$

$$= 2\pi\left(\frac{1084}{35}\right)\left(1-\frac{1}{3}\right)$$

$$= 2\pi\left(\frac{1084}{35}\right)\frac{2}{3}$$ 1

Put $\cos\varphi = u \Rightarrow$
$-\sin\varphi\, d\varphi = du$
$= -(1-u^2)\,du$
$= -u+\frac{u^3}{3}+c$