جامعة
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Basics of Engineering Measurements

(AGE 2340)

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Chapter 3:

Measurement System Behaviour

System Dynamics

Basic System Models

- **Modelling** is the process of representing the behavior of a system by a collection of mathematical equations & logics.

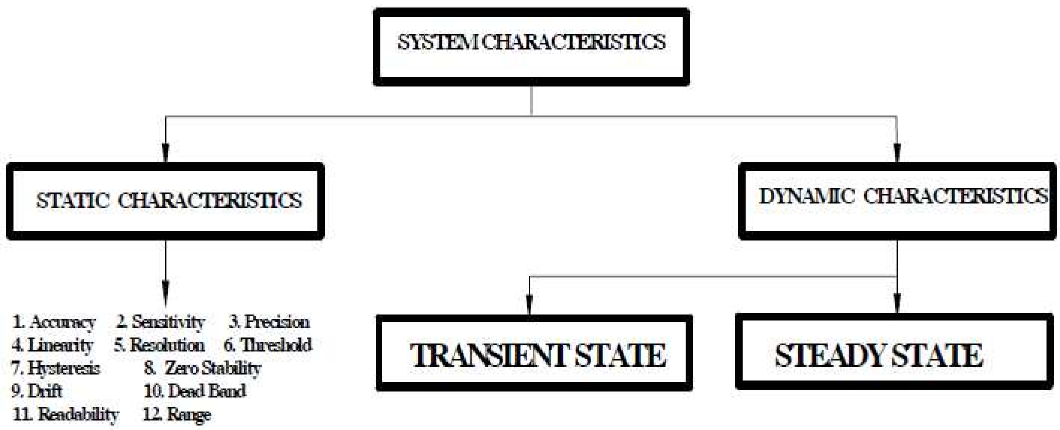
  Modelling is comprehensively utilized to study the response of any system.
- **Response** of a system is a measure of its fidelity to its purpose.
- **Simulation** is the process of solving the model and it performed on a computer.
- **Equations** are used to describe the relationship between the input and output of a system.

Input ⟹ Governing Equations ⟹ *Output*

- **Analogy** approach is also widely used to study system response.

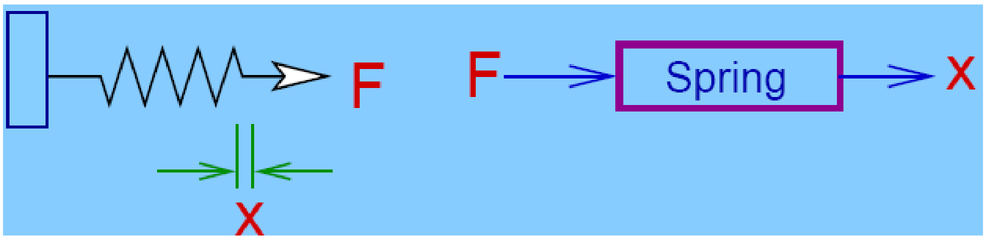
Measuring System

Characteristics of Measurement Systems:



- Static characteristics: Define the performance criteria for the measurement of quantities that remain constant or vary quite slowly.
- Dynamic characteristics: Concern the relationship between the system input and output when the measurand is varying rapidly.

Mechanical System Elements: (a) Spring



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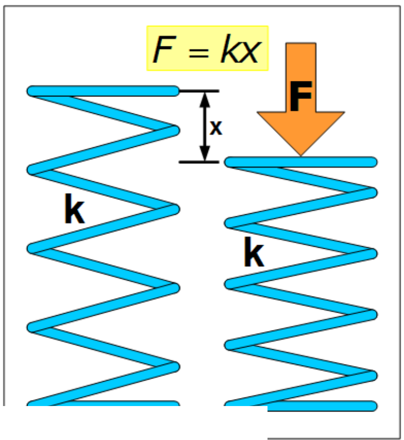
$$F = k\ x$$

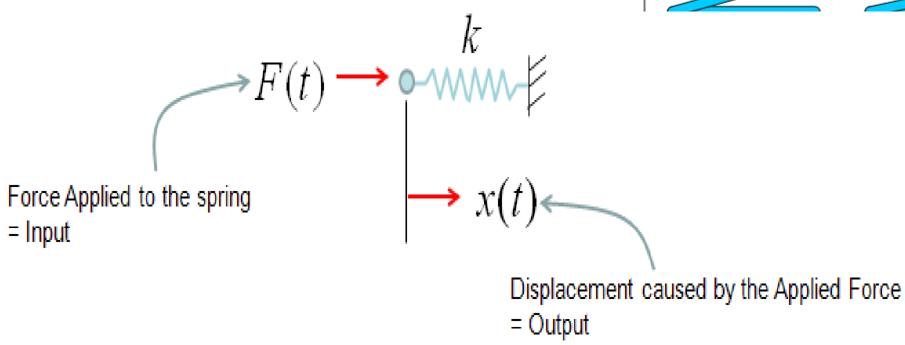
$F \equiv$ Force (tension or compression),

$x \equiv$ Displacement (extension or compression),

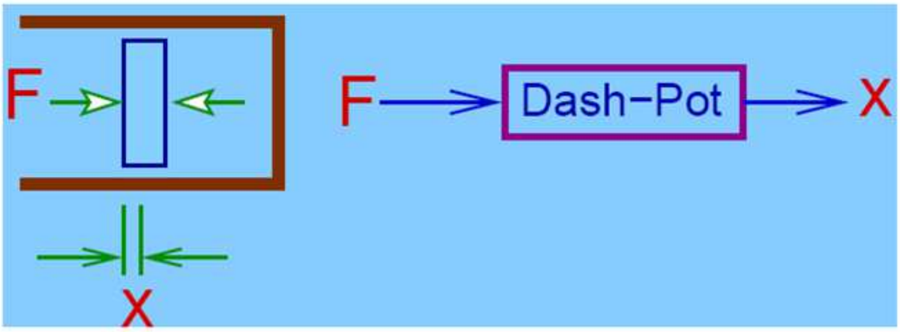
$k \equiv$ Spring constant. The bigger the value of k the greater the forces required to stretch or compress the spring & so the greater the stiffness.

Mechanical System Elements: (a) Spring





Mechanical System Elements: (b) Dashpot

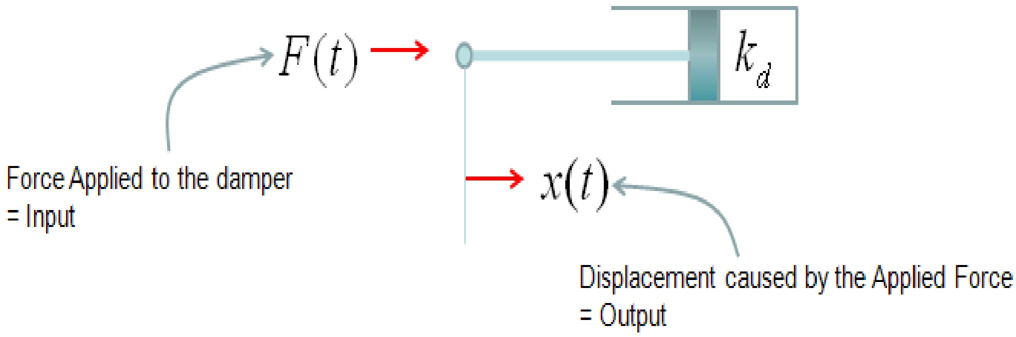


$$F = -c\,v = -c\,\frac{dx}{dt}$$

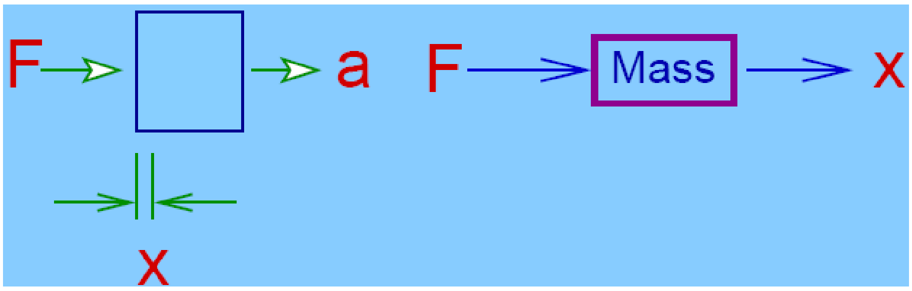
$F \equiv$ Force opposing the motion at velocity v,

$c \equiv$ Damping coefficient.

Mechanical System Elements: (b) Dashpot



Mechanical System Elements: (c) Mass

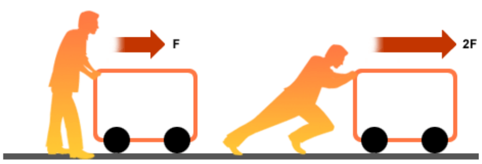


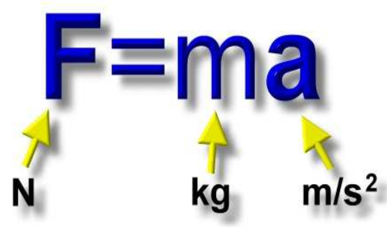
$$F = m\,a = m\,\frac{dv}{dt} = m\,\frac{d^2x}{dt^2}$$

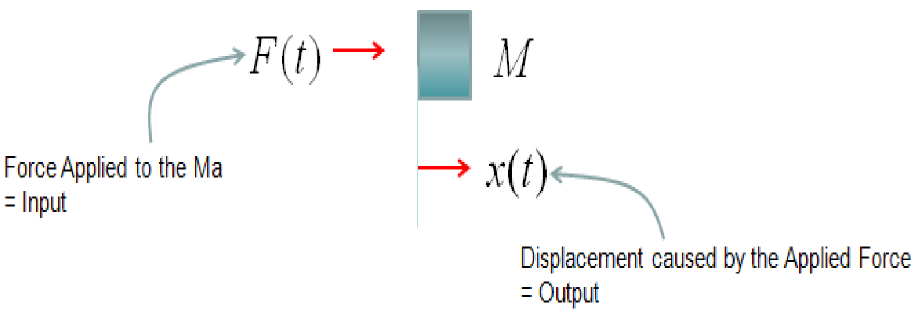
$F \equiv$ Force required to cause acceleration, a,

$m \equiv$ Mass of the element that is distributed throughout some volume. However, in many cases, it is assumed to be concentrated at a point.

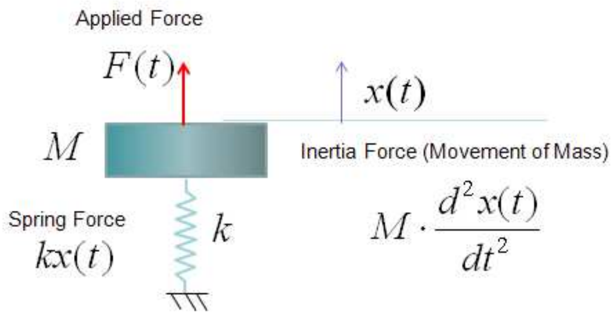
Mechanical System Elements: (c) Mass







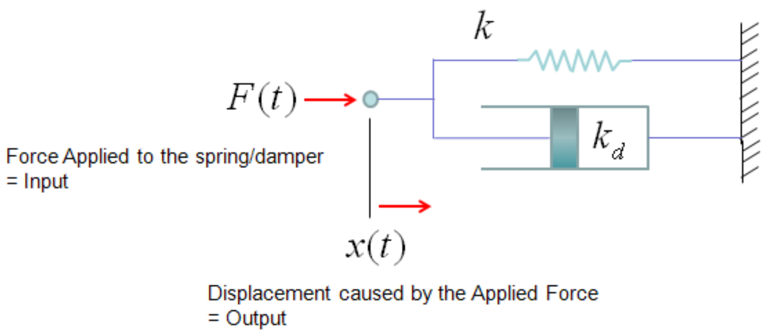
Mechanical System Elements: Mass + Spring



Applied Force, Spring Force, Movement of Mass

$$F(t) = kx(t) + M \cdot \frac{d^2x(t)}{dt^2}$$

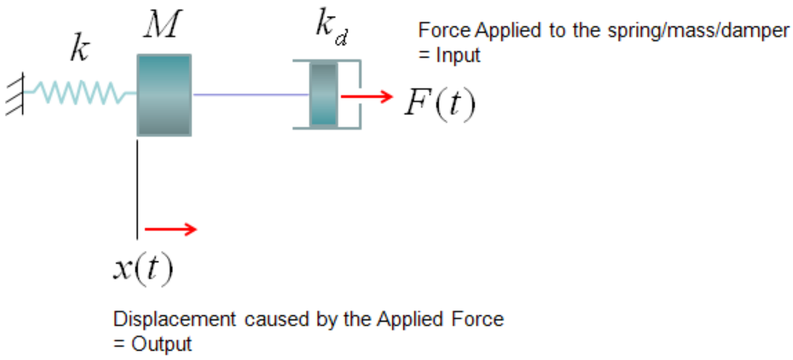
Mechanical System Elements: Damper + Spring



Applied Force, Spring Force, Damping Force

$$F(t) = kx(t) + k_d \cdot \frac{dx(t)}{dt}$$

Mechanical System Elements: Mass+ Damper + Spring



Applied Force — Spring Force — Movement of Mass — Damping Fource

$$F(t) = kx(t) + M\frac{d^2x(t)}{dt^2} + k_d \cdot \frac{dx(t)}{dt}$$

Modelling of a General System

- **A dynamic system can be represented in general by the differential equation:**

$$a_n\frac{d^nx}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + \underbrace{a_2\frac{d^2x}{dt^2} + \underbrace{a_1\frac{dx}{dt} + \overbrace{a_0x = F(t)}^{0^{th}\ order}}_{1^{st}\ order}}_{2^{nd}\ order}$$

$F(t)$ ≡ Input or the forcing function,

$x(t)$ ≡ Output or the response of the system,

a's ≡ Constants, physical system parameters

↬ **Order of a system is designated by the order of the D.E.**

First Order System

$$\tau \frac{dx}{dt} + x = \Bbbk\, F(t)$$

$\tau = [a_1/a_o] \iff$ time constant of the system,

$a_o \iff$ dissipation (electric or thermal resistance).

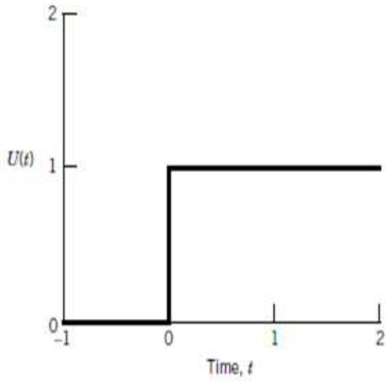
$a_1 \iff$ storage (electric or thermal capacitance).

↬ Example: Thermometer, capacitor etc.

Step Function Input

The step function, $AU(t)$, is defined as

$$AU(t) = 0 \quad t \le 0^-$$
$$AU(t) = A \quad t \ge 0^+$$



Response of a 1st Order System: Step Input

$$x = x_o,\ F = 0 : t = 0; \qquad F(t) = A : t > 0$$

$$\tau \frac{dx}{dt} + x = \Bbbk\, F(t)$$

$$\Longrightarrow \quad \tau \frac{dx}{dt} + x = A\Bbbk$$

$$\Longrightarrow \underbrace{x(t)}_{\text{Time response}} = \underbrace{\left[x_o - A\Bbbk\right] e^{-t/\tau}}_{\text{Transient response}} + \underbrace{A\Bbbk}_{\text{Steady response}}$$

- ▶ $x(t \to \infty) = A\Bbbk = x_\infty \Longleftarrow$ Steady State Response
- ▶ Error, $e_m = x_\infty - x(t) = (x_\infty - x_o)e^{-t/\tau}$
- ▶ Non-dimensional Error, $e_m/(x_\infty - x_o) = e^{-t/\tau}$

Response of a 1st Order System: Step Input

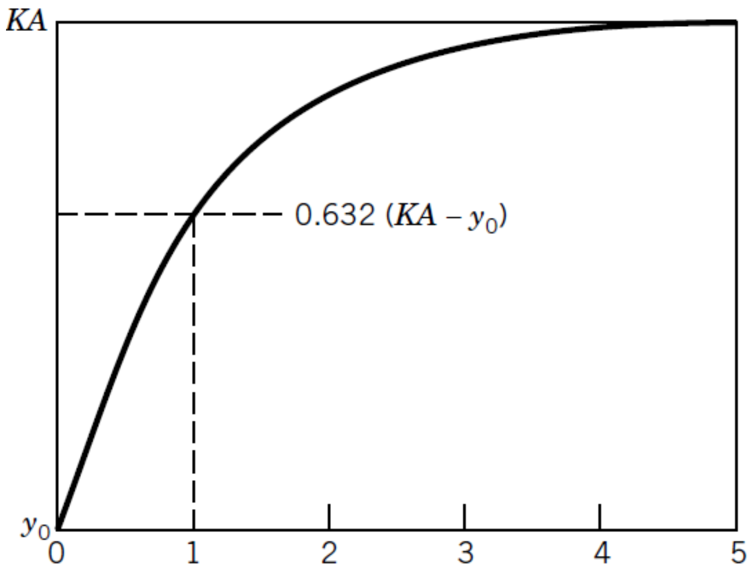
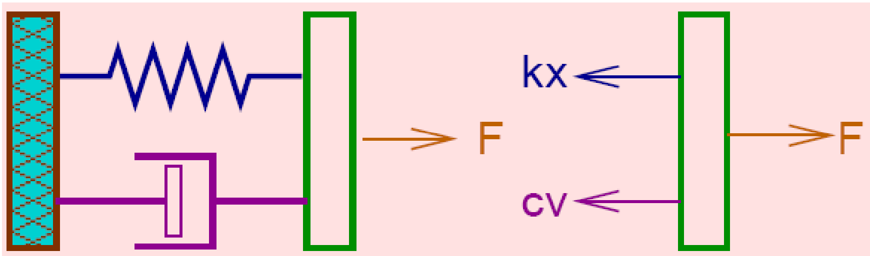


Figure 3.6 First-order system time response to a step function input: the time response, $y(t)$.

Response of a 1st Order System: Step Input

- **Time Constant,** τ - time required to complete 63.2% of the process.
- **Rise Time,** T_r - time required to achieve response from 10% to 90% of final value.
  - For first order system, $T_r = 2.31\tau - 0.11\tau = 2.2\tau$.
- **Settling Time,** T_s - the time for the response to reach, and stay within 2% of its final value.
  - For first order system, $T_s = 4\tau$.
- Process is assumed to be completed when $t \geq 5\tau$.
- Faster response is associated with shorter τ.

Second Order System



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$$F - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2} \Longrightarrow m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$$

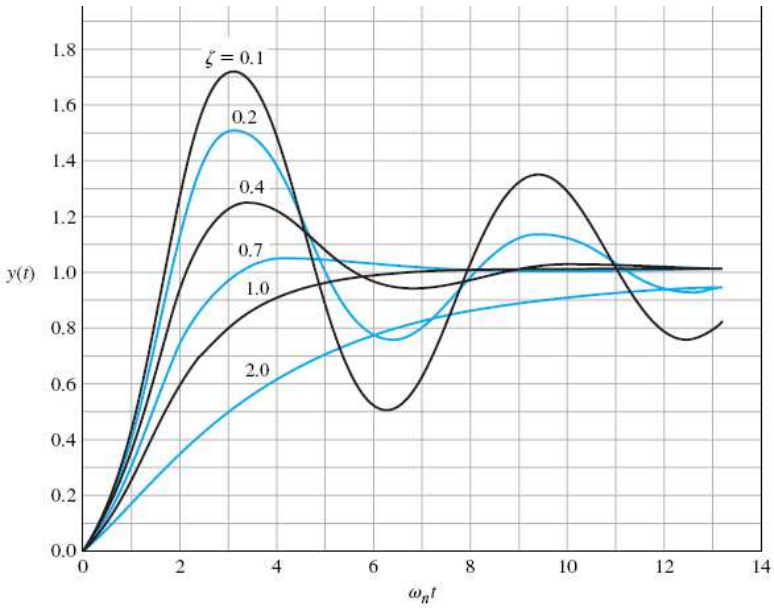
$$\omega_n \equiv \sqrt{\frac{k}{m}} \Longleftrightarrow \text{undamped natural frequency (rad/s)}$$

$$c_c \equiv 2\sqrt{mk} \Longleftrightarrow \text{critical damping coefficient}$$

$$\zeta \equiv c/c_c \Longleftrightarrow \text{damping ratio}$$

Response of a 2nd Order System

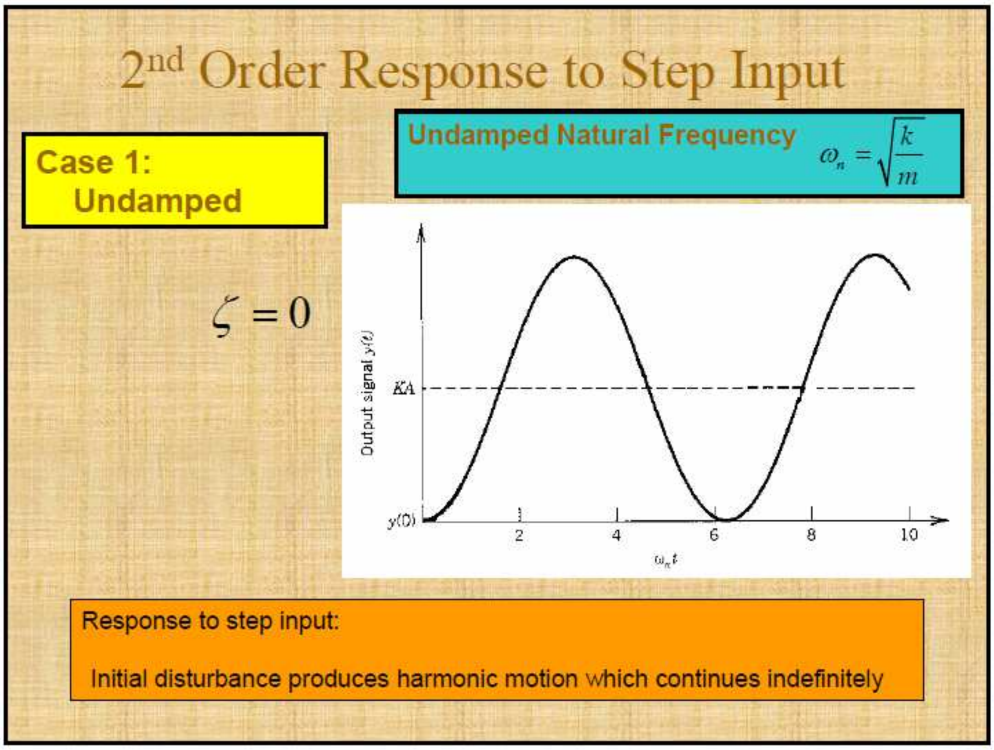
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \Longrightarrow \frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + 2\frac{\zeta}{\omega_n}\frac{dx}{dt} + x = \frac{F(t)}{k}$$



Response of a 2nd Order System

- Steady state position is obtained after a long period of time.
- Underdamped system ($\zeta < 1$): response overshoots the steady-state value initially, & then eventually decays to the steady-state value. The smaller the value of ζ, the larger the overshoot.
- Critical damping ($\zeta = 1$): an exponential rise occurs to approach the steady-state value without any overshoot.
- Overdamped ($\zeta > 1$): the system approaches the steady-state value without overshoot, but at a slower rate.

Measuring System Response



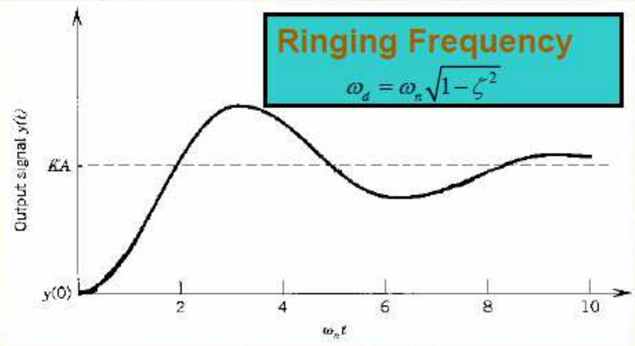
Measuring System Response

2nd Order Response to Step Input

Case 2:
Underdamped

$$0 < \zeta < 1$$

$$\zeta = 0.25$$



Ringing Frequency of System $[\omega_d]$

Defn: The frequency of free oscillations of a damped system. A function of natural frequency and damping ratio

Response : Displacement response overshoots the steady-state value initially and then eventually decays to steady state value

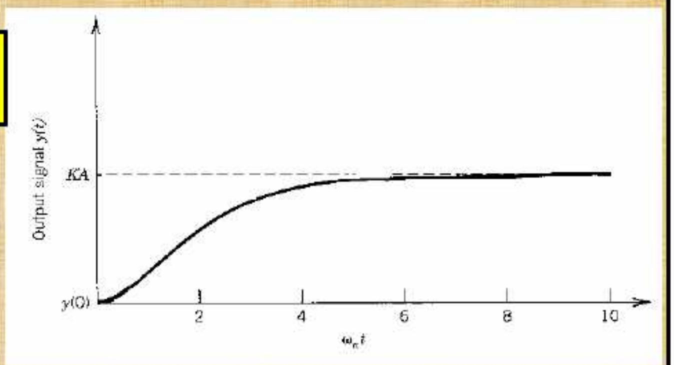
$$y(t) = KA - KAe^{-\zeta\omega_n t} \cdot \left[\frac{\zeta}{(1-\zeta^2)^{1/2}} \sin(\omega_n\sqrt{1-\zeta^2} \cdot t) + \cos(\omega_n\sqrt{1-\zeta^2} \cdot t) \right]$$

Measuring System Response

2nd Order Response to Step Input

Case 3:
Critically Damped

$$\zeta = 1$$



Response: An exponential rise occurs to approach the steady state value without overshooting it

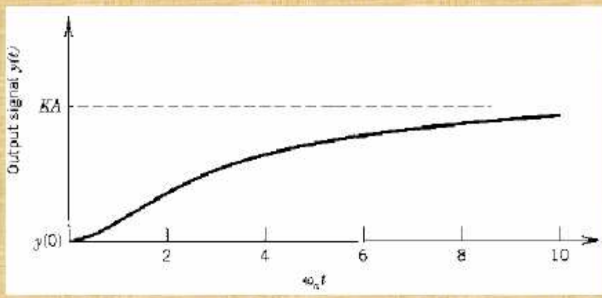
$$y(t) = KA - KA(1+\omega_n t)e^{-\omega_n t}$$

Measuring System Response

2nd Order Response to Step Input

Case 4: Overdamped

$$\zeta > 1$$



Response: System approaches the steady state value at a slow rate with no overshoot

$$y(t) = KA - KA \cdot \left[\frac{\zeta + \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}} e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + \frac{\zeta - \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}} e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \right]$$