KING SAUD UNIVERSITY **COLLEGE OF SCIENCE**

M203 **DEPARTMENT OF MATHEMATICS** **TIME: 90 Minutes**

(SEMESTER 2, 1439-1440)

First Mid-term Exam

Note: All questions carry equal Marks.

Q1. Determine whether the following series converges or diverges and if it converges, find its sum: $\sum_{n=1}^{\infty} \frac{e^{n}-e^{-n}}{e^{2n}}$.

Q2. Use integral test to determine the convergence or divergence of the series: $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$.

Q3. Determine the convergence or divergence of the series:

$$\sum_{n=1}^{\infty} \frac{2n^2+5}{n^3 \cos^2(n)}.$$

Q4. Find the interval of convergence and the radius of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{(-2x)^n}{n^2+1}.$$

Q5. Find the MacLaurin series of $\sin(x)$ and use its first two non-zero terms to approximate the integral $\int_0^{0.5} \frac{\sin(x^2)}{x}\, dx$.

M- 203

I Mid-term Exam (II Semester 1439/1440)

Max. Marks: 25 Time: 90 Mts

Q#1) Determine whether the following Series converges or diverges and if it converges, find its sum:

$$\sum_{n=1}^{\infty} \frac{e^{n}-e^{-n}}{e^{2n}}$$ [Marks: 5]

Soln. $\sum_{n=1}^{\infty} \frac{e^{n}}{e^{2n}} - \sum_{n=1}^{\infty} \frac{e^{-n}}{e^{2n}} = \sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^{n} - \sum_{n=1}^{\infty}\left(\frac{1}{e^{3}}\right)^{n}$ and

both are convergent Geometric Series. 3

Sum: $\frac{1/e}{1-\frac{1}{e}} - \frac{\frac{1}{e^3}}{1-\frac{1}{e^3}} = \frac{1}{e-1} - \frac{1}{e^3-1}$ 2

Q#2) Use integral test to determine the convergence or divergence of the Series: $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$. [Marks: 5]

Soln. Clearly, $f(x) = \frac{1}{x(\ln x)^3}$ is positive-valued,

continuous and decreasing function on $[2,\infty)$ 2

Hence, we apply the Integral test: $\int_2^{\infty} \frac{1}{x(\ln x)^3}dx$

$= \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln x)^3}dx$ Put $\ln x = u \Rightarrow \frac{1}{x}dx = du$

$\therefore \int \frac{1}{u^3}du = -\frac{1}{2}\frac{1}{u^2}$

$\therefore \lim_{t\to\infty} -\frac{1}{2}\left[\frac{1}{(\ln x)^2}\right] = \lim_{t\to\infty} -\frac{1}{2}\left[\frac{1}{(\ln \infty)^{\infty}} - \frac{1}{(\ln 2)^2}\right] = \frac{1}{2}\cdot\frac{1}{(\ln 2)^2}$

3 Conv.

Q#3) Determine the convergence or divergence of the Series: $\sum_{n=1}^{\infty} \frac{2n^2+5}{n^3\cos^2(n)}$. [Marks: 5] (2)

Soln. we know $\cos^2(n) \le 1$. we have $\frac{2n^2+5}{n^3\cos^2(n)} \geq \frac{2n^2+5}{n^3}$ $\forall n \geq 1$

Next, $\frac{2n^2+5}{n^3} \times \frac{n}{1} = 2 > 0$, as we choose $\sum b_n = \sum \frac{1}{n}$ which is div.

and apply the Limit comparison test (LCT)

Hence, by by LCT, (and BCT) $\sum_{n=1}^{\infty} \frac{2n^2+5}{n^3\cos^2(n)}$ is divergent. (3)

Q#4) Find the Interval of convergence and the radius of convergence of the power series: $\sum_{n=0}^{\infty} \frac{(-2x)^n}{n^2+1}$. [Marks: 5

Soln. we apply Absolute Ratio-test

$$\lim_{n\to\infty} \left| \frac{(-2x)^{n+1}}{(n+1)^2+1} \times \frac{n^2+1}{(-2x)^n} \right| = 2|x|$$

For absolute convergence: $2|x| < 1 \iff |x| < \frac{1}{2}$

$\iff -\frac{1}{2} < x < \frac{1}{2}$ (2)

At $x = -\frac{1}{2}$, we have $\sum_{n=0}^{\infty} \frac{[-\cancel{2}(-\frac{1}{\cancel{2}})]^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$

$\frac{1}{n^2+1} \le \frac{1}{n^2}$ $\forall n$ and $\sum \frac{1}{n^2}$ is conv. by p-series test (1)

Hence by BCT, $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ is conv.

At $x = \frac{1}{2}$, we have $\sum_{n=0}^{\infty} \frac{(-\cancel{2} \times \frac{1}{\cancel{2}})^n}{n^2+1} = (-1)^n \frac{1}{n^2+1}$ which is (1)

conv. by AST. Hence it ...

Q#5) Find the Maclaurin Series of $\sin x$ and use its first two non-zero terms to approximate the integral

$$\int_0^{0.5} \frac{\sin(x^2)}{x}\,dx.$$ [Marks: 5]

Soln. We know the Maclaurin Series as

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots \quad \text{--- (1)}$$

Here

$$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$$
$$f'(x) = \cos x \Rightarrow f'(0) = 1$$
$$f''(x) = -\sin x \Rightarrow f''(0) = 0$$
$$f'''(x) = -\cos x \Rightarrow f'''(0) = 1$$
$$f^{(iv)}(x) = \sin x \Rightarrow f^{(iv)}(0) = 0$$

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Putting these values in Maclaurin Series (1), we get

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

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Now, $\int_0^{0.5} \frac{\sin(x^2)}{x}\,dx$ Replacing x by x^2, we get

$$= \int_0^{0.5} \frac{1}{x}\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\right) dx$$ 1

$$= \int_0^{0.5} \frac{1}{x}\left[\frac{x^2}{\cdot} - \frac{x^6}{(3!)}\right] dx$$

$$= \int_0^{0.5} \left(\frac{x}{} - \frac{x^5}{3!} + \cdots\right) dx = \left[\frac{x^2}{2} - \frac{x^6}{6(3!)}\right]_0^{0.5}$$

$$= \frac{(0.5)^2}{2} - \frac{(0.5)^6}{\ldots} \approx 0.125 - 0.0004311$$