Faculty of Engineering
Mechanical Engineering Department

MATH 2140
Numerical Methods

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B.1. Fixed Point Iteration

- Also known as **one-point iteration** or **successive substitution**

- To find the root for $f(x) = 0$, we reformulate $f(x) = 0$ so that there is an x on one side of the equation.

$$f(x) = 0 \quad \Leftrightarrow \quad g(x) = x$$

- If we can solve $g(x) = x$, we solve $f(x) = 0$.
  - x is known as the fixed point of $g(x)$.
- We solve $g(x) = x$ by computing

$$x_{i+1} = g(x_i) \quad \text{with } x_0 \text{ given}$$

until x_{i+1} converges to x.

Fixed Point Iteration – Example

$$f(x) = x^2 + 2x - 3 = 0$$

$$x^2 + 2x - 3 = 0 \quad \Rightarrow \quad 2x = 3 - x^2 \quad \Rightarrow \quad x = \frac{3 - x^2}{2}$$

$$\Rightarrow x_{i+1} = g(x_i) = \frac{3 - x_i^2}{2}$$

Reason: If x converges, i.e. $x_{i+1} \rightarrow x_i$

$$x_{i+1} = \frac{3 - x_i^2}{2} \rightarrow x_i = \frac{3 - x_i^2}{2}$$

$$\Rightarrow x_i^2 + 2x_i - 3 = 0$$

- There are infinite ways to construct $g(x)$ from $f(x)$.

For example, $f(x) = x^2 - 2x - 3 = 0$ (ans: $x = 3$ or -1)

Case a:

$$x^2 - 2x - 3 = 0$$
$$\Rightarrow x^2 = 2x + 3$$
$$\Rightarrow x = \sqrt{2x+3}$$
$$\Rightarrow g(x) = \sqrt{2x+3}$$

Case b:

$$x^2 - 2x - 3 = 0$$
$$\Rightarrow x(x-2) - 3 = 0$$
$$\Rightarrow x = \frac{3}{x-2}$$
$$\Rightarrow g(x) = \frac{3}{x-2}$$

Case c:

$$x^2 - 2x - 3 = 0$$
$$\Rightarrow 2x = x^2 - 3$$
$$\Rightarrow x = \frac{x^2-3}{2}$$
$$\Rightarrow g(x) = \frac{x^2-3}{2}$$

So which one is better?

Case a

$$x_{i+1} = \sqrt{2x_i + 3}$$

1. $x_0 = 4$
2. $x_1 = 3.31662$
3. $x_2 = 3.10375$
4. $x_3 = 3.03439$
5. $x_4 = 3.01144$
6. $x_5 = 3.00381$

Converge!

Case b

$$x_{i+1} = \frac{3}{x_i - 2}$$

1. $x_0 = 4$
2. $x_1 = 1.5$
3. $x_2 = -6$
4. $x_3 = -0.375$
5. $x_4 = -1.263158$
6. $x_5 = -0.919355$
7. $x_6 = -1.02762$
8. $x_7 = -0.990876$
9. $x_8 = -1.00305$

Converge, but slower

Case c

$$x_{i+1} = \frac{x_i^2 - 3}{2}$$

1. $x_0 = 4$
2. $x_1 = 6.5$
3. $x_2 = 19.625$
4. $x_3 = 191.070$

Diverge!

How to choose $g(x)$?

- Can we know which $g(x)$ would converge to solution before we do the computation?

Convergence of Fixed Point Iteration

According to the derivative mean-value theorem, if $g(x)$ and $g'(x)$ are continuous over an interval $x_i \le x \le \alpha$, there exists a value $x = c$ within the interval such that

$$g'(c) = \frac{g(\alpha) - g(x_i)}{\alpha - x_i} \qquad (7)$$

From (1) and (6), we have $\delta_i = \alpha - x_i$ *and* $\delta_{i+1} = g(\alpha) - g(x_i)$

Thus $(7) \Rightarrow g'(c) = \dfrac{\delta_{i+1}}{\delta_i} \Rightarrow \boxed{\delta_{i+1} = g'(c)\delta_i}$

- Therefore, if $|g'(c)| < 1$, the error decreases with each iteration. If $|g'(c)| > 1$, the error increase.
- If the derivative is positive, the iterative solution will be monotonic.
- If the derivative is negative, the errors will oscillate.

(a) $|g'(x)| < 1$, $g'(x)$ is +ve

$\Rightarrow$ converge, monotonic

(b) $|g'(x)| < 1$, $g'(x)$ is -ve

$\Rightarrow$ converge, oscillate

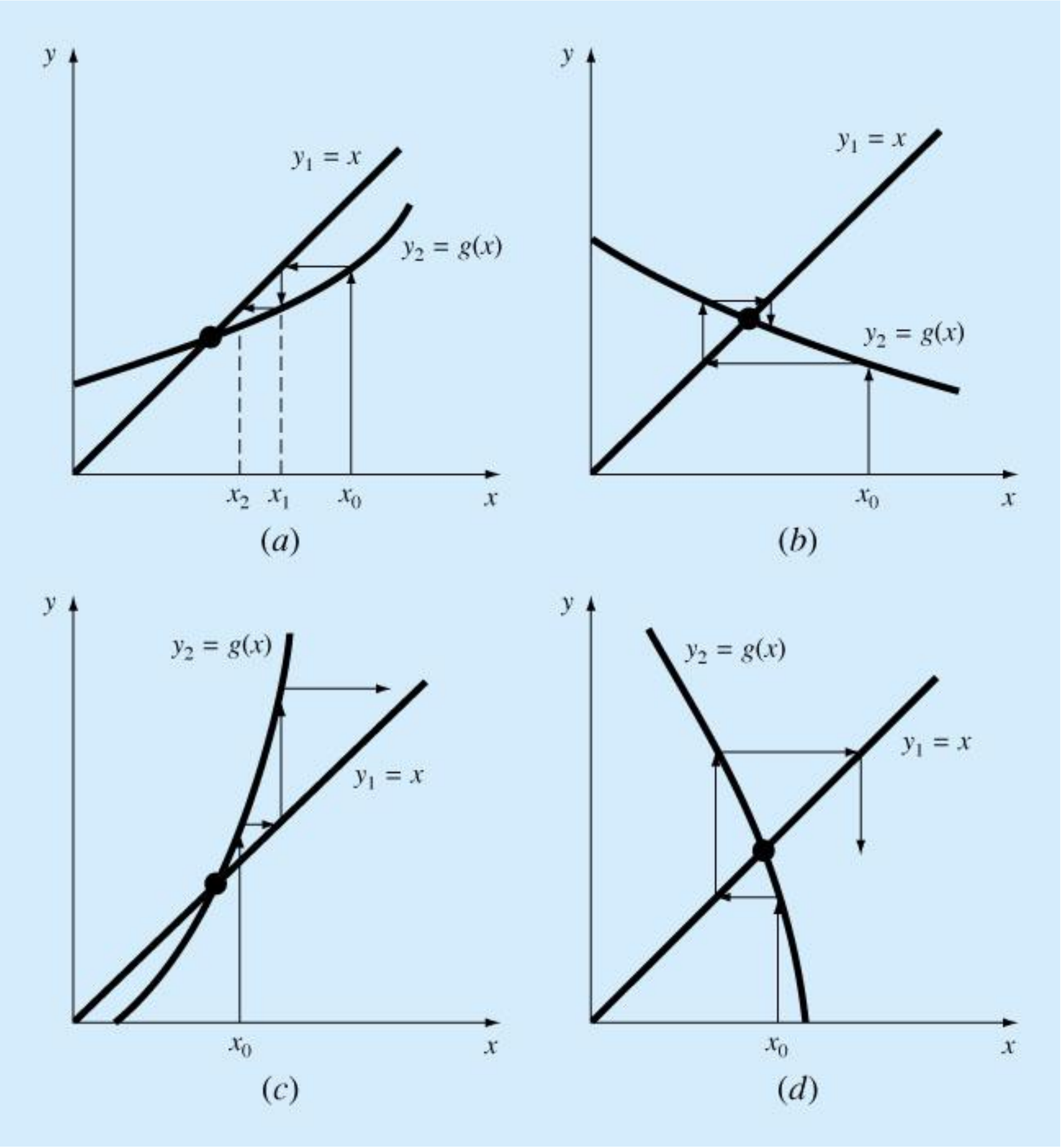
(c) $|g'(x)| > 1$, $g'(x)$ is +ve

$\Rightarrow$ diverge, monotonic

(d) $|g'(x)| > 1$, $g'(x)$ is -ve

$\Rightarrow$ diverge, oscillate

Demo



1. Find the root of **(cos[x])-(x * exp[x]) = 0**

Consider **g(x) = cos[x]/exp[x]**

The graph of **g(x)** and **x** are given in the figure.

let the initial guess $\mathbf{x_0}$ be

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...	31	32
x_i	1	0.199	0.803	0.311	0.698	0.381	0.634	0.427	0.594	0.458	0.567	0.478	0.551	0.491	...	0.518	**0.518**

2. Find the root of $\mathbf{x^4 - x - 10 = 0}$

Consider $\mathbf{g(x) = (x + 10)^{1/4}}$

The graph of **g(x)** and **x** are given in the figure.

let the initial guess $\mathbf{x_0}$ be **4.0**

i	0	1	2	3	4	5	6
x_i	4.0	1.93434	1.85866	1.8557	1.85559	1.85558	**1.85558**

That is for $\mathbf{g(x) = (x + 10)^{1/4}}$ the itirative process is converged to **1.85558**.

3. Find the root of **x-exp[-x] = 0**

Consider **g(x) = exp[-x]**

The graph of **g(x)** and **x** are given in the figure..

let the initial guess $\mathbf{x_0}$ be **3.0**

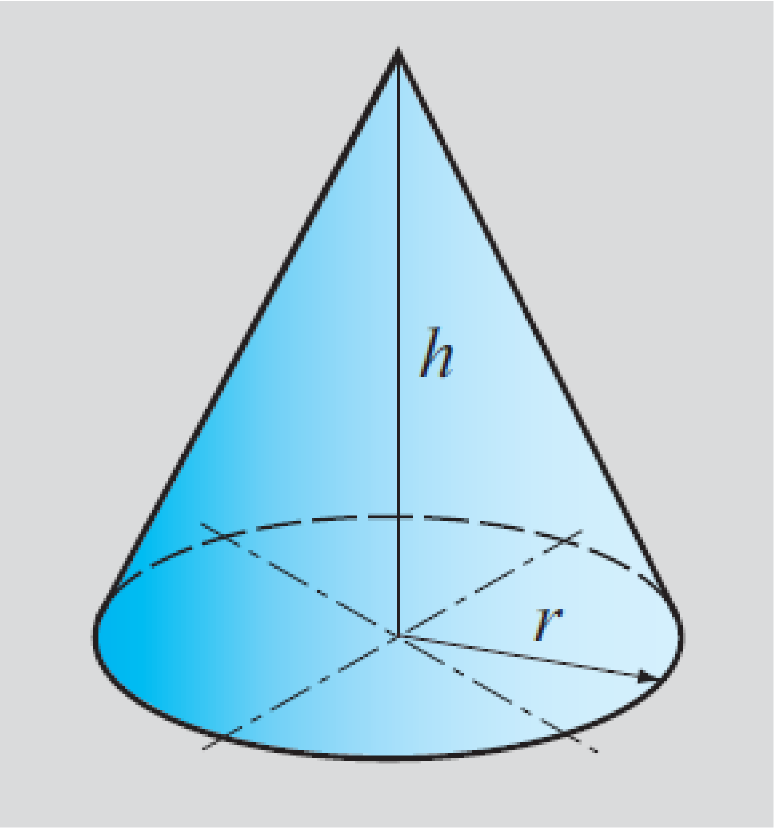
i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x_i	3.0	0.05	0.951	0.386	0.68	0.507	0.602	0.548	0.578	0.561	0.571	0.565	0.568	0.567	**0.567**

That is for **g(x) = exp[-x]** the itirative process is converged to **0.567**.

3.4 The lateral surface area, S, of a cone is given by:

$$S = \pi r\sqrt{r^2 + h^2}$$

where r is the radius of the base and h is the height. Determine the radius of a cone that has a surface area of 1800 m^2 and a height of 25 m. Solve by using the fixed-point iteration method with $r = S/(\pi\sqrt{r^2 + h^2})$ as the iteration function. Start with $r = 17$ m and calculate the first four iterations.



Solution

Using 5 significant digits.

$$r_{i+1} = \frac{S}{\pi\sqrt{r_i^2 + h^2}}, \quad S = 1800\,\text{m}^2, \quad h = 25\ \text{m}$$

$$i = 1, \quad r_1 = 17, \quad r_2 = \frac{1800}{\pi\sqrt{17^2 + 25^2}} = 18.952$$

$$i = 2, \quad r_2 = 18.952, \quad r_3 = \frac{1800}{\pi\sqrt{18.952^2 + 25^2}} = 18.264$$

$$i = 3, \quad r_3 = 18.264, \quad r_4 = \frac{1800}{\pi\sqrt{18.264^2 + 25^2}} = 18.506$$

$$i = 4, \quad r_4 = 18.506, \quad r_5 = \frac{1800}{\pi\sqrt{18.506^2 + 25^2}} = 18.421$$