Answer the following questions:

Q1: [3+2+3]

An oil drilling company drills at a large number of locations in search of oil. The probability of success at any location is 0.25 and the locations may be regarded as independent.

a) What is the probability that the driller will experience 1 success if 10 locations are drilled?

b) The driller feels that he will go bankrupt if he drills 10 times before experiencing his first success. What is the probability that he will go bankrupt?

c) What is the probability that he will get the first success on the 10^{th} trial?

Q2: [5+4]

a) The joint probability density function of the two random variables X and Y

is $f(x,y)=8xy$, $0 \le x \le y \le 1$. Find $f_{X|Y}(x|\frac{1}{2})$ and $\rho(X,Y)$

b) Let X and Y two random variables have the joint normal (bivariate normal) distribution. What value of α that minimizes the variance of $Z=\alpha X+(1-\alpha)Y$? Simplify your result when X and Y are independent.

Q3: [5+3]

a) Let $X=\begin{cases} 0 & \text{if } N=0 \\ \xi_1+\xi_2+\ \dots\ +\xi_N & \text{if } N>0 \end{cases}$ be a random sum and assume that $E(\xi_k)=\mu$, $E(N)=\upsilon$

and $Var(\xi_k)=\sigma^2$, $Var(N)=\tau^2$

Prove that $E(X)=\mu\upsilon$ and $Var(X)=\upsilon\sigma^2+\mu^2\tau^2$

b) The number of accidents occurring in a factory in a week is a Poisson random variable with mean 2. The number of individuals injured in different accidents is independently distributed, each with mean 3 and variance 4. Determine the mean and variance of the number of individuals injured in a weak.

The Model Answer

Q1: [3+2+3]

a) This implies that n=10, p=0.25 and X=1

$$\therefore \ \text{pr}(x=1)=\binom{10}{1}p^1q^9$$

$$=10\times 0.25\times 0.75^9$$

$$=0.1877$$

b) The probability that he will go bankrupt is given by

$$\text{pr}(x=0)=\binom{10}{0}p^0q^{10}$$

$$=0.25^0\times 0.75^{10}$$

$$=0.0563$$

c) What is the probability that he will get the first success on the 10th trial?

$$\text{pr}(x=10)=p(1-p)^9$$

$$=0.25(0.75)^9$$

$$=0.0188$$

Q2: [5+4]

a) $f_{X|Y}(x,y)=\dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

$$f_{X,Y}(x,y)=8xy\ ,\ \ 0\le x\le y\le 1$$

$$\because\ f_Y(y)=\int_{-\infty}^{\infty} f(x,y)dx$$

$$=\int_0^y 8xydx$$

$$=8y\left[\frac{x^2}{2}\right]_0^y$$

$$\therefore\ f_Y(y)=4y^3,\qquad 0\le y\le 1$$

$$\therefore\ f_{X|Y}(x|y)=\frac{8xy}{4y^3}$$

$$=\frac{2x}{y^2}$$

$$\therefore\ f_{X|Y}(x|\tfrac{1}{2})=8x,\quad 0\le x\le 1$$

$$\because\ f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy$$

$$=\int_x^1 8xydy,\quad x\le y\le 1$$

$$=8x\left[\frac{y^2}{2}\right]_x^1$$

$$\therefore\ f_X(x)=4x(1-x^2),\quad 0\le x\le 1$$

$$E(X)=\int_{-\infty}^{\infty} xf(x)dx$$

$$=\int_0^1 x.\ 4x(1-x^2)dx$$

$$=4\int_0^1 (x^2-x^4)dx$$

$$\therefore\ E(X)=4[\tfrac{x^3}{3}-\tfrac{x^5}{5}]_0^1=\tfrac{8}{15}$$

$$E(X^2)=\int_0^1 x^2.\ 4x(1-x^2)dx$$

$$=4\int_0^1 (x^3-x^5)dx$$

$$\therefore\ E(X^2)=\tfrac{1}{3}$$

$$Var(X)=E(X^2)-E^2(X)$$

$$\therefore Var(X)=\tfrac{1}{3}-(\tfrac{8}{15})^2=\tfrac{11}{225}$$

Similarly,

$$E(Y)=\int_0^1 y(4y^3)dy$$

$$\therefore\ E(Y)=\tfrac{4}{5}$$

$$E(Y^2)=\int_0^1 y^2(4y^3)dy$$

$$\therefore\ E(Y^2)=\tfrac{2}{3}$$

$$E(XY)=\int_0^1\int_0^y xy(8xy)dxdy$$

$$=8\int_0^1 y^2[\tfrac{x^3}{3}]_0^y dy$$

$$\therefore\ E(XY)=\tfrac{8}{3}\int_0^1 y^5 dy=\tfrac{4}{9}$$

Cov(X,Y)=E(XY) -E(X)E(Y)

$$\therefore \text{Var(X)}=\tfrac{4}{9}-(\tfrac{8}{15})(\tfrac{4}{5})=\tfrac{4}{225}$$

$$\rho(X,Y)=\frac{\text{Cov(X,Y)}}{\sigma_X\sigma_Y}$$

$$=\frac{4/225}{\sqrt{11/225}\sqrt{2/75}}=\frac{4}{\sqrt{66}}$$

$$\therefore \rho(X,Y)\approx 0.49$$

b) $Z=\alpha X+(1-\alpha)Y$

$$\text{Var(Z)}=\alpha^2\sigma_X^2+2\alpha(1-\alpha)\rho\sigma_X\sigma_Y+(1-\alpha)^2\sigma_Y^2$$

$$\therefore \text{Var(Z)}=\alpha^2\sigma_X^2+(2\alpha-2\alpha^2)\rho\sigma_X\sigma_Y+(1-2\alpha+\alpha^2)\sigma_Y^2$$

To get α^* that minimizes Var(Z) let $\frac{\partial V}{\partial\alpha}=0$

$\Rightarrow$

$$2\alpha\sigma_X^2+(2-4\alpha)\rho\sigma_X\sigma_Y+(-2+2\alpha)\sigma_Y^2=0$$

$$\therefore \alpha=\alpha^*=\frac{\sigma_Y^2-\rho\sigma_X\sigma_Y}{\sigma_X^2-2\rho\sigma_X\sigma_Y+\sigma_Y^2},\ -1<\rho<1$$

For independent random variables X and Y, $\rho=0$

Consequently, $\alpha^*=\frac{\sigma_Y^2}{\sigma_X^2+\sigma_Y^2}$

Q3: [5+3]

i) To prove that $E(X)=\mu\upsilon$

a) $\because\ E(X)=\sum_{n=0}^{\infty} E[X|N=n]P_N(n)$ Def. of Total Expectation

$\therefore\ E(X)=\sum_{n=1}^{\infty} E[\xi_1+\xi_2+\ ...\ +\xi_N|N=n]P_N(n)$ Def. of Random Sum

$\therefore\ E(X)=\sum_{n=1}^{\infty} E[\xi_1+\xi_2+\ ...\ +\xi_n|N=n]P_N(n)$ Prop. of Conditional Expectation

$\therefore\ E(X)=\sum_{n=1}^{\infty} E[\xi_1+\xi_2+\ ...\ +\xi_n]P_N(n)$ where N is independent of $\xi_1,\ \xi_2,\ ...$

$\because\ E(\xi_k)=\mu,\quad k=1,2,\ ...\ ,n$

$\therefore\ E(X)=\sum_{n=1}^{\infty} n\mu P_N(n)$

$\therefore\ E(X)=\mu\sum_{n=1}^{\infty} nP_N(n)$

$\therefore\ E(X)=\mu E(N)=\mu\upsilon$

i) To prove that $Var(X)=\upsilon\sigma^2+\mu^2\tau^2$

$Var(X)=E[(X-\mu\upsilon)^2]$

$\quad\quad =E[X-N\mu+N\mu-\upsilon\mu]^2$

$Var(X)=E[(X-N\mu)^2]+E[\mu^2(N-v)^2]+2E[\mu(X-N\mu)(N-v)] \quad (1)$

$\because E[(X-N\mu)^2]=\sum_{n=0}^{\infty} E[(X-N\mu)^2|N=n]P_N(n)$

$\quad\quad =\sum_{n=1}^{\infty} E[(\xi_1+\xi_2+\ ...\ +\xi_n-n\mu)^2|N=n]P_N(n)$

$\therefore E[(X-N\mu)^2]=\sum_{n=1}^{\infty} E(\xi_1+\xi_2+\ ...\ +\xi_n-n\mu)^2]P_N(n)$

$\because Var(\xi_k)=E(\xi_k-\mu)^2=\sigma^2,\quad k=1,2,...,n$

$$\therefore \mathrm{E}[(\mathrm{X}-N\mu)^2] = \sum_{n=1}^{\infty} n\sigma^2 P_N(n)$$

$$= \sigma^2 \sum_{n=1}^{\infty} nP_N(n)$$

$$\therefore \mathrm{E}[(\mathrm{X}-N\mu)^2] = \upsilon\sigma^2, \text{ where } \sum_{n=1}^{\infty} nP_N(n) = \upsilon \qquad (2)$$

$$\mathrm{E}[\mu^2 (N-v)^2] = \mu^2 \mathrm{E}[(N-v)^2]$$

$$\therefore \mathrm{E}[\mu^2 (N-v)^2] = \mu^2 Var(N) = \mu^2\tau^2 \qquad (3)$$

Also,

$$\mathrm{E}[\mu(\mathrm{X}-N\mu)(N-v)] = \mu\sum_{n=1}^{\infty} \mathrm{E}[(\mathrm{X}-n\mu)(n-\upsilon)\,|\,N=n]P_N(n)$$

$$= \mu\sum_{n=1}^{\infty}(n-\upsilon)\mathrm{E}[(\mathrm{X}-n\mu)\,|\,N=n]P_N(n)$$

$$= 0 \qquad (4)$$

where $\mathrm{E}[(\mathrm{X}-n\mu)\,|\,N=n] = \mathrm{E}(\mathrm{X}-n\mu)$ independent prop.

$$= \mathrm{E}(\xi_1 + \xi_2 + \ \ldots + \xi_n - n\mu)$$

$$= n\mu - n\mu = 0$$

Substitute (2), (3) and (4) in (1), we get

Var(X)= $\upsilon\sigma^2 + \mu^2\tau^2$

b)

N ~ Poisson (2)

N is the # of accidents in a week

ξ_k is the # of individuals injured for kth accident

$E(\xi_k) = 3$, $\mathrm{var}(\xi_k) = 4$

$E(N) = 2$, $\mathrm{var}(N) = 2$

$\therefore$ $E(X) = \mu v = 3(2) = 6$

$\mathrm{var}(X) = v\sigma^2 + \mu^2\tau^2$

$\therefore$ var(X)=2(4)+9(2)=26