Solution of the midterm exam

Question 1 (2+2+4):

a) Find the values of λ for which the matrix $\begin{bmatrix} 1 & 0 & \lambda \\ 2 & 1 & 2+\lambda \\ 2 & 3 & \lambda^2 \end{bmatrix}$ is invertible.

$\begin{bmatrix} 1 & 0 & \lambda \\ 2 & 1 & 2+\lambda \\ 2 & 3 & \lambda^2 \end{bmatrix}$ is invertible if and only if $\begin{vmatrix} 1 & 0 & \lambda \\ 2 & 1 & 2+\lambda \\ 2 & 3 & \lambda^2 \end{vmatrix} \neq 0.$

$$0 \neq \begin{vmatrix} 1 & 0 & \lambda \\ 2 & 1 & 2+\lambda \\ 2 & 3 & \lambda^2 \end{vmatrix} \underset{=}{\overset{-2R_{12}}{-2R_{13}}} \begin{vmatrix} 1 & 0 & \lambda \\ 0 & 1 & 2-\lambda \\ 0 & 3 & \lambda^2-2\lambda \end{vmatrix} \overset{-3R_{23}}{=} \begin{vmatrix} 1 & 0 & \lambda \\ 0 & 1 & 2-\lambda \\ 0 & 0 & \lambda^2+\lambda-6 \end{vmatrix} = \lambda^2+\lambda-6.$$

So $0 \neq \lambda^2+\lambda-6 = (\lambda-2)(\lambda+3)$. Hence, $\lambda \in \mathbb{R} - \{2,-3\}$.

b) By using properties of the determinants, show that: $\begin{vmatrix} a+b+c & b & a \\ d+e+f & e & d \\ g+h+i & h & g \end{vmatrix} = \begin{vmatrix} c & b & a \\ f & e & d \\ i & h & g \end{vmatrix}$.

$$\begin{vmatrix} a+b+c & b & a \\ d+e+f & e & d \\ g+h+i & h & g \end{vmatrix} = \begin{vmatrix} a & b & a \\ d & e & d \\ g & h & g \end{vmatrix} + \begin{vmatrix} b & b & a \\ e & e & d \\ h & h & g \end{vmatrix} + \begin{vmatrix} c & b & a \\ f & e & d \\ i & h & g \end{vmatrix} = 0+0+\begin{vmatrix} c & b & a \\ f & e & d \\ i & h & g \end{vmatrix}$$

c) Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. Find adj(A) and A^{-1}.

$adj(A) = C^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix}$. Det(A) = $(-1)\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = (-1)(-1) = 1.$

$A^{-1} = \frac{1}{|A|} adj(A) = adj(A).$

Question 2 (2+4+3):

a) Let $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$. Show that the linear system AX = B has a unique solution for any fixed $\alpha, \beta, \gamma \in \mathbb{R}$.

Since $|\boldsymbol{A}| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 3 \end{vmatrix} \underset{-2R_{13}}{\overset{-R_{12}}{=}} \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = 1 \neq \mathbf{0}$, $\boldsymbol{A}^{-1}$ exists. So, the linear system has the unique solution X= $\boldsymbol{A}^{-1}\boldsymbol{B} = \boldsymbol{A}^{-1}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ for any fixed $\alpha, \boldsymbol{\beta}, \gamma \in \mathbb{R}$.

b) Solve the following system of linear equations by using the Cramer's rule:

$$\boldsymbol{x - y + z = 0}$$
$$\boldsymbol{x + y + z = 2}$$
$$\boldsymbol{x + 2y + 4z = 3}.$$

$|\boldsymbol{A}| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = \mathbf{6}$, $|\boldsymbol{A}_x| = \begin{vmatrix} 0 & -1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{vmatrix} = \mathbf{6}$, $|\boldsymbol{A}_y| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 4 \end{vmatrix} = \mathbf{6}$ and $|\boldsymbol{A}_z| = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \mathbf{0}$.

Hence, $\boldsymbol{x} = \frac{|A_x|}{|A|} = \frac{\mathbf{6}}{\mathbf{6}} = \mathbf{1}$, $\boldsymbol{y} = \frac{|A_y|}{|A|} = \frac{\mathbf{6}}{\mathbf{6}} = \mathbf{1}$ and $\boldsymbol{z} = \frac{|A_z|}{|A|} = \frac{\mathbf{0}}{\mathbf{6}} = \mathbf{0}$.

c) Use any of the elimination methods to show that the following system of linear equations is inconsistent:

$$-x + 2y - 5z = 3$$
$$x - 3y + z = 4$$
$$5x - 13y + 13z = 8.$$

Since

$$\left[\begin{array}{ccc|c} -1 & 2 & -5 & 3 \\ 1 & -3 & 1 & 4 \\ 5 & -13 & 13 & 8 \end{array}\right] \xrightarrow{R_{12}} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ -1 & 2 & -5 & 3 \\ 5 & -13 & 13 & 8 \end{array}\right] \underset{-5R_{13}}{\overset{1R_{12}}{\rightarrow}} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -1 & -4 & 7 \\ 0 & 2 & 8 & -12 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 1 & 4 & -7 \\ 0 & 2 & 8 & -12 \end{array}\right]$$
$$\xrightarrow{-2R_{23}} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The last row gives us the equation $0x + 0y + 0z = 2$ which is impossible. Hence, the system is inconsistent.

Question 3 (3+5+5):

a) Let $\{v_1, v_2, v_3\}$ be a linearly independent subset of vector space V. Show that the subset $\{w_1, w_2, w_3\}$ is linearly independent in V, where $w_1 = v_1 + 2v_3$, $w_2 = v_1 + v_2 + v_3$ and $w_3 = v_2 + v_3$.

Let $0 = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 = \alpha_1(v_1 + 2v_3) + \alpha_2(v_1 + v_2 + v_3) + \alpha_3(v_2 + v_3)$
$= (\alpha_1 + \alpha_2)v_1 + (\alpha_2 + \alpha_3)v_2 + (2\alpha_1 + \alpha_2 + \alpha_3)v_3$.

By the linear independence of $\{v_1, v_2, v_3\}$, we get $\alpha_1 + \alpha_2 = 0$, $\alpha_2 + \alpha_3 = 0$ and $2\alpha_1 + \alpha_2 + \alpha_3 = 0$. Hence,

$$\alpha_1 + \alpha_2 = 0$$
$$\alpha_2 + \alpha_3 = 0$$
$$2\alpha_1 + \alpha_2 + \alpha_3 = 0$$

Since the determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 2(1 - 0) = 2 \neq 0$$

So $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\{w_1, w_2, w_3\}$ is linearly independent in V.

b) Show that $F = \{(x, y, z) \in \mathbb{R}^3 | \; y - z = 0, \; y + z = 0\}$ is a vector subspace of Euclidean space $\mathbb{R}^3$. Then find a basis and dimension of F.

$(x, y, z) \in F \Leftrightarrow y - z = 0$ and $y + z = 0 \Leftrightarrow y = z = 0 \Leftrightarrow (x, y, z) = (x, 0,0) = x(1,0,0)$.

So, $F = span(\{(1,0,0)\})$; which is a vector subspace of $\mathbb{R}^3$.

Hence, $\{(1,0,0)\}$ is a basis of F (a nonzero vector, so it is linearly independent) and so $\dim(F) = 1$.

Another way to prove that F is a vector subspace of $\mathbb{R}^3$: For every $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3) \in F$ and $k \in \mathbb{R}$, we have:

1) F is not empty since (0,0,0) belongs to it (0−0=0 and 0+0=0).
2) $u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$. Now,
   $(u_2 + v_2) + (u_3 + v_3) = (u_2 + u_3) + (v_2 + v_3) = 0 + 0 = 0$,
   $(u_2 + v_2) - (u_3 + v_3) = (u_2 - u_3) + (v_2 - v_3) = 0 + 0 = 0$,
   So $u + v \in F$.
3) $ku = (ku_1, ku_2, ku_3)$. Now,
   $ku_2 + ku_3 = k(u_2 + u_3) = k(0) = 0$,
   $ku_2 - ku_3 = k(u_2 - u_3) = k(0) = 0$,
   So $ku \in F$.
   1), 2) & 3) $\Rightarrow$ F is a vector subspace of $\mathbb{R}^3$.

c) Show that $B := \{t^2 + 2, -t + 1, 2t - 1\}$ is a basis of the real vector space $P_2(t)$ of all polynomials in real variable t having degree ≤ 2. Then find the coordinate vector of the polynomial $t^2 + 3t + 3$ with respect to the basis B.

Firstly, we know that $\{t^2, t, 1\}$ is another basis of $P_2(t)$.

If $0 = \alpha(t^2 + 2) + \beta(-t + 1) + \gamma(2t - 1) = \alpha t^2 + (2\gamma - \beta)t + 2\alpha + \beta - \gamma$, then

$$\alpha = 0$$
$$2\gamma - \beta = 0$$
$$2\alpha + \beta - \gamma = 0$$

and hence

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & -1 & 2 & 0\\ 2 & 1 & -1 & 0\end{array}\right] \xrightarrow{-2R_{12}} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & -1 & 2 & 0\\ 0 & 1 & -1 & 0\end{array}\right] \xrightarrow{1R_{23}} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & -1 & 2 & 0\\ 0 & 0 & 1 & 0\end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & 1 & 0\end{array}\right] \xrightarrow{2R_{32}} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

which gives us $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma} = \mathbf{0}$. So, the set $\boldsymbol{B}$ is linearly independent in the vector space $\boldsymbol{P_2(t)}$.
However, $\mathbf{dim}(\boldsymbol{P_2(t)}) = \mathbf{3}$. So, $\boldsymbol{B}$ is a basis of $\boldsymbol{P_2(t)}$.
Now, if $\boldsymbol{t^2 + 3t + 3 = \alpha(t^2 + 2) + \beta(-t + 1) + \gamma(2t - 1) = \alpha t^2 + (2\gamma - \beta)t + 2\alpha + \beta - \gamma}$, then

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & -1 & 2 & 3\\ 2 & 1 & -1 & 3\end{array}\right] \xrightarrow{-2R_{12}} \left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & -1 & 2 & 3\\ 0 & 1 & -1 & 1\end{array}\right] \xrightarrow{1R_{23}} \left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & -1 & 2 & 3\\ 0 & 0 & 1 & 4\end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & -2 & -3\\ 0 & 0 & 1 & 4\end{array}\right] \xrightarrow{2R_{32}} \left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 5\\ 0 & 0 & 1 & 4\end{array}\right]$$

So $\boldsymbol{\alpha} = \mathbf{1}$, $\boldsymbol{\beta} = \mathbf{5}$ and $\boldsymbol{\gamma} = \mathbf{4}$. Hence, $[\boldsymbol{t^2 + 3t + 3}]\boldsymbol{B} = \begin{bmatrix}\mathbf{1}\\ \mathbf{5}\\ \mathbf{4}\end{bmatrix}$.