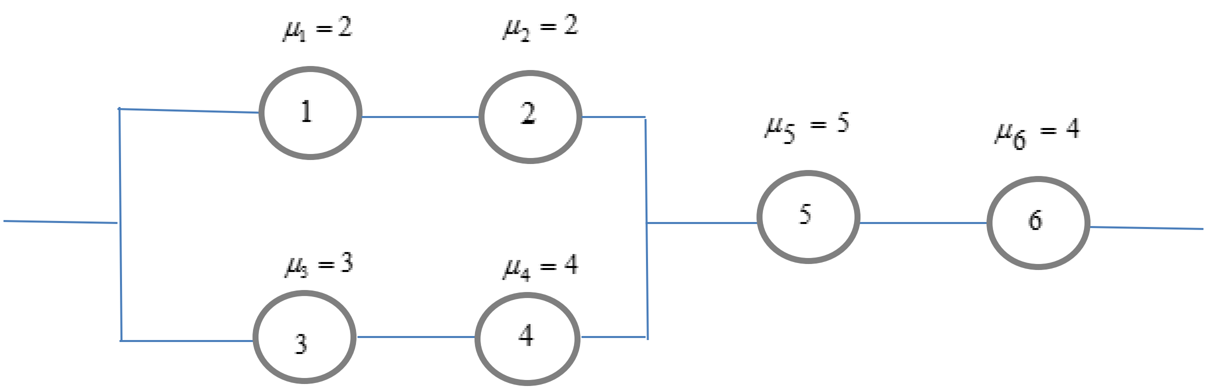
King Saud University
College of Sciences
Department of Mathematics

Mid-Term Exam, S1 1442
M 507 - Advanced Operation Research
Time: 2 hours

Answer the following questions

Q1: [8]

Consider the following system configuration diagram for 6 components, their lifetimes follow exponential distributions with means μ_i, $i = 1, 2, \ldots, 6$ that measured in thousand hours.



Find each of the following:

(a) The system reliability

(b) What is the reliability for the system to achieve a life at least 2000 hours?

Q2: [12]

(a) Let $T > 0$ be a random variable, its hazard function is $h(t)$ Prove that each of the following:

i) The reliability function is given by $R(t) = \exp\left\{-\int_0^t h(u)du\right\}$, then determine $R(t)$ if $T \sim \exp(\lambda)$.

ii) The average failure rate over the interval (t_1, t_2) is $\bar{\lambda} = \frac{\ln[R(t_1)] - \ln[R(t_2)]}{t_2 - t_1}$, then derive a formula for $\bar{\lambda}$ if T follows a two-parameter Weibull distribution.

(b) The life of a product follows a Weibull distribution with a shape parameter of 3.5 and a scale parameter of 500 hours. Find each of the following:

i) The probability that the product will perform satisfactory for at least 100 hours?

ii) Compute the instantaneous failure rate at its characteristic value and the average failure rate over the time interval $(400, 600)$.

iii) Compute the mode, the tenth percentile, the median, the MTTF and the variance for this distribution.

Q3: [10]

a) The reliability of each of 10 identical components is 0.95. If these components are part of a system for which at least six components must function for the system to function, compute the system reliability. If this system could be replaced by a parallel combination of five identical components, what would the reliability of those components have to be to give the same system reliability as the 6 out of 10 system?

b) The life of a product follows a lognormal distribution. The median life is 1000 hours. The probability that the product will survive a life of 2000 hours is 10%. Compute the expected life.

Model Answer

Q1: [8]

(a)

For parallel components, $R_{sys} = 1 - \prod_{i=1}^{n}(1 - R_i)$

For series components, $R_{sys} = \prod_{i=1}^{n} R_i$

For the given diagram,

$$\because R_{1234}(t) = 1 - [1 - R_{12}(t)][1 - R_{34}(t))]$$
$$and \quad \because \; R_{sym}(t) = R_{1234}(t) R_5(t) R_6(t)$$
$$\therefore \; R_{sym}(t) \; = [1 - (1 - e^{-t})(1 - e^{-7t/12})]e^{-9t/20}$$

(b)

$$R_{sym}(2) = [1 - (1 - e^{-2})(1 - e^{-7/6})]e^{-0.9}$$
$$\approx 0.1645$$

Q2: [12]

(a)

(i)

To prove that $\therefore R(t) = \exp\left\{-\int_0^t h(u)du\right\}$

The hazard function or failure rate is given by $h(\text{t}) = \dfrac{f_T(T)}{R(t)}$

$$\Rightarrow \quad \begin{aligned} h(t) &= \frac{d}{dt}F_T(t).\frac{1}{R(t)} \\ h(t) &= -\frac{d}{dt}R(t).\frac{1}{R(t)} \end{aligned}$$

$$\Rightarrow \int_0^t \frac{dR(u)}{R(u)} = -\int_0^t h(u)du$$

$$\therefore \; [\ln R(u)]_0^t = -\int_0^t h(u)du$$

$$\therefore \; \ln R(t) - \ln R(0) = -\int_0^t h(u)du \;, \;\; \because \;\; \ln R(0) = \ln(1) = 0$$

$$\therefore \; \ln R(t) = -\int_0^t h(u)du$$

$$\therefore \; R(t) = e^{-\int_0^t h(u)du}$$

$$\therefore R(t) = \exp\left\{-\int_0^t h(u)du\right\}$$

$$\because T \sim \exp(\lambda)$$

$$\therefore R(t) = \exp\left\{-\int_0^t \lambda du\right\}$$

$$= \exp(-\lambda t)$$

(ii)

$$\because \; R(t) = e^{-\int_0^t h(u)du}$$

$$\therefore \; R(t) = e^{-\Lambda(t)}, \; \Lambda(t) = \int_0^t h(u)du$$

The average failure rate over the interval (t_1, t_2) is given by

$$\bar{\lambda} = \frac{\int_{t_1}^{t_2} h(u)du}{t_2 - t_1}$$

$$\therefore \;\; \bar{\lambda} = \frac{\int_0^{t_2} h(u)du - \int_0^{t_1} h(u)du}{t_2 - t_1} = \frac{\Lambda(t_2) - \Lambda(t_1)}{t_2 - t_1} \qquad (1)$$

$$\therefore\ \bar{\lambda} = \frac{\ln[R(t_1)] - \ln[R(t_2)]}{t_2 - t_1} \qquad (2)$$

For 2p Weibull

$$\therefore\ R(t) = e^{-\Lambda(t)},\ \Lambda(t) = \int_0^t \lambda(u)du$$

$$R(t) = \exp[-(\tfrac{t}{\eta})^{\beta}],\ R(t) = e^{-\Lambda(t)}$$

$$\therefore\ \Lambda(t) = (\tfrac{t}{\eta})^{\beta} \qquad (3)$$

$$\therefore\ (1), (3) \Rightarrow \quad \bar{\lambda} = \frac{t_2{}^{\beta} - t_1{}^{\beta}}{\eta^{\beta}(t_2 - t_1)}$$

(b)

i)

$$\begin{aligned} \Pr(T > 100) &= \mathrm{R}(100) \\ &= \exp[-(\tfrac{t}{\eta})^{\beta}] \\ &= e^{-(\frac{100}{500})^{3.5}} \\ &= 0.9964 \end{aligned}$$

ii)

The instantaneous failure rate is given by

$$\lambda(t) = \frac{\beta}{\eta^{\beta}} t^{\beta-1}$$

$$\begin{aligned} \lambda(500) &= \frac{3.5}{500^{3.5}} \times 500^{2.5} = \frac{3.5}{500} \\ &= 0.007 \end{aligned}$$

and the average failure rate is

$$\bar{\lambda} = \frac{t_2^{\beta} - t_1^{\beta}}{\eta^{\beta}(t_2 - t_1)}$$
$$= \frac{600^{3.5} - 400^{3.5}}{500^{3.5}(600 - 400)}$$
$$= 7.1749 \times 10^{-3}$$

iii) The Mode is given by

$$x_m = \eta\left(\frac{\beta - 1}{\beta}\right)^{1/\beta}$$
$$= 500\left(\frac{2.5}{3.5}\right)^{1/3.5}$$
$$= 454.17$$

The tenth percentile is

$$x_p = \left(\ln\left(\frac{1}{1-p}\right)\right)^{1/\beta} .\eta$$
$$= \left(\ln\left(\frac{1}{0.9}\right)\right)^{1/3.5} \times 500$$
$$= 262.87$$

Also, the median is

$$x_{0.50} = \left(\ln\left(\frac{1}{1-0.50}\right)\right)^{1/3.5} \times 500$$
$$= (\ln 2)^{2/7} \times 500$$
$$= 450.29$$

The MTTF for 2p Weibull is given by

$$\mu = \eta\, \Gamma\left(\frac{1}{\beta} + 1\right)$$
$$= \eta\, B_1$$
$$= 500 \times 0.8997$$
$$= 449.85$$

The variance is

$$\sigma^2 = \eta^2[B_2 - B_1^2]$$
$$=500^2 \times 0.0811$$
$$=20275$$

Q3: [10]

(a)

$$R_{sys} = \sum_{i=k}^{n} \binom{n}{i} R^i (1-R)^{n-i}$$
$$= \sum_{i=6}^{10} \binom{10}{i} 0.95^i (0.05)^{10-i}$$
$$= 0.0009648 + 0.010475 + 0.0746348 + 0.3151247 + 0.5987369$$
$$= 0.9999363$$

For parallel system consists of 5 identical components, $\mathrm{R}_{sys} = 1-(1-\mathrm{R})^5$

For $R_{sys} = 0.9999363$

$$\Rightarrow (1-R)^5 = 0.0000637$$
$$\Rightarrow R = 0.85518$$

(b)

$$x_{0.50} = 1000$$
$$P(X > 2000) = 0.1$$

$$x_{0.50} = \exp(\mu + z_{0.50}\sigma)$$
$$= \exp(\mu + 0 \times \sigma)$$
$$= \exp(\mu)$$

$$\exp(\mu) = 1000$$
$$\Rightarrow \mu = \ln(1000)$$
$$= 6.908$$

$$P(X > 2000) = 1 - F(2000)$$
$$= 1 - \Phi\left(\frac{\ln(2000) - 6.908}{\sigma}\right)$$
$$= 0.1$$

For $\Phi(z) = 0.9, \quad z \approx 1.28$

$$\therefore \quad \frac{\ln(2000) - 6.908}{\sigma} = 1.28$$
$$\therefore \quad \boldsymbol{\sigma} = 0.54$$

$$\therefore \quad E(X) = \exp(\boldsymbol{\mu} + 0.5\boldsymbol{\sigma}^2)$$
$$= \exp(6.908 + 0.5 \times 0.54^2)$$
$$= 1157.25$$