King Saud University
College of Sciences
Department of Mathematics

First Mid Term, S1-1441H
M 380 – Stochastic Processes
Time: 90 minutes - Female Sec.

Answer the following questions:

Q1: [4+4]

a) If $T \sim \exp(\lambda)$ prove that: $pr(T > t+s | T > s) = pr(T > t) \quad \forall\ t,\ s \geq 0$

b) The lifetime T of a certain component has an exponential distribution with parameter $\lambda = 0.03$. Find $\mathrm{pr}(T \leq 110 | T > 100)$

Q2: [4+5]

a) The joint probability density function of the two random variables X and Y is $f(x,y) = 8xy$, $0 \leq x \leq y \leq 1$. Find $f_{X|Y}(x|\frac{1}{2})$

b) Given the following joint distribution.

X \ Y	1	2
1	$\frac{1}{6}$	$\frac{1}{3}$
2	$\frac{1}{12}$	$\frac{1}{3}$
3	$\frac{1}{12}$	0

Calculate $\rho(X,Y)$.

Q3: [4+4]

a) Determine the distribution function, mean and variance corresponding to the probability density function $f(x)=\begin{cases} Rx^{R-1} & \text{for } 0\le x\le 1 \\ 0 & \text{elsewhere} \end{cases}$

where $R>0$ is a fixed parameter.

b) Given independent exponentially distributed random variables S and T with common parameter λ, determine the probability density function of the sum $R=S+T$ and identify its type by name.

The Model Answer

Q1: [4+4]

a) If $T \sim \exp(\lambda)$ prove that: $pr(T > t+s | T > s) = pr(T > t) \quad \forall\ t,\ s \geq 0$

Proof:

$$pr(T > t+s | T > s) = \frac{pr(T > t+s, T > s)}{pr(T > s)}$$

$$= \frac{pr(T > t+s)}{pr(T > s)}$$

$\because\ T \sim \exp(\lambda)$

$$\therefore\ pr(T > t+s | T > s) = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = R(t)$$

$$= pr(T > t)$$

b)

$$pr(T \leq 110 | T > 100) = 1 - pr(T > 110 | T > 100)$$

$$= 1 - pr(T > 10 + 100 | T > 100)$$

$$= 1 - pr(T > 10)$$

$$= pr(T \leq 10)$$

$$= 1 - e^{-\lambda t}$$

$$= 1 - e^{-0.03(10)}$$

$$\therefore pr(T \leq 110 | T > 100) = 1 - e^{-0.3} \simeq 0.26$$

Q2: [4+5]

a) $f_{X|Y}(x,y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

$f_{X,Y}(x,y) = 8xy\ ,\ \ 0 \leq x \leq y \leq 1$

$$\because\ f_Y(y)=\int_{-\infty}^{\infty} f(x,y)dx$$

$$=\int_0^y 8xydx$$

$$=8y\left[\frac{x^2}{2}\right]_0^y$$

$$\therefore\ f_Y(y)=4y^3, \quad 0\le y\le 1$$

$$\therefore\ f_{X|Y}(x|y)=\frac{8xy}{4y^3}$$

$$=\frac{2x}{y^2}$$

$$\therefore\ f_{X|Y}(x|\tfrac{1}{2})=8x, \quad 0\le x\le 1$$

b)

X \ Y	1	2	$P_X(x)$
1	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{6}{12}$
2	$\frac{1}{12}$	$\frac{4}{12}$	$\frac{5}{12}$
3	$\frac{1}{12}$	0	$\frac{1}{12}$
$P_Y(y)$	$\frac{4}{12}$	$\frac{8}{12}$	Sum=1

$E(X)=\frac{19}{12}$, $E(X^2)=\frac{35}{12}$, $Var(X)=\frac{59}{144}\simeq 0.4097$

$E(Y)=\frac{20}{12}$, $E(Y^2)=\frac{36}{12}$, $Var(Y)=\frac{32}{144}\simeq 0.2222$

$E(XY)=\frac{31}{12}\simeq 0.4097$

Cov(X,Y)=E(XY)-E(X)E(Y)=-0.0556

$$\rho(X,Y)=\frac{Cov(X,Y)}{\sigma_X\sigma_Y}=-0.184$$

Q3: [4+4]

a)

$$\because\ f(x)=\begin{cases} Rx^{R-1} & \text{for } 0\le x\le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$F_X(x)=R\int_0^x t^{R-1}dx$$

$$F_X(x)=x^R\ ,\qquad 0\le x\le 1$$

$$\therefore\ F_X(x)=\begin{cases} 0\ , & x<0 \\ x^R, & 0\le x\le 1 \\ 1\ , & x>1 \end{cases}$$

$$\because\ E(X)=R\int_0^1 x^R dx$$

$$\therefore\ \mu=E(X)=\frac{R}{R+1}$$

$$E(X^2)=R\int_0^1 x^{R+1}dx$$

$$=\frac{R}{R+2}$$

$$\because\ Var(X)=E(X^2)-\mu^2$$

$$=\frac{R}{R+2}-\frac{R^2}{(R+1)^2}$$

$$\therefore\ Var(X)=\frac{R}{(R+2)(R+1)^2}$$

b)

$$\because\ S, T\sim exp(\lambda)\ , R=S+T$$

$\therefore\ R \sim \text{Gamma}(2,\lambda)$

$\Rightarrow f_R(r) = \frac{\lambda^2}{\Gamma(2)} r^{2-1} e^{-\lambda r}$

$\therefore\ f_R(r) = \lambda^2 r e^{-\lambda r}, \quad r \geq 0$

Which is the Gamma probability density function.