King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1 1439/1440
M 507 – Advanced Operation Research
Time: 3 hours

Answer the following questions:

Q1: [6]

The reliability of each of 10 identical components is 0.95. If these components are part of a system for which at least six components must function for the system to function, compute the system reliability. If this system could be replaced by a parallel combination of five identical components, what would the reliability of those components have to be to give the same system reliability as the 6 out of 10 system?

Q2: [4+4]

a) Prove that each of the following:

i) $R(t) = e^{-\int_0^t \lambda(u)\,du}$ ii) $MTTF = \int_0^\infty R(t)\,dt$

b) The hazard function for a certain non-repairable product is given by $\lambda(t) = kt,\ k>0$. Compute its reliability function and the mean time to failure.

Q3: [6+3]

a) An item is randomly drawn from a two-parameter Weibull population having a shape parameter $\beta = 1.4$ and a scale parameter $\eta = 100$ hours.

i) What is the probability that the item fails before achieving a life of $x=30$ hours?

ii) Compute the mode, the tenth percentile, the median, the MTTF and the variance for this distribution.

b) The life of a product follows a lognormal distribution. The median life is 1000 hours. The probability that the product will survive a life of 2000 hours is 10%. Compute the expected life.

Q4: [4+4]

(a) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{n-1} i_n}$ where $p_{i_0} = \mathrm{pr}\{X_0 = i_0\}$

(b) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.7 & 0.2 & 0.1 \\ 1 & 0 & 0.6 & 0.4 \\ 2 & 0.5 & 0 & 0.5 \end{array}$$

Find $pr\{X_1 = 1, X_2 = 1 \mid X_0 = 0\}$.

Q5: [6+3]

(a) Using the differential equations

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - \lambda p_n(t), \; n=1,2,3, \ldots \qquad (2)$$

where all birth parameters are the same constant λ with initial condition X(0)=(

Show that $p_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!}$, $n = 0, 1, 2, \ldots$

(b) Suppose that customers arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let X(t) be the number of customers that have arrived up to t. Determine the following conditional probabilities

$pr\{X(3) = 6 \mid X(1) = 2\}$ and $pr\{X(1) = 2 \mid X(3) = 6\}$.

Model answer of final Exam 1439/1440 – S1

M 507 – Advanced operation Research

Q1 $\boxed{6}$

$$R_{sys} = \sum_{i=1}^{n} \binom{n}{i} R^i (1-R)^{n-i}$$

$$R_{sys} = pr(X \geq 6) = \sum_{x=6}^{10} \binom{10}{x} R^x (1-R)^{10-x}$$

$$= \binom{10}{6}(0.95)^6(0.05)^4 + \binom{10}{7}(0.95)^7(0.05)^3$$

$$+ \binom{10}{8}(0.95)^8(0.05)^2 + \binom{10}{9}(0.95)^9(0.05)$$

$$+ \binom{10}{10}(0.95)^{10}$$

$$\therefore R_{sys} = 0.99994$$

/ For parallel 5 identical components

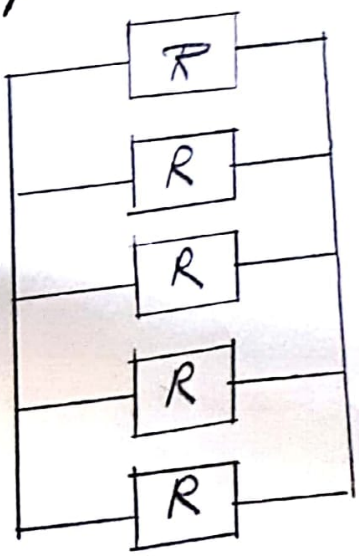
$$R_{sys} = 1 - \prod_{i=1}^{5}(1-R_i)$$

$$1-(1-R)^5 = 0.99994$$

$$(1-R)^5 = 6 \times 10^{-5}$$

$$(1-R) = (6 \times 10^{-5})^{0.2}$$

$$\therefore R \simeq 0.86$$



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Q2 a) (i) To prove that

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$$R(t) = e^{-\int_0^t \lambda(u)\,du}$$

$$\lambda(t) = \frac{f_T(t)}{R(t)}$$ (the hazard fn or failure rate)

$$\lambda(t) = \frac{d}{dt} F_T(t) \cdot \frac{1}{R(t)}$$

$$\lambda(t) = -\frac{dR(t)}{dt} \cdot \frac{1}{R(t)}$$

$$\Rightarrow \int_0^t \frac{dR(u)}{R(u)} = -\int_0^t \lambda(u)\,du$$

$$\therefore \left[\ln R(u)\right]_0^t = -\int_0^t \lambda(u)\,du$$

$$\therefore \ln R(t) - \ln R(0) = -\int_0^t \lambda(u)\,du$$

$$\&\because \ln R(0) = \ln 1 = 0$$

$$\therefore \ln R(t) = -\int_0^t \lambda(u)\,du$$

$$\therefore R(t) = e^{-\int_0^t \lambda(u)\,du}$$

$$\therefore R(t) = \exp\left[-\int_0^t \lambda(u)\,du\right]$$

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3 ii) To prove that $MTTF = \int_0^\infty R(t)\,dt$

$$MTTF = \int_0^\infty t\, f(t)\, dt$$

$$MTTF = -\int_0^\infty t\, \frac{dR(t)}{dt}\, dt$$

$$MTTF = -\int_0^\infty t\, dR(t)$$

By using Integration by parts, we can deduce that

$$MTTF = -\left[t R(t)\right]_0^\infty + \int_0^\infty R(t)\,dt$$

at $t = 0 \Rightarrow t R(t) = 0\, R(0) = 0(1) = 0$

$$\lim_{t\to\infty} t R(t) = \lim_{t\to\infty} \frac{t}{1/R(t)} \rightarrow \frac{\infty}{\infty}$$

when $R(t) \rightarrow 0$ as $t \rightarrow \infty$

$$= \lim_{t\to\infty} \frac{1}{-R'(t)/R^2(t)}$$ by using L'Hôpital's rule

$$= \lim_{t\to\infty} \frac{R(t)}{-R'(t)/R(t)}$$

$$= \lim_{t\to\infty} \frac{R(t)}{\lambda(t)} = 0$$

when $R(t) = 0$ as $t \rightarrow \infty$ and $\lambda(t) \neq 0$ as $t \rightarrow \infty$

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$$\therefore MTTF = \int_0^\infty R(t)\,dt$$

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b) $\lambda(t) = kt, \quad k > 0$

$$R(t) = e^{-\int_0^t \lambda(u)\,du}$$

$$R(t) = e^{-k\int_0^t u\,du}$$

$$R(t) = e^{-kt^2/2} = e^{-\frac{1}{2}kt^2}$$

$$MTTF = \int_0^\infty R(t)\,dt$$

$$MTTF = \frac{1}{k}\int_0^\infty e^{-\frac{1}{2}kt^2}\cdot k\,dt$$

$$MTTF = \frac{1}{k}\sqrt{\frac{\pi}{2}}$$

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Note that: $\int_0^\infty e^{-\frac{1}{2}z^2}\,dz = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}$

Q3 let the lifetime of the item X be a r.v.

i) $pr(X < 30) = F(30) = 1 - e^{\left(\frac{-30}{100}\right)^{1.4}}$

$= 1 - e^{-(0.3)^{1.4}}$ 1

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$\simeq 0.1692$

ii) the mode is $x_m = \eta\left(\frac{\beta-1}{\beta}\right)^{1/\beta}$

$= 100\left(\frac{1.4-1}{1.4}\right)^{1/1.4} \simeq \boxed{40.9}$ hours 1

the tenth percentile $x_p = \left[\ln\left(1/(1-p)\right)\right]^{1/\beta}\cdot\eta$

$x_{0.10} = \left[\ln\left(1/0.9\right)\right]^{1/1.4}\cdot 100$

$x_{0.10} \simeq \boxed{20.04}$ 2 1

5, the median $x_{0.50} = [\ln(1/(1-0.5))]^{1/1.4} \cdot 100$

$x_{0.50} = [\ln 2]^{1/1.4}(100) \simeq 76.97$ ③ 1

k-th moment $E(X^k)$ is defined as

$$E(X^k) = \eta^k \Gamma\left(\frac{k}{\beta}+1\right)$$

$$MTTF = E(X) = \eta \Gamma\left(\frac{1}{1.4}+1\right) = 100(0.9114)$$ 1

$\boxed{MTTF = 91.14}$ ④

and the variance is given by

$$\sigma^2 = E(X^2) - \mu^2 = \eta^2[B_2 - B_1^2]$$ ⑤ 1

$\therefore \sigma^2 = (100)^2(0.4351) = \boxed{4351}$

b) $x_p = \exp(\mu + z_p\sigma)$ ..... see p. 66 text

the median value of X is

$x_{0.50} = \exp(\mu + 0\sigma)$, $z_p = 0$ for $p = 0.50$

$1000 = e^{\mu} \Rightarrow \mu = \ln 1000$

$\mu = \boxed{6.9078}$ 1

$pr(X \geq 2000) = 1 - pr(X < 2000)$

$\therefore 0.10 = 1 - F(2000) \Rightarrow \Phi\left(\frac{\ln(2000) - 6.9078}{\sigma}\right) = 0.90$

$\Phi\left(\frac{0.6931}{\sigma}\right) = 0.90 \Rightarrow \frac{0.6931}{\sigma} = 1.29$ using stand. Normal prob. table

$\therefore \sigma = \boxed{0.5373}$ 1

$\therefore$ The expected life is $E(X) = e^{\mu + \sigma^2/2}$

$= e^{6.9078 + (0.5373)^2/2}$

$\simeq \boxed{1155.3}$ hours. 1 #

$\frac{6}{8}$

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Q4

a) $pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$

$= pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}$
$\cdot pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}$ (1)

$\because pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}$

$= pr\{X_n = i_n | X_{n-1} = i_{n-1}\} = P_{i_{n-1} i_n}$ Markov def$\underline{n}$ (2)

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Subs. (2) in (1)

$pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$

$= pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \cdot P_{i_{n-1} i_n}$

By repeating this argument $n-1$ times, we have

$pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$ (3)

$= P_{i_0} P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$

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where $P_{i_0} = pr\{X_0 = i_0\}$ for initial dist$\underline{n}$

b) $pr\{X_1 = 1, X_2 = 1 | X_0 = 0\}$

$= pr\{X_2 = 1 | X_1 = 1, \cancel{X_0 = 0}\} \cdot pr\{X_1 = 1 | X_0 = 0\}$

$= P_{11} \cdot P_{01} = 0.6(0.2) = \boxed{0.12}$

7 Q5 $\boxed{9}$

a) let $X(t)$ represents the size of population with initial

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Condition $X(0)=0$

$$\Rightarrow P_n(0)=\begin{cases}1 & , n=0\\ 0 & , \text{otherwise}\end{cases}$$

$$① \Rightarrow \frac{dP_0(t)}{dt}=-\lambda P_0(t)$$

$$\frac{dP_0(t)}{P_0(t)}=-\lambda dt \Rightarrow \int_0^t \frac{dP_0(u)}{P_0(u)}=-\lambda\int_0^t du$$

$$\therefore \left[\ln P_0(u)\right]_0^t=-\lambda\left[u\right]_0^t$$

$$\therefore \ln P_0(t)-\cancel{\ln P_0(0)}=-\lambda t$$

$$\Rightarrow \ln P_0(t)\overset{0}{=}-\lambda t \quad \therefore \boxed{P_0(t)=e^{-\lambda t}} \; ③$$

$$② \Rightarrow \frac{dP_n(t)}{dt}=\lambda P_{n-1}(t)-\lambda P_n(t)$$

$$\frac{dP_n(t)}{dt}+\lambda P_n(t)=\lambda P_{n-1}(t) \quad , n=1,2,3,\cdots$$

Multiply both sides by $e^{\lambda t}$

$$e^{\lambda t}\left[\frac{dP_n(t)}{dt}+\lambda P_n(t)\right]=\lambda P_{n-1}(t)e^{\lambda t}$$

$$\therefore \frac{d}{dt}\left[e^{\lambda t}P_n(t)\right]=\lambda P_{n-1}(t)e^{\lambda t}$$

$$\Rightarrow \int_0^t d\left[e^{\lambda t}P_n(t)\right]=\lambda\int_0^t P_{n-1}(x)e^{\lambda x}\,dx$$

$$e^{\lambda t}P_n(t)-\cancel{P_n(0)}_{0}=\lambda\int_0^t P_{n-1}(x)e^{\lambda x}\,dx \quad , n=1,2,\cdots$$

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$$P_n(t) = \lambda e^{-\lambda t} \int_0^t P_{n-1}(x)\, e^{\lambda x}\, dx \quad , n = 1, 2, \dots \quad (4)$$

which is a recurrence relation

at $n=1$ $\quad P_1(t) = \lambda e^{-\lambda t} \int_0^t P_0(x)\, e^{\lambda x}\, dx$

$(3) \Rightarrow P_0(x) = e^{-\lambda x}$

$$\therefore P_1(t) = \lambda e^{-\lambda t} \int_0^t dx$$

$$\boxed{P_1(t) = \lambda t\, e^{-\lambda t}} \quad (5)$$

, at $n=2$

$(4) \Rightarrow P_2(t) = \lambda e^{-\lambda t} \int_0^t P_1(x)\, e^{\lambda x}\, dx$

$(6) \Rightarrow P_1(x) = \lambda x e^{-\lambda x}$

$$\therefore P_2(t) = \lambda e^{-\lambda t} \int_0^t \lambda x\, dx$$

$$\therefore P_2(t) = \lambda^2 e^{-\lambda t} \left[\frac{x^2}{2}\right]_0^t = \boxed{\frac{(\lambda t)^2}{2!} e^{-\lambda t}} \quad (6)$$

$(3), (5)$ and $(6) \Rightarrow P_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!}$, $n = 0, 1, 2, \dots$ (Poisson process)

b) i) $pr\{X(3)=6 \mid X(1)=2\}$

$= pr\{X(3) - X(1) = 4\}$

$= \dfrac{4^4 e^{-4}}{4!} = \dfrac{64 e^{-4}}{6}$ $\quad$ 1.5

when $\lambda t = 4$, $n = 4$

ii) $pr\{X(1)=2 \mid X(3)=6\}$

$= \binom{n}{x} p^x q^{n-x}$

$= \binom{6}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$

1.5

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