King Saud University
College of Sciences
Department of Mathematics

First Mid Term, S1 1439/1440
M 380 – Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [6+3]

a) Find the mean and variance of X, where $X \sim$ Uniform (a,b)

b) Twelve independent random variables, each uniformly distributed over the interval $(0, 1]$, are added, and 6 is subtracted from the total. Determine the mean and variance of the resulting random variable.

Q2: [4+4]

a) Given independent exponentially distributed random variables S and T with common parameter λ, determine the probability density function of the sum $R=S+T$ and identify its type by name.

b) The lifetime T of a certain component has an exponential distribution with parameter $\lambda=0.02$. Find $pr(T \leq 110 | T > 100)$

Q3: [4+4]

a) Let X and Y two random variables have the joint normal (bivariate normal) distribution. What value of α minimizes the variance of $Z=\alpha X+(1-\alpha)Y$? Simplify your result when X and Y are independent.

b) Let $X_1, X_2, \ldots, X_n$ be independent random variables that are exponentially distributed with respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. Identify the distribution of the minimum $V=\min\{X_1, X_2, \ldots, X_n\}$.

Model Answer

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Q1 9

a) mean $\Rightarrow \mu = E(X)$

$\mu = \int_a^b x f(x) dx$

$\mu = \int_a^b \frac{x\,dx}{b-a}$ 3

$\mu = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$

$\mu = \frac{1}{2}\left(\frac{1}{b-a}\right)(b^2-a^2)$

$\therefore \boxed{\mu = \frac{1}{2}(a+b)}$

, Variance $\Rightarrow \sigma^2 = E(X^2) - \mu^2$

$E(X^2) = \int_a^b \frac{x^2}{b-a} dx$

$E(X^2) = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$

$E(X^2) = \frac{1}{3}\cdot\frac{1}{b-a}(b^3-a^3)$ 3

$E(X^2) = \frac{1}{3}(b^2+ab+a^2)$

$\Rightarrow \sigma^2 = \frac{1}{3}(b^2+ab+a^2) - \left(\frac{a+b}{2}\right)^2$

$\sigma^2 = \frac{1}{12}b^2 - \frac{1}{6}ab + \frac{1}{12}a^2$

$\boxed{\sigma^2 = \frac{1}{12}(b-a)^2}$, $b > a$

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b) $X = X_1 + X_2 + \cdots + X_{12}$

$X_i \sim$ uniform $(0, 1)$, with $a = 0$, $b = 1$

$E(X_i) = \frac{1}{2}(a+b) = \boxed{\frac{1}{2}}$

$E(X_1) = E(X_2) = \cdots = E(X_{12}) = \frac{1}{2}$

$\therefore E(X) = 12\left(\frac{1}{2}\right) = 6$ 3

$E(X-6) = E(X) - 6$

$= 6 - 6 = 0$

, $Var(X_1) = Var(X_2) = \cdots = Var(X_{12}) = \frac{1}{12}$

, $Var(X-6) = Var(X) - Var(6)$

$= 1 - 0 = 1$

Q2 a) $S, T \sim exp(\lambda)$ 8

, $R = S + T$

$\therefore R \sim Gamma(2, \lambda)$

$\Rightarrow f_R(r) = \frac{\lambda^2}{\Gamma(2)} \cdot r^{2-1} e^{-\lambda r}$

$f_R(r) = \frac{\lambda^2}{1!} r e^{-\lambda r}$, $r \geq 0$

$\therefore \boxed{f_R(r) = \lambda^2 r e^{-\lambda r}}$, $r \geq 0$

which is the Gamma prob. density fn 4

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b) $T \sim exp(0.02)$

$pr(T \le 110 \mid T > 100)$

$= \frac{pr(100 < T \le 110)}{pr(T > 100)}$ 4

$= \frac{F(110) - F(100)}{1 - F(100)}$

$= \frac{e^{-0.02(100)} - e^{-0.02(110)}}{e^{-0.02(100)}}$

$= 1 - e^{-0.2} \simeq \boxed{0.18}$

OK

$pr(T \le 110 \mid T > 100)$

$= 1 - pr(T > 110 \mid T > 100)$

$= 1 - pr(T > \overset{s}{100} + \overset{t}{10} \mid T > \overset{s}{100})$

$= 1 - pr(T > 10)$

memoryless prop.

$= 1 - e^{-0.02(10)}$

$= 1 - e^{-0.2} \simeq \boxed{0.18}$

Q3 $Z = \alpha X + (1-\alpha) Y$ 8

$V = Var(Z)$
$= \alpha^2 \sigma_X^2 + 2\alpha(1-\alpha)\rho\sigma_X\sigma_Y + (1-\alpha)^2\sigma_Y^2$

To minimize V by using α

let $\frac{\partial V}{\partial \alpha} = 0$

$\Rightarrow 2\alpha\sigma_X^2 + (2-4\alpha)\rho\sigma_X\sigma_Y - 2\sigma_Y^2 + 2\alpha\sigma_Y^2 = 0$

$\Rightarrow 2\alpha(\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2)$

$= 2(\sigma_Y^2 - \rho\sigma_X\sigma_Y)$

$\rightarrow \alpha^* = \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}$, $-1 \le \rho \le 1$

for Independent r.v.s X and Y

, $\rho = 0$

$\therefore \boxed{\alpha^* = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}}$

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b) $pr(V > v)$, $\min V = \min(X_1, X_2, \dots, X_n)$

$= pr(X_1 > v, X_2 > v, \dots, X_n > v)$

$= pr(X_1 > v)\, pr(X_2 > v) \dots pr(X_n > v)$

for Independent r.v's

$= e^{-\lambda_1 v} \cdot e^{-\lambda_2 v} \cdot \dots \; e^{-\lambda_n v}$

$(X_1, X_2, \dots, X_n$ are exp. distributed

$= e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_n) v}$

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$\therefore pr(V > v) = e^{-(\sum_i \lambda_i) v}$

$\Rightarrow V \sim \exp(\sum_i \lambda_i)$

$\therefore V$ is exponentially distributed with parameter $\sum_i \lambda_i$

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