



Answer the following questions:

Q1: [4+3]

(a) Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{matrix}$$

(i) Starting in state 2, determine the probability that the Markov chain ends in state 0.

(ii) Determine the mean time to absorption.

(iii) Sketch, the Markov chain diagram, and determine whether it's an absorbing chain or not.

b) Let  $X_n$  denote the quality of the nth item that produced in a certain factory with  $X_n = 0$  meaning "good" and  $X_n = 1$  meaning "defective". Suppose that  $\{X_n\}$  be a Markov chain whose transition matrix is

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \left\| \begin{array}{cc} 0.89 & 0.11 \\ 0.02 & 0.98 \end{array} \right\| \end{matrix}$$

i) What is the probability that the fourth item is good given that the first item is defective?

ii) In the long run, what is the probability that an item produced by this system is good or it's defective?

Q2: [8]

(a) A Markov chain  $X_0, X_1, X_2, \dots$  has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.5 & 0.1 \\ 0.3 & 0.2 & 0.5 \end{vmatrix} \end{matrix}$$

and initial distribution  $p_0 = 0.3$  and  $p_1 = 0.7$ . Determine the following probabilities

- i)  $pr\{X_2 = 0\}$
- ii)  $pr\{X_3 = 0\}$
- iii)  $pr\{X_0 = 1, X_1 = 1, X_2 = 0\}$
- iv)  $pr\{X_1 = 1, X_2 = 1, X_3 = 0\}$

**Q3: [3+3]**

(a) Suppose that the weather on any day depends on the weather conditions for the previous 2 days. Suppose also that if it was sunny today but cloudy yesterday, then it will be sunny tomorrow with probability 0.5; if it was cloudy today but sunny yesterday, then it will be sunny tomorrow with probability 0.4; if it was sunny today and yesterday, then it will be sunny tomorrow with probability 0.7; if it was cloudy for the last 2 days, then it will be sunny tomorrow with probability 0.2. Transform this model into a Markov chain, and then find the transition probability matrix. Find also the limiting distribution and the long run fraction of days in which it is sunny.

(b) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with  $\Pr\{\xi_n = 0\} = 0.3$ ,  $\Pr\{\xi_n = 1\} = 0.5$ ,  $\Pr\{\xi_n = 2\} = 0.2$ . and suppose  $s=0$  and  $S=2$ . Determine the transition probability matrix for the Markov chain  $\{X_n\}$ , where  $X_n$  is defined to be the quantity on hand at the end of period  $n$ .

**Q4: [3+3]**

(a) The number of accidents occurring in a factory in a week is a Poisson random variable with mean 3. The number of individuals injured in different accidents is independent exponentially distributed, each with mean 0.5. Determine the mean and variance of the number of individuals injured in a week.

(b) The probability of the thrower winning in the dice game is  $p=0.5071$ . Suppose player A is the thrower and begins the game with \$10, and player B, his opponent, begins with \$5. What is the probability that player A goes bankrupt before player B? Assume that the bet is \$1 per round.

Q5: [3+3]

(a) A pure birth process starting from  $X(0) = 0$  has birth parameters  $\lambda_0 = 1$ ,  $\lambda_1 = 2$  and  $\lambda_2 = 3$ . Determine  $P_n(t)$  for  $n = 0, 1, 2$ .

(b) A pure death process starting from  $X(0) = 3$  has death parameters  $\mu_0 = 0$ ,  $\mu_1 = 2$ ,  $\mu_2 = 3$  and  $\mu_3 = 4$ . Determine  $P_n(t)$  for  $n = 0, 1, 2, 3$ .

Q6: [3+4]

(a) Demands on a first aid facility in a certain location occur according to a nonhomogeneous Poisson process having the rate function

$$\lambda(t) = \begin{cases} 2-t & \text{for } 0 \leq t < 1 \\ 3 & \text{for } 1 \leq t < 2 \\ 0.25t & \text{for } 2 \leq t \leq 4 \end{cases}$$

where  $t$  is measured in hours from the opening time of the facility. What is the probability that two demands occur in the first 2h of operation and two in the second 2h?

(b)

Let  $p_n(t) = pr\{X(t) = n\}$

$$= \lambda_{n-1} e^{-\lambda_n t} \int_0^t e^{\lambda_n x} p_{n-1}(x) dx, \quad n = 1, 2, 3, \dots$$

and  $p_0(t) = e^{-\lambda_0 t}$

where  $\lambda_0, \lambda_1, \lambda_2, \dots$  are distinct birth parameters.

Calculate  $p_1(t)$  and  $p_2(t)$

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## Model Answer

Q1: [4+3]

(a)

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{matrix}$$

$$u_i = pr\{X_T = 0 | X_0 = i\} \text{ for } i=1,2,$$

$$\text{and } v_i = E[T | X_0 = i] \text{ for } i=1,2.$$

(i)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$

$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.2 + 0.4u_1 + 0.3u_2$$

$$u_2 = 0.1 + 0.5u_1 + 0.3u_2$$

$\Rightarrow$

$$6u_1 - 3u_2 = 2 \quad (1)$$

$$5u_1 - 7u_2 = -1 \quad (2)$$

Solving (1) and (2), we get

$$u_1 = \frac{17}{27} \text{ and } u_2 = \frac{16}{27}$$

Starting in state 2, the probability that the Markov chain ends in state 0 is

$$u_2 = u_{20} = \frac{16}{27} = 0.5926$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$

$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

⇒

$$v_1 = 1 + 0.4v_1 + 0.3v_2$$

$$v_2 = 1 + 0.5v_1 + 0.3v_2$$

⇒

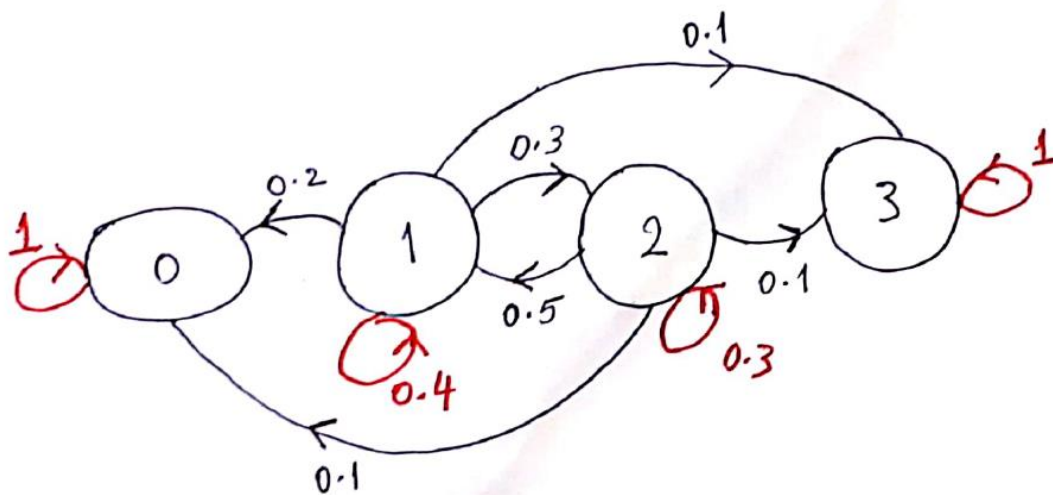
$$6v_1 - 3v_2 = 10 \quad (1)$$

$$5v_1 - 7v_2 = -10 \quad (2)$$

Solving (1) and (2), we get

$$v_2 = v_{20} = \frac{110}{27} \\ \approx 4.0741$$

(iii) It's an absorbing Markov Chain.



Markov Chain Diagram

(b)

i)

$$P^2 = \begin{bmatrix} 0.89 & 0.11 \\ 0.02 & 0.98 \end{bmatrix} \begin{bmatrix} 0.89 & 0.11 \\ 0.02 & 0.98 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.7943 & 0.2057 \\ 0.0374 & 0.9626 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.7110 & 0.2890 \\ 0.0525 & 0.9475 \end{bmatrix}$$

$$\begin{aligned} \text{pr}\{X_3 = 0 | X_0 = 1\} &= p_{10}^3 = 0.0525 \\ &= 5.25\% , \end{aligned}$$

which is the probability that the fourth item is good given that the first item is defective

ii) In the long run, the probability that an item produced by this system is good is given by:

$$\begin{aligned} b / (a + b) &= \frac{0.02}{0.02 + 0.11} \\ &= \frac{2}{13} = 15.38\% \end{aligned}$$

In the long run, the probability that an item produced by this system is defective is given by:

$$\begin{aligned} a / (a + b) &= \frac{0.11}{0.02 + 0.11} \\ &= \frac{11}{13} = 84.62\% , \end{aligned}$$

$$\text{where } \lim_{n \rightarrow \infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

$$\text{Clearly, } 1 - \frac{2}{13} = \frac{11}{13} = 84.62\%$$

Q2: [8]

$$\begin{aligned} \text{i) } \because pr\{X_2 = 0\} &= pr\{X_2 = 0|X_0 = 0\}pr\{X_0 = 0\} \\ &+ pr\{X_2 = 0|X_0 = 1\}pr\{X_0 = 1\} \\ &= P_{00}^2 P_0 + P_{10}^2 P_1, \quad P_0 = 0.3, P_1 = 0.7 \end{aligned}$$

$$\begin{aligned} P^2 &= \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.5 & 0.1 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.5 & 0.1 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.31 & 0.31 & 0.38 \\ 0.31 & 0.39 & 0.30 \\ 0.29 & 0.29 & 0.42 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \therefore pr\{X_2 = 0\} &= 0.31(0.3) + 0.31(0.7) \\ &= 0.31 \end{aligned}$$

$$\begin{aligned} \text{ii) } \because pr\{X_3 = 0\} &= pr\{X_3 = 0|X_0 = 0\}pr\{X_0 = 0\} \\ &+ pr\{X_3 = 0|X_0 = 1\}pr\{X_0 = 1\} \\ &= P_{00}^3 P_0 + P_{10}^3 P_1, \quad P_0 = 0.3, P_1 = 0.7 \end{aligned}$$

$$\begin{aligned} P^3 &= \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.5 & 0.1 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.31 & 0.31 & 0.38 \\ 0.31 & 0.39 & 0.30 \\ 0.29 & 0.29 & 0.42 \end{bmatrix} \\ &= \begin{bmatrix} 0.3000 & 0.3240 & 0.3760 \\ 0.3080 & 0.3480 & 0.3440 \\ 0.3000 & 0.3160 & 0.3840 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \therefore pr\{X_3 = 0\} &= 0.3(0.3) + 0.3080(0.7) \\ &= 0.09 + 0.2156 \\ &= 0.3056 \end{aligned}$$

$$\begin{aligned} \text{iii) } pr\{X_0 = 1, X_1 = 1, X_2 = 0\} &= p_1 P_{11} P_{10} \quad , \quad p_1 = pr\{X_0 = 1\} \\ &= 0.7(0.5)(0.4) \\ &= 0.14 \end{aligned}$$

$$\text{iv) } pr\{X_1=1, X_2=1, X_3=0\} = p_1 P_{11} P_{10} \quad , \quad p_1 = pr\{X_1=1\}$$

to find  $pr\{X_1=1\}$

$$\begin{aligned} pr\{X_1=1\} &= pr\{X_1=1|X_0=0\}pr\{X_0=0\} \\ &\quad + pr\{X_1=1|X_0=1\}pr\{X_0=1\} \\ &\quad + pr\{X_1=1|X_0=2\}pr\{X_0=2\} \\ &= P_{01}p_0 + P_{11}p_1 + P_{21}p_2 \end{aligned}$$

$$\begin{aligned} \therefore pr\{X_1=1\} &= 0.3(0.3) + 0.5(0.7) \\ &= 0.09 + 0.35 = 0.44 \end{aligned}$$

$$\begin{aligned} \therefore pr\{X_1=1, X_2=1, X_3=0\} &= 0.44(0.5)(0.4) \\ &= 0.088 \end{aligned}$$

**Q3: [3+3]**

(a)

|          | $(S, S)$ | $(S, C)$ | $(C, S)$ | $(C, C)$ |
|----------|----------|----------|----------|----------|
| $(S, S)$ | 0.7      | 0.3      | 0        | 0        |
| $(S, C)$ | 0        | 0        | 0.4      | 0.6      |
| $(C, S)$ | 0.5      | 0.5      | 0        | 0        |
| $(C, C)$ | 0        | 0        | 0.2      | 0.8      |

In the long run, the limiting distribution is  $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$

$$0.7\pi_0 + 0.5\pi_2 = \pi_0 \Rightarrow \pi_2 = \frac{3}{5}\pi_0 \quad (1)$$

$$0.3\pi_0 + 0.5\pi_2 = \pi_1 \Rightarrow \pi_1 = \frac{3}{5}\pi_0 \quad (2)$$

$$0.6\pi_1 + 0.8\pi_3 = \pi_3 \Rightarrow \pi_3 = \frac{9}{5}\pi_0 \quad (3)$$

$$\text{And } \therefore \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \quad (4)$$

$$\therefore \pi_0 = 0.25$$

$$\Rightarrow \pi = (0.25, 0.15, 0.15, 0.45)$$

The long run fraction of days in which it is sunny is

$$\begin{aligned} \pi_0 + \pi_1 &= 0.25 + 0.15 \\ &= 0.4 \end{aligned}$$



(b) Inventory Model

$$\begin{array}{c} -1 \quad 0 \quad 1 \quad 2 \\ -1 \left\| \begin{array}{cccc} 0 & 0.2 & 0.5 & 0.3 \end{array} \right\| \\ 0 \left\| \begin{array}{cccc} 0 & 0.2 & 0.5 & 0.3 \end{array} \right\| \\ 1 \left\| \begin{array}{cccc} 0.2 & 0.5 & 0.3 & 0 \end{array} \right\| \\ 2 \left\| \begin{array}{cccc} 0 & 0.2 & 0.5 & 0.3 \end{array} \right\| \end{array}$$

where

$$P_{ij} = \Pr\{X_{n+1} = j | X_n = i\} \\ = \begin{cases} \Pr(\xi_{n+1} = 2 - j), i \leq 0 & \text{replenishment} \\ \Pr(\xi_{n+1} = i - j), 0 < i \leq 2 & \text{without replenishment} \end{cases}$$

**Q4: [3+3]**

(a)

$N \sim \text{Poisson}(3)$

$N$  is the # of accidents in a week

$\xi_k$  is the # of individuals injured for  $k$ th accident

$$E(\xi_k) = \mu = 0.5, \quad \text{var}(\xi_k) = \sigma^2 = 0.25$$

$$E(N) = v = 3, \quad \text{var}(N) = \tau^2 = 3$$

$$\therefore E(X) = \mu v = 0.5(3) = 1.5$$

$$\text{var}(X) = v\sigma^2 + \mu^2\tau^2$$

$$\therefore \text{var}(X) = 3(0.25) + 0.25(3) = 1.5$$

(b)

$i = \$10$  fortune for player A

$$N = \$10 + \$5 = \$15$$

$$p = 0.5071 \Rightarrow q = 0.4929$$

$u_i = pr \{X_n \text{ reaches state 0 before state } N | X_0 = i\}$

$$u_i = \frac{(q/p)^i - (q/p)^N}{1 - (q/p)^N}, \quad p \neq q$$

$$u_i = \frac{\left[ \left( \frac{0.4929}{0.5071} \right)^{10} - \left( \frac{0.4929}{0.5071} \right)^{15} \right]}{\left[ 1 - \left( \frac{0.4929}{0.5071} \right)^{15} \right]}$$

$$= 0.2873$$

**Q5: [3+3]**

(a)

For pure birth process,

$$p_0(t) = e^{-\lambda_0 t}, \quad (1)$$

$$p_1(t) = \lambda_0 \left[ \frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} \right], \quad (2)$$

and  $p_n(t) = pr \{X(t) = n | X(0) = 0\}$

$$= \lambda_0 \lambda_1 \dots \lambda_{n-1} \left[ B_{0,n} e^{-\lambda_0 t} + \dots + B_{k,n} e^{-\lambda_k t} + \dots + B_{n,n} e^{-\lambda_n t} \right], \quad n > 1, \quad (3)$$

where

$$B_{k,n} = \prod_{i=0}^n \left( \frac{1}{\lambda_i - \lambda_k} \right) \quad i \neq k, \quad 0 < k < n,$$

$$B_{0,n} = \prod_{i=1}^n \left( \frac{1}{\lambda_i - \lambda_0} \right)$$

and

$$B_{n,n} = \prod_{i=0}^{n-1} \left( \frac{1}{\lambda_i - \lambda_n} \right)$$

at  $n = 0$  (1)  $\Rightarrow p_0(t) = e^{-\lambda_0 t}$ ,  $\lambda_0 = 1$

$$\therefore p_0(t) = e^{-t}$$

at  $n=1$  (2)  $\Rightarrow p_1(t) = e^{-t} - e^{-2t}$

at  $n=2$  (3)  $\Rightarrow p_2(t) = \lambda_0 \lambda_1 [B_{0,2} e^{-\lambda_0 t} + B_{1,2} e^{-\lambda_1 t} + B_{2,2} e^{-\lambda_2 t}]$ ,

$$\text{where, } B_{0,2} = \frac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)}$$

$$= \frac{1}{2},$$

$$B_{1,2} = \frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)}$$

$$= -1$$

and

$$B_{2,2} = \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}$$

$$= \frac{1}{2}$$

$$\therefore p_2(t) = 2 \left[ \frac{1}{2} e^{-t} - e^{-2t} + \frac{1}{2} e^{-3t} \right]$$

$$= e^{-t} - 2e^{-2t} + e^{-3t}$$

(b) The transition probabilities are given by

$$p_N(t) = e^{-\mu_N t} \quad (1)$$

and for  $n < N$

$$p_n(t) = pr \{ X(t) = n | X(0) = N \}$$

$$= \mu_{n+1} \mu_{n+2} \dots \mu_N \left[ A_{n,n} e^{-\mu_n t} + \dots + A_{k,n} e^{-\mu_k t} + \dots + A_{N,n} e^{-\mu_N t} \right] \quad (2)$$

$$\text{where } A_{k,n} = \prod_{i=N}^n \frac{1}{(\mu_i - \mu_k)}, \quad i \neq k, \quad n \leq k \leq N, \quad i = N, N-1, \dots, n \quad (3)$$

For  $N=3$  (1)  $\Rightarrow p_3(t) = e^{-\mu_3 t}$

$$\therefore p_3(t) = e^{-4t} \quad (I)$$

For  $n=2$  (2)  $\Rightarrow p_2(t) = \mu_3 [A_{2,2} e^{-\mu_2 t} + A_{3,2} e^{-\mu_3 t}]$

$$(3) \Rightarrow A_{2,2} = \prod_{i=3}^2 \frac{1}{(\mu_i - \mu_2)}, \quad i \neq 2$$

$$= \frac{1}{\mu_3 - \mu_2} = 1,$$

$$A_{3,2} = \prod_{i=3}^2 \frac{1}{(\mu_i - \mu_3)}, \quad i \neq 3$$

$$= \frac{1}{\mu_2 - \mu_3} = -1$$

$$\therefore p_2(t) = 4[e^{-3t} - e^{-4t}] \quad (\text{II})$$

$$\text{For } n=1 \quad (2) \Rightarrow p_1(t) = \mu_2 \mu_3 [A_{1,1} e^{-\mu_1 t} + A_{2,1} e^{-\mu_2 t} + A_{3,1} e^{-\mu_3 t}]$$

$$(3) \Rightarrow A_{1,1} = \prod_{i=3}^1 \frac{1}{(\mu_i - \mu_1)}, \quad i \neq 1$$

$$= \frac{1}{(\mu_3 - \mu_1)(\mu_2 - \mu_1)} = \frac{1}{2},$$

$$A_{2,1} = \prod_{i=3}^1 \frac{1}{(\mu_i - \mu_2)}, \quad i \neq 2$$

$$= \frac{1}{(\mu_3 - \mu_2)(\mu_1 - \mu_2)} = -1,$$

$$A_{3,1} = \prod_{i=3}^1 \frac{1}{(\mu_i - \mu_3)}, \quad i \neq 3$$

$$= \frac{1}{(\mu_2 - \mu_3)(\mu_1 - \mu_3)} = \frac{1}{2}$$

$$p_1(t) = 12 \left[ \frac{1}{2} e^{-2t} - e^{-3t} + \frac{1}{2} e^{-4t} \right]$$

$$\therefore p_1(t) = 6[e^{-2t} - 2e^{-3t} + e^{-4t}] \quad (\text{III})$$

Using (I), (II) and (III) we can get  $p_0(t)$  as follows

$$\begin{aligned}
\therefore p_0(t) &= 1 - [p_1(t) + p_2(t) + p_3(t)] \\
&= 1 - [6e^{-2t} - 12e^{-3t} + 6e^{-4t} + 4e^{-3t} - 4e^{-4t} + e^{-4t}] \\
&= 1 - 6e^{-2t} + 8e^{-3t} - 3e^{-4t} \quad (\text{IV})
\end{aligned}$$

Q6: [6]

(a)

i)

$$\begin{aligned}
\mu_1 &= \int_0^2 \lambda(u) du \\
&= \int_0^1 (2-t) dt + \int_1^2 3 dt \\
&= \left[ 2t - \frac{t^2}{2} \right]_0^1 + 3 \left[ t \right]_1^2 \\
&= 4.5
\end{aligned}$$

The prob. that two demands occur in the first 2h of operation is

$$\begin{aligned}
\Pr\{X(2) = 2\} &= \Pr\{X(2) - X(0) = 2\} \\
&= \frac{e^{-\mu_1} \mu_1^k}{k!} \\
&= \frac{e^{-4.5} \times 4.5^2}{2!} \\
&= 0.1125
\end{aligned}$$

ii)

$$\begin{aligned}
\mu_2 &= \int_2^4 \lambda(u) du \\
&= \int_2^4 (0.25t) dt \\
&= \frac{1}{4} \left[ \frac{t^2}{2} \right]_2^4 \\
&= 1.5
\end{aligned}$$

The prob. that two demands occur in the second 2h of operation is

$$\begin{aligned}
\Pr\{X(4) - X(2) = 2\} &= \frac{e^{-\mu_2} \mu_2^k}{k!} \\
&= \frac{e^{-1.5} \times 1.5^2}{2!} \\
&\approx 0.25
\end{aligned}$$

(b)

at  $n = 1$

$$\begin{aligned}
P_1(t) &= \lambda_0 e^{-\lambda_1 t} \int_0^t e^{\lambda_1 x} p_0(x) dx \\
&= \lambda_0 e^{-\lambda_1 t} \int_0^t e^{\lambda_1 x} e^{-\lambda_0 x} dx \\
&= \lambda_0 e^{-\lambda_1 t} \int_0^t e^{-(\lambda_0 - \lambda_1)x} dx \\
&= \lambda_0 e^{-\lambda_1 t} \left[ \frac{e^{-(\lambda_0 - \lambda_1)x}}{-(\lambda_0 - \lambda_1)} \right]_0^t \\
&= \lambda_0 e^{-\lambda_1 t} \left[ \frac{e^{-(\lambda_0 - \lambda_1)t}}{(\lambda_1 - \lambda_0)} + \frac{1}{\lambda_0 - \lambda_1} \right] \\
\therefore P_1(t) &= \lambda_0 \left[ \frac{1}{(\lambda_1 - \lambda_0)} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} \right] \tag{1}
\end{aligned}$$

at  $n = 2$

$$\begin{aligned}
P_2(t) &= \lambda_1 e^{-\lambda_2 t} \int_0^t e^{\lambda_2 x} p_1(x) dx \\
&= \lambda_0 \lambda_1 e^{-\lambda_2 t} \int_0^t e^{\lambda_2 x} \left[ \frac{1}{(\lambda_1 - \lambda_0)} e^{-\lambda_0 x} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 x} \right] dx \\
&= \lambda_0 \lambda_1 e^{-\lambda_2 t} \int_0^t \left[ \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 x} + \frac{1}{(\lambda_1 - \lambda_0)} e^{-\lambda_0 x} \right] e^{\lambda_2 x} dx \\
P_2(t) &= \lambda_0 \lambda_1 e^{-\lambda_2 t} \int_0^t \left[ \frac{1}{\lambda_0 - \lambda_1} e^{-(\lambda_1 - \lambda_2)x} + \frac{1}{(\lambda_1 - \lambda_0)} e^{-(\lambda_0 - \lambda_2)x} \right] dx
\end{aligned}$$

$$P_2(t) = \lambda_0 \lambda_1 e^{-\lambda_2 t} \left[ -\frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} e^{-(\lambda_1 - \lambda_2)x} - \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)} e^{-(\lambda_0 - \lambda_2)x} \right]_0^t$$

$$P_2(t) = \lambda_0 \lambda_1 e^{-\lambda_2 t} \times$$

$$\left[ -\frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} e^{-(\lambda_1 - \lambda_2)t} + \frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} - \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)} e^{-(\lambda_0 - \lambda_2)t} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)} \right]$$

$$\begin{aligned} \therefore P_2(t) &= \lambda_0 \lambda_1 \left[ \frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} e^{-\lambda_1 t} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} e^{-\lambda_0 t} \right] \\ &\quad + \lambda_0 \lambda_1 \left[ \frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)} \right] e^{-\lambda_2 t} \end{aligned} \quad (2)$$

$$\therefore P_2(t) = \lambda_0 \lambda_1 \left[ \frac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} e^{-\lambda_0 t} + \frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} e^{-\lambda_1 t} + \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)} e^{-\lambda_2 t} \right]$$

$$\text{where } \frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)} = \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}$$

$$\therefore P_2(t) = \lambda_0 \lambda_1 \left[ B_{0,2} e^{-\lambda_0 t} + B_{1,2} e^{-\lambda_1 t} + B_{2,2} e^{-\lambda_2 t} \right] \quad (3)$$

$$\text{where } B_{0,2} = \frac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)}, B_{1,2} = \frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} \text{ and } B_{2,2} = \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}$$


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