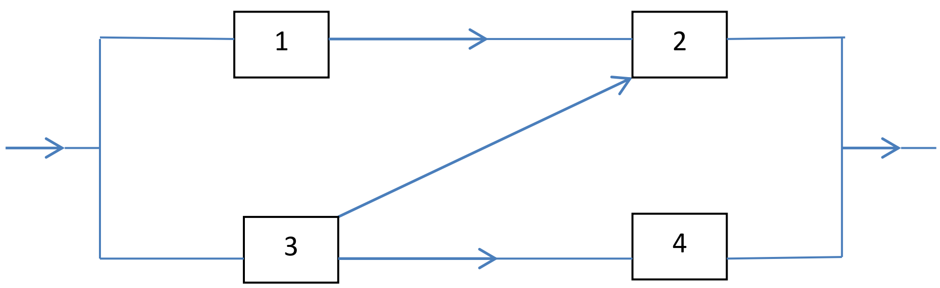
King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1 1441
M 507 - Advanced Operation Research
Time: 3 hours

Answer the following questions:

Q1: [7]

Consider the following system configuration diagram where each component i, i=1,2,3,4 follows the exponential distribution and the mean time to failure for each component is $\frac{1}{\lambda_i}$. Compute the reliability and the MTTF for this system. Also, determine them if the system components are identical.



Q2: [3+3]

Suppose that a series of experiments have shown that the current gains of certain transistors (which are proportional to the logarithm of I_o/I_i, the ratio of the output to the input current) are normally distributed. If current gain is measured in units for which it equals in (I_o/I_i), and if it is normally distributed with $\mu = 2$ and $\sigma^2 = 0.01$, then find each of the following:

a) $pr(6.1 \le \frac{I_o}{I_i} \le 8.2)$

b) The mean and the standard deviation of this distribution.

Q3: [11]

a) Derive the formulas for the mean and variance of the Weibull distribution.

b) The life of a product follows a Weibull distribution with a shape parameter of 2.5 and a scale parameter of 600 hours. Find each of the following:

i) The probability that the product will perform satisfactory for at least 100 hours?

ii) Compute the instantaneous failure rate at its characteristic value and the average failure rate over the time interval $(400, 600)$.

iii) Compute the mode, the tenth percentile, the median, the MTTF and the variance for this distribution.

Q4: [4+4]

a) Solve the following linear programming problems by using Simplex method.

$$\begin{aligned} \max \quad & z = 4x_1 + x_2 \\ \text{s.t} \quad & -x_1 + 2x_2 \leq 4 \\ & 2x_1 + 3x_2 \leq 12 \\ & x_1 - x_2 \leq 3 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

b)

$$\begin{aligned} \min \quad & z = 4x_1 - x_2 \\ \text{s.t} \quad & 2x_1 + x_2 \leq 8 \\ & x_2 \leq 5 \\ & x_1 - x_2 \leq 4 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Q5: [8]

a) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \text{pr}\{X_0 = i_0\}$

b) Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a good item is followed by another good item with probability α and is followed by a defective item with probability $1-\alpha$.

Similarly, a defective item is followed by another defective item with probability β and is followed by a good item with probability $1-\beta$. Answer each of the following:

i) If the first item is good, what is the probability that the first defective item to appear is the fifth item?

ii) If the first item is bad, what is the probability that the first good item to appear is the fifth item?

Model Answer

Q1: [8]

For parallel components, $R_{sys} = 1 - \prod_{i=1}^{n}(1 - R_i)$

For series components, $R_{sys} = \prod_{i=1}^{n} R_i$

For the given diagram,

$$\because R_{13}R_2 = [1-(1-R_1)(1-R_3)]R_2$$
$$R_{24}R_3 = [1-(1-R_2)(1-R_4)]R_3$$

$\therefore$ The system reliability where the components are independent is given by

$$R_{sys} = R_1R_2 + R_2R_3 + R_3R_4 - R_1R_2R_3 - R_2R_3R_4$$
$$= e^{-(\lambda_1+\lambda_2)t} + e^{-(\lambda_2+\lambda_3)t} + e^{-(\lambda_3+\lambda_4)t} - e^{-(\lambda_1+\lambda_2+\lambda_3)t} - e^{-(\lambda_2+\lambda_3+\lambda_4)t}$$

$$\text{MTTF} = \int_0^\infty R(t)dt$$
$$= \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_2+\lambda_3} + \frac{1}{\lambda_3+\lambda_4} - \frac{1}{\lambda_1+\lambda_2+\lambda_3} - \frac{1}{\lambda_2+\lambda_3+\lambda_4}$$

Also, you can determine R_{sys} by using the Decomposition Method where the component 2 will be chosen as the pivot element as follows.

$$R_{sys} = R_2R^+ + (1 - R_2)R^-$$
$$= R_2(R_1 + R_3 - R_1R_3) + R_3R_4(1 - R_2)$$
$$= R_1R_2 + R_2R_3 + R_3R_4 - R_1R_2R_3 - R_2R_3R_4$$

For identical components, $\lambda_i = \lambda$, i=1,2,3,4

$\Rightarrow$

$$R_{sys} = 3R^2 - 2R^3$$
$$= 3e^{-(2\lambda)t} - 2e^{-(3\lambda)t}$$

$$\text{MTTF} = \frac{3}{2\lambda} - \frac{2}{3\lambda}$$
$$= \frac{5}{6\lambda}$$

Q2: [3+3]

a)

$\because \ \frac{I_o}{I_i} \sim$ Log-Norm(2, 0.01)

$$\therefore \ pr(6.1 \le \frac{I_o}{I_i} \le 8.2) = F\left(\frac{\ln 8.2 - 2}{0.1}\right) - F\left(\frac{\ln 6.1 - 2}{0.1}\right)$$

=F(1.04) − F(-1.92)

=0.8508 − 0.0274

=0.8234

b)

Mean=exp $(\mu + \frac{1}{2}\sigma^2)$

Variance=exp$(2\mu + \sigma^2)$.[exp$(\sigma^2) - 1$]

Mean=7.4261, Var=0.5542 and Stand. Dev=0.7445

Q3: [3+8]

a) The p. d. f. for $T \sim \text{Weibull}(\eta, \beta)$ is

$$f(t) = \frac{\beta}{\eta}\left[\frac{t}{\eta}\right]^{\beta-1} \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right]$$

Let, $\alpha = \left(\frac{1}{\eta}\right)^{\beta} = \eta^{-\beta}$

$\Rightarrow f(t) = \alpha\beta\, t^{\beta-1} e^{-\alpha t^{\beta}}$

The Mean, $\mu = \int_0^{\infty} t \ \alpha\beta\, t^{\beta-1} e^{-\alpha t^{\beta}} dt$

Let, $u = \alpha t^{\beta}$

$\Rightarrow$ du=$\alpha\beta t^{\beta-1} dt$, $t = \left(\frac{u}{\alpha}\right)^{\frac{1}{\beta}}$

The Mean, $\mu = \int_0^\infty \left(\frac{u}{\alpha}\right)^{\frac{1}{\beta}} e^{-u} du$

$$\mu = \alpha^{\frac{-1}{\beta}} \int_0^\infty (u)^{\frac{1}{\beta}} e^{-u} du$$

$$\mu = \alpha^{\frac{-1}{\beta}} \Gamma(1+\frac{1}{\beta})$$

$$\therefore \ \mu = \eta\Gamma(1+\frac{1}{\beta}), \ \eta = \alpha^{\frac{-1}{\beta}} \qquad (1)$$

Variance, $\sigma^2 = E(T^2) - \mu^2 \qquad (2)$

$$E(T^2) = \int_0^\infty t^2 \ \alpha\beta \ t^{\beta-1} e^{-\alpha t^\beta} dt$$

Again, let $u = \alpha t^\beta$

$\Rightarrow$ du$=\alpha\beta t^{\beta-1} dt$, $t = \left(\frac{u}{\alpha}\right)^{\frac{1}{\beta}}$

$$E(T^2) = \alpha^{\frac{-2}{\beta}} \int_0^\infty (u)^{\frac{2}{\beta}} e^{-u} du$$

$$= \alpha^{\frac{-2}{\beta}} \Gamma(1+\frac{2}{\beta})$$

$$= \eta^2 \Gamma(1+\frac{2}{\beta}) \qquad (3)$$

Subs. (1), (3) in (2)

$$\sigma^2 = \eta^2 \left[\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta}) \right]$$

$$= \eta^2 \left[B_2 - B_1^2 \right]$$

b)

i)

$$\Pr(T > 100) = \text{R}(100)$$

$$= \exp[-(\tfrac{t}{\eta})^\beta]$$

$$= \text{e}^{-(\frac{100}{600})^{2.5}}$$

$$= 0.9887$$

ii)

The instantaneous failure rate is given by

$$\lambda(t) = \frac{\beta}{\eta^{\beta}} t^{\beta-1}$$

$$\lambda(600) = \frac{2.5}{600^{2.5}} \times 600^{1.5}$$

$$= \frac{2.5}{600}$$

$$= 4.1667 \times 10^{-3}$$

And the average failure rate is

$$\bar{\lambda} = \frac{t_2^{\beta} - t_1^{\beta}}{\eta^{\beta}(t_2 - t_1)}$$

$$= \frac{600^{2.5} - 400^{2.5}}{600^{2.5}(600 - 400)}$$

$$= 3.18556 \times 10^{-3}$$

iii) The Mode is given by

$$x_m = \eta \left(\frac{\beta - 1}{\beta} \right)^{1/\beta}$$

$$= 600 \left(\frac{1.5}{2.5} \right)^{1/2.5}$$

$$\simeq 489.12$$

The tenth percentile is

$$x_p = \left(\ln \left(\frac{1}{1-p} \right) \right)^{1/\beta} . \eta$$

$$= \left(\ln \left(\frac{1}{0.9} \right) \right)^{1/2.5} \times 600$$

$$\simeq 243.91$$

Also, the median is

$$x_{0.50} = \left(\ln\left(\frac{1}{1-0.50}\right)\right)^{1/2.5} \times 600$$
$$= (\ln 2)^{2/5} \times 600$$
$$\simeq 518.18$$

The MTTF for 2p Weibull is given by

$$\mu = \eta\, \Gamma\left(\frac{1}{\beta} + 1\right)$$
$$= \eta\, B_1$$
$$= 600 \times 0.8873$$
$$= 532.38$$

The variance is

$$\sigma^2 = \eta^2 [B_2 - B_1^2]$$
$$= 600^2 \times 0.1441$$
$$= 51876$$

Q4: [8]

a)

Step 1: Standard form $\Rightarrow$

$$\begin{aligned} \max \quad & z - 4x_1 - x_2 = 0 \\ s.t \quad & -x_1 + 2x_2 + s_1 = 4 \\ & 2x_1 + 3x_2 + s_2 = 12 \\ & x_1 - x_2 + s_3 = 3 \\ & x_i, s_j \geq 0 \quad \forall i = 1,2 \quad , \quad j = 1,2,3 \end{aligned}$$

Step 2: Simplex table (canonical form) $\Rightarrow$

z	x_1	x_2	s_1	s_2	s_3	Rhs	BV
1	−4	−1	0	0	0	0	$z = 0$
	−1	2	1	0	0	4	$s_1 = 4$
	2	3	0	1	0	12	$s_2 = 12$
	1	−1	0	0	1	3	$s_3 = 3$

Step 3: Pivot element $\Rightarrow$

	z	$[[x_1]]$	x_2	s_1	s_2	s_3	Rhs	BV	Ratio
Ro	1	-4	-1	0	0	0	0	$z=0$	$-$
R1		-1	2	1	0	0	4	$s_1=4$	$-$
R2		2	3	0	1	0	12	$s_2=12$	6
R3		[[1]]	-1	0	0	1	3	$[[s_3]]=3$	3

Step 4: Row operations $\Rightarrow$

Row operations	z	$[[x_1]]$	x_2	s_1	s_2	s_3	Rhs	BV	Ratio
4R3+Ro	1	0	-5	0	0	4	12	$z=12$	$-$
R3+R1		0	1	1	0	1	7	$s_1=7$	7
-2R3+R2		0	[[5]]	0	1	-2	6	$[[s_2]]=6$	$6/5$
R3		1	-1	0	0	1	3	$x_1=3$	$-$

Repeat steps 3&4 until reaching optimization table $\Rightarrow$

Row operations	z	$[[x_1]]$	x_2	s_1	s_2	s_3	Rhs	BV
5R2+R0		0	0	0	1	2	18	$z=18$
-R2+R1		0	0	1	$-1/5$	$7/5$	$29/5$	$s_1=29/5$
R2		0	[[1]]	0	$1/5$	$-2/5$	$6/5$	$x_2=6/5$
R2+R3		1	0	0	$1/5$	$3/5$	$21/5$	$x_1=21/5$

$\therefore$ The optimal solution is $\max z=18,\ x_1=4.2,\ x_2=1.2$

b) Ans: The optimal solution is $\min z=-5, x_1=0,\ x_2=5$

Q5: [3+4]

a)

$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\} . \Pr\{X_n = i_n \mid X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\} . P_{i_{n-1} i_n}$ Definition of Markov

By repeating this argument $n-1$ times

$\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

i)

$\Pr\{X_2 = G,\ X_3 = G, X_4 = G, X_5 = D \mid X_1 = G\}$

$= \Pr\{X_5 = D,\ X_4 = G, X_3 = G, X_2 = G \mid X_1 = G\}$

$= \Pr\{X_5 = D \mid X_4 = G, X_3 = G, X_2 = G, X_1 = G\} \cdot \Pr\{X_4 = G \mid X_3 = G, X_2 = G, X_1 = G\} \cdot \Pr\{X_3 = G \mid X_2 = G, X_1 = G$

$= \Pr\{X_5 = D \mid X_4 = G\} \cdot \Pr\{X_4 = G \mid X_3 = G\} \cdot \Pr\{X_3 = G \mid X_2 = G\} \cdot \Pr\{X_2 = G \mid X_1 = G\}$

$= p_{GD} p_{GG}^3$

$= (1-\alpha)\alpha^3$

$= \alpha^3(1-\alpha)$

Also, you can solve it as follows.

$p_1 p_{12} p_{23} p_{34} p_{45}$, $p_1 = \Pr(X_1 = G) = 1$

$= p_G p_{GG}^3 p_{GD}$

$= \alpha^3(1-\alpha)$

ii)

Similarly,

$$\Pr\left\{X_2 = D,\ X_3 = D, X_4 = D, X_5 = G \middle| X_1 = D\right\}$$

$$= \Pr\left\{X_5 = G,\ X_4 = D, X_3 = D, X_2 = D \middle| X_1 = D\right\}$$

$$= \Pr\left\{X_5 = G \middle| X_4 = D, X_3 = D, X_2 = D, X_1 = D\ \right\} \cdot \Pr\left\{X_4 = D \middle|\ X_3 = D, X_2 = D, X_1 = D\ \right\} \cdot \Pr\left\{X_3 = D \middle|\ X_2 = D, X_1 =\right.$$

$$= \Pr\left\{X_5 = G \middle| X_4 = D\right\} \cdot \Pr\left\{X_4 = D \middle|\ X_3 = D\right\} \cdot \Pr\left\{X_3 = D \middle|\ X_2 = D\right\} \cdot \Pr\left\{X_2 = D \middle| X_1 = D\ \right\}$$

$$= p_{DG} p_{DD}^3$$

$$= (1-\beta)\beta^3$$

$$= \beta^3(1-\beta)$$

Also, you can solve it as follows.

$$p_1 p_{12} p_{23} p_{34} p_{45},\ p_1 = \Pr(X_1 = D) = 1$$

$$= p_D p_{DD}^3 p_{DG}$$

$$= \beta^3(1-\beta)$$