Math 481 Midterm 2

Time: 90 minutes (Total: 25 points)

Question 1. (5 pts) Let

$$f_n(x) = \frac{x}{1+n^2x}, \quad x \in [0,1].$$

(a) Find the pointwise limit $f(x)$.

(b) Determine whether the convergence is uniform on $[0,1]$. Prove or disprove using the ε–N definition.

(c) Decide whether

$$\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx.$$

Question 2. (3 pts) Show that

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}\,dx = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^4}.$$

Justify carefully the interchange between the integral and the infinite series, citing an appropriate convergence theorem.

Question 3. (3 pts) Prove that

$$\lim_{x\to 0} \sum_{n=0}^{\infty} \frac{e^{-n^2x^2}}{3^n} = \frac{1}{2}.$$

Justify the interchange between the limit and the infinite sum, stating the relevant theorem.

Question 4. (7 pts) Find the radius of convergence R of each of the following power series:

$$\sum_{n=0}^{\infty} a_n x^n.$$

(a)

$$a_n = \begin{cases} 2, & n = k^3 \text{ for some } k \in \mathbb{N}, \\ \dfrac{1}{n}, & \text{otherwise.} \end{cases}$$

(b) Suppose there exist constants $b, c > 0$ such that for all n,

$$nb \leq |a_n| \leq n^2 c.$$

Determine R and justify your answer.

Question 5. (7 pts) Consider the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

(a) Determine the radius of convergence R, and deduce that the series converges uniformly on every interval $[-c, c]$ with $0 \leq c < 1$. Show that f is differentiable on $(-1, 1)$.

(b) Identify explicitly the function represented by the series on $(-1, 1)$. *(Hint: first compute $f'(x)$, then use the condition $f(0) = 0$).*

(c) Apply *Abel's test for uniform convergence* to discuss the uniform convergence of the series on the closed interval $[0, 1]$.