King Saud University
College of Sciences
Department of Mathematics

First Mid Term, S2 1439/1440
M 380 – Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [3+6]

a) A fraction $p=0.05$ of the items coming off of a production process are defective. The output of the process is sampled, one by one, in a random manner. What is the probability that the first defective item found is the tenth item sampled?

b) The lifetime, in years, of a certain class of light bulbs has an exponential distribution with parameter $\lambda = 2$. What is the probability that a bulb selected at random from this class will last more than 1.5 years? What is the probability that a bulb selected at random will last exactly 1.5 years?

Q2: [3+6]

a) Let X and Y two random variables have the joint normal (bivariate normal) distribution. What value of α that minimizes the variance of $Z=\alpha X+(1-\alpha)Y$? Simplify your result when X and Y are independent.

b) Given the following joint distribution. Calculate $E(X)$, $E(Y)$, $Var(X)$, $Var(Y)$, $Cov(X,Y)$, $\rho(X,Y)$, and verify $E(X)$ using the law of total Expectation.

Y \ X	0	1
0	0.1	0.3
1	0.4	0.2

Q3: [3+4]

a) Let $X=\begin{cases} 0 & \text{if } N=0 \\ \xi_1+\xi_2+\ \dots\ +\xi_N & \text{if } N>0 \end{cases}$ be a random sum and assume that $E(\xi_k)=\mu$, $E(N)=\upsilon$

Prove that $E(X)=\mu\upsilon$

b) The following experiment is performed: An observation is made of a Poisson random variable N with parameter λ. Then N independent Bernoulli trials are performed, each with probability p of success. Let Z be the total number of successes observed in the N trials.

(i) Formulate Z as a random sum and thereby determine its mean and variance.

(ii) What is the distribution of Z?

The Model Answer

Q1: [3+6]

a) X ~ geom (0.05)

Pr(X=x)=p(1-p)$^{k-1}$, k=1,2, ...

Pr(X=10)=0.05(0.95)^9 $= 0.0315$

b) X ~ exp (2)

i) Pr(T>1.5)=$e^{-3} = 0.0498$

ii) Pr(T=1.5)=0

Q2: [3+6]

a) $Z = \alpha X + (1-\alpha)Y$

Var(Z)=$\alpha^2\sigma_X^2 + 2\alpha(1-\alpha)\rho\sigma_X\sigma_Y + (1-\alpha)^2\sigma_Y^2$

To get α^* that minimizes Var(Z) let $\dfrac{\partial V}{\partial \alpha} = 0$

$$\alpha^* = \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}$$

For independent random variables X and Y, ρ=0

Consequently, $\alpha^* = \dfrac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$

b)

Y \ X	0	1	$P_Y(y)$
0	0.1	0.3	0.4
1	0.4	0.2	0.6
$P_X(x)$	0.5	0.5	Sum=1

$E(X)=0.5,\ E(X^2)=0.5,\ \ Var(X)=0.25$

$E(Y)=0.6,\ E(Y^2)=0.6,\ \ Var(Y)=0.24$

$E(XY)=0.2,\ Cov(X,Y)=-0.10,\ \ \rho(X,Y)=-0.4$

$$P(X|Y=y)=\frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$P(X=0|Y=0)=\frac{0.1}{0.4}=\frac{1}{4},\quad P(X=1|Y=0)=\frac{0.3}{0.4}=\frac{3}{4}$$

$$P(X=0|Y=1)=\frac{0.4}{0.6}=\frac{2}{3},\quad P(X=1|Y=1)=\frac{0.2}{0.6}=\frac{1}{3}$$

$X\|Y$	0	1	$E[X\|Y]$
y=0	1/4	3/4	3/4
y=1	2/3	1/3	1/3

$$E(X)=\sum_y E(X|Y=y)P_Y(y)$$

$$E(X)=\frac{3}{4}P_Y(0)+\frac{1}{3}P_Y(1)$$

$$E(X)=\frac{3}{4}(0.4)+\frac{1}{3}(0.6)=0.5$$

Q3: [3+4]

a) $\because\ E(X)=\sum_{n=0}^{\infty}E[X|N=n]P_N(n)$ Def. of Total Expectation

$\therefore\ E(X)=\sum_{n=1}^{\infty}E[\xi_1+\xi_2+\ ...\ +\xi_N|N=n]P_N(n)$ Def. of Random Sum

$\therefore\ E(X)=\sum_{n=1}^{\infty}E[\xi_1+\xi_2+\ ...\ +\xi_n|N=n]P_N(n)$ Prop. of Conditional Expectation

$\therefore\ E(X)=\sum_{n=1}^{\infty}E[\xi_1+\xi_2+\ ...\ +\xi_n]P_N(n)$ where N is independent of $\xi_1,\ \xi_2,\ ...$

$\because\ E(\xi_k)=\mu,\quad k=1,2,\ldots,n$

$\therefore\ E(X)=\sum_{n=1}^{\infty} n\mu P_N(n)$

$\therefore\ E(X)=\mu\sum_{n=1}^{\infty} nP_N(n)$

$\therefore\ E(X)=\mu E(N)=\mu\upsilon$

b) $N \sim$ Poisson (λ)

$E(N)=\lambda,\quad Var(N)=\lambda$

$E(\xi_k)=p,\quad Var(\xi_k)=p(1-p)$

$E(Z)=\lambda p$

$Var(Z)=\lambda p(1-p)+p^2\lambda=\lambda p$

$\therefore\ Z \sim Poisson(\lambda p)$
