Phys 201

Matrices and Determinants

1.1 Matrices

1.2 Operations of matrices

1.3 Types of matrices

1.4 Properties of matrices

1.5 Determinants

1.6 Inverse of a 3×3 matrix

1.1 Matrices

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix}$$

Both A and B are examples of matrix. A matrix is a rectangular array of numbers enclosed by a pair of bracket.

Why matrix?

1.1 Matrices

Consider the following set of equations:

$$\begin{cases} x+y=7, \\ 3x-y=5. \end{cases}$$

It is easy to show that $x=3$ and $y=4$.

How about solving

$$\begin{cases} x+y-2z=7, \\ 2x-y-4z=2, \\ -5x+4y+10z=1, \\ 3x-y-6z=5. \end{cases}$$

Matrices can help…

1.1 Matrices

In the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

- numbers a_{ij} are called *elements*. First subscript indicates the row; second subscript indicates the column. The matrix consists of mn elements
- It is called "the $m \times n$ matrix $A = [a_{ij}]$" or simply "the matrix A " if number of rows and columns are understood.

1.1 Matrices

Square matrices

- When $m = n$, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

- A is called a "square matrix of order n" or "n-square matrix"
- elements a_{11}, a_{22}, a_{33},…, a_{nn} called diagonal elements.
- $\sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}$ is called the *trace* of A.

1.1 Matrices

Equal matrices

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal $(A = B)$ iff each element of A is equal to the corresponding element of B, i.e., $a_{ij} = b_{ij}$ for $1 \le i \le m$, $1 \le j \le n$.

- *iff* pronouns "if and only if"

  if $A = B$, it implies $a_{ij} = b_{ij}$ for $1 \le i \le m$, $1 \le j \le n$;

  if $a_{ij} = b_{ij}$ for $1 \le i \le m$, $1 \le j \le n$, it implies $A = B$.

1.1 Matrices

Equal matrices

Example: $A=\begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Given that $A=B$, find a, b, c and d.

if $A=B$, then $a=1$, $b=0$, $c=$ -4 and $d=2$.

1.1 Matrices

Zero matrices

- Every element of a matrix is zero, it is called a zero matrix, i.e.,

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix}$$

1.2 Operations of matrices

Sums of matrices

- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then $A + B$ is defined as a matrix $C = A + B$, where $C = [c_{ij}]$, $c_{ij} = a_{ij} + b_{ij}$ for $1 \le i \le m$, $1 \le j \le n$.

Example: if $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$

Evaluate $A + B$ and $A - B$.

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0+(-1) & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0-(-1) & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

1.2 Operations of matrices

Sums of matrices

- Two matrices of the <u>same</u> order are said to be *conformable* for addition or subtraction.

- Two matrices of <u>different</u> orders cannot be added or subtracted, e.g.,

$$\begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix}$$

are NOT conformable for addition or subtraction.

1.2 Operations of matrices

Scalar multiplication

- Let λ be any scalar and $A = [a_{ij}]$ is an $m \times n$ matrix. Then $\lambda A = [\lambda a_{ij}]$ for $1 \le i \le m$, $1 \le j \le n$, i.e., each element in A is multiplied by λ.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$. Evaluate $3A$.

$$3A = \begin{bmatrix} 3\times1 & 3\times2 & 3\times3 \\ 3\times0 & 3\times1 & 3\times4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 12 \end{bmatrix}$$

- In particular, $\lambda = -1$, i.e., $-A = [-a_{ij}]$. It's called the *negative* of A. Note: $A - A = 0$ is a zero matrix

1.2 Operations of matrices

Properties

Matrices A, B and C are conformable,

- $A + B = B + A$ (commutative law)
- $A + (B + C) = (A + B) + C$ (associative law)
- $\lambda(A + B) = \lambda A + \lambda B$, where λ is a scalar (distributive law)

Can you prove them?

1.2 Operations of matrices

Properties

Example: Prove $\lambda(A+B) = \lambda A + \lambda B$.

Let $C = A + B$, so $c_{ij} = a_{ij} + b_{ij}$.

Consider $\lambda c_{ij} = \lambda\,(a_{ij} + b_{ij}) = \lambda a_{ij} + \lambda b_{ij}$, we have,
$\lambda C = \lambda A + \lambda B$.

Since $\lambda C = \lambda(A+B)$, so $\lambda(A+B) = \lambda A + \lambda B$

1.2 Operations of matrices

Matrix multiplication

- If $A = [a_{ij}]$ is a $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix, then AB is defined as a $m \times n$ matrix $C = AB$, where $C = [c_{ij}]$ with

$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj}$ for $1 \le i \le m$, $1 \le j \le n$.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$ and $C = AB$.
Evaluate c_{21}.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \qquad c_{21} = 0 \times (-1) + 1 \times 2 + 4 \times 5 = 22$$

1.2 Operations of matrices

Matrix multiplication

Example: $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix}$, Evaluate $C = AB$.

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix} \Rightarrow \begin{cases} c_{11} = 1\times(-1)+2\times 2+3\times 5 = 18 \\ c_{12} = 1\times 2+2\times 3+3\times 0 = 8 \\ c_{21} = 0\times(-1)+1\times 2+4\times 5 = 22 \\ c_{22} = 0\times 2+1\times 3+4\times 0 = 3 \end{cases}$$

$$C = AB = \begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix} = \begin{bmatrix}18 & 8\\ 22 & 3\end{bmatrix}$$

1.2 Operations of matrices

Matrix multiplication

- In particular, A is a $1 \times m$ matrix and B is a $m \times 1$ matrix, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}$$

then $C = AB$ is a scalar. $C = \sum_{k=1}^{m} a_{1k} b_{k1} = a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1m}b_{m1}$

1.2 Operations of matrices

Matrix multiplication

- BUT BA is a $m \times m$ matrix!

$$BA = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} & b_{11}a_{12} & \dots & b_{11}a_{1m} \\ b_{21}a_{11} & b_{21}a_{12} & & b_{21}a_{1m} \\ \vdots & & \ddots & \\ b_{m1}a_{11} & b_{m1}a_{12} & & b_{m1}a_{1m} \end{bmatrix}$$

- So $AB \neq BA$ in general !

1.2 Operations of matrices

Properties

Matrices A, B and C are conformable,

- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- $A(BC) = (AB)\,C$

However

- $AB \neq BA$ in general
- $AB = 0$ NOT necessarily imply $A = 0$ or $B = 0$
- $AB = AC$ NOT necessarily imply $B = C$

1.2 Operations of matrices

Properties

Example: Prove $A(B + C) = AB + AC$ where A, B and C are n-square matrices

Let $X = B + C$, so $x_{ij} = b_{ij} + c_{ij}$. Let $Y = AX$, then

$$y_{ij} = \sum_{k=1}^{n} a_{ik} x_{kj} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$

$$= \sum_{k=1}^{n} (a_{ik} b_{kj} + a_{ik} c_{kj}) = \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj}$$

So $Y = AB + AC$; therefore, $A(B + C) = AB + AC$

1.3 Types of matrices

- Identity matrix
- The inverse of a matrix
- The transpose of a matrix
- Symmetric matrix
- Orthogonal matrix

1.3 Types of matrices

Identity matrix

- A square matrix whose elements $a_{ij} = 0$, for $i > j$ is called upper triangular, i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

- A square matrix whose elements $a_{ij} = 0$, for $i < j$ is called lower triangular, i.e.,

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & & 0 \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

1.3 Types of matrices

Identity matrix

- Both upper and lower triangular, i.e., $a_{ij} = 0$, for $i \neq j$, i.e.,

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

is called a diagonal matrix, simply

$$D = \mathrm{diag}[a_{11}, a_{22}, \ldots, a_{nn}]$$

1.3 Types of matrices

Identity matrix

- In particular, $a_{11} = a_{22} = \ldots = a_{nn} = 1$, the matrix is called identity matrix.
- Properties: $AI = IA = A$

Examples of identity matrices: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1.3 Types of matrices

Special square matrix

- $AB \neq BA$ in general. However, if two square matrices A and B such that $AB = BA$, then A and B are said to be *commute*.

Can you suggest two matrices that must commute with a square matrix A?

Ans: A itself, the identity matrix, ..

- If A and B such that $AB = -BA$, then A and B are said to be *anti-commute*.

1.3 Types of matrices

The inverse of a matrix

- If matrices A and B such that $AB = BA = I$, then B is called the inverse of A (symbol: A^{-1}); and A is called the inverse of B (symbol: B^{-1}).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Show B is the the inverse of matrix A.

Ans: Note that $AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Can you show the details?

1.3 Types of matrices

The transpose of a matrix

- The matrix obtained by interchanging the rows and columns of a matrix A is called the transpose of A (write A^T).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The transpose of A is $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- For a matrix $A = [a_{ij}]$, its transpose $A^T = [b_{ij}]$, where $b_{ij} = a_{ji}$.

1.3 Types of matrices

Symmetric matrix

- A matrix A such that $A^T = A$ is called symmetric, i.e., $a_{ji} = a_{ij}$ for all i and j.
- $A + A^T$ must be symmetric. Why?

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric.

- A matrix A such that $A^T = -A$ is called skew-symmetric, i.e., $a_{ji} = -a_{ij}$ for all i and j.
- $A - A^T$ must be skew-symmetric. Why?

1.3 Types of matrices

Orthogonal matrix

- A matrix A is called orthogonal if $AA^T = A^TA = I$, i.e., $A^T = A^{-1}$

Example: prove that $A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$ is orthogonal.

Since, $A^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$. Hence, $AA^T = A^TA = I$.

Can you show the details?

We'll see that orthogonal matrix represents a rotation in fact!

1.4 Properties of matrix

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^T = A$ and $(\lambda A)^T = \lambda A^T$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

1.4 Properties of matrix

Example: Prove $(AB)^{-1} = B^{-1}A^{-1}$.

Since $(AB)\,(B^{-1}A^{-1}) = A(B\,B^{-1})A^{-1} = I$ and

$(B^{-1}A^{-1})\,(AB) = B^{-1}(A^{-1}\,A)B = I$.

Therefore, $B^{-1}A^{-1}$ is the inverse of matrix AB.

1.5 Determinants

Determinant of order 2

Consider a 2×2 matrix: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

- Determinant of A, denoted $|A|$, is a number and can be evaluated by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

1.5 Determinants

Determinant of order 2

- easy to remember (for order 2 only)..

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11}a_{22} - a_{12}a_{21}$$

\- +

Example: Evaluate the determinant: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

1.5 Determinants

The following properties are true for determinants of *any* order.

- If every element of a row (column) is zero, e.g., $\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 1\times 0 - 2\times 0 = 0$, then $|A| = 0$.

- $|A^T| = |A|$ ← determinant of a matrix = that of its transpose

- $|AB| = |A||B|$

1.5 Determinants

Example: Show that the determinant of any orthogonal matrix is either $+1$ or -1.

For any orthogonal matrix, $A\,A^T = I$.

Since $|AA^T| = |A||A^T| = 1$ and $|A^T| = |A|$, so $|A|^2 = 1$ or $|A| = \pm 1$.

1.5 Determinants

For any 2x2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Its inverse can be written as $A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Example: Find the inverse of $A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$

The determinant of A is -2

Hence, the inverse of A is $A^{-1} = \begin{bmatrix} -1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

How to find an inverse for a 3x3 matrix?

1.5 Determinants of order 3

Consider an example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Its determinant can be obtained by:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + 9\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 3(-3) - 6(-6) + 9(-3) = 0$$

You are encouraged to find the determinant by using other rows or columns

1.6 Inverse of a 3×3 matrix

Cofactor matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$

The cofactor for each element of matrix A:

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = \begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

1.6 Inverse of a 3×3 matrix

Cofactor matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is then given by:

$$\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

1.6 Inverse of a 3×3 matrix

Inverse matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is given by:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}^T = \frac{1}{22}\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 12/11 & -6/11 & -1/11 \\ 5/22 & 3/22 & -5/22 \\ -2/11 & 1/11 & 2/11 \end{bmatrix}$$