



Answer the following questions:

**Q1: [8]**

Suppose that the summands  $\xi_1, \xi_2, \dots$  are continuous random variables having a probability

$$\text{density function } f(z) = \begin{cases} \lambda e^{-\lambda z} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

and  $P_N(n) = \beta(1-\beta)^{n-1}$  for  $n=1, 2, \dots$

Find the probability density function for  $X = \xi_1 + \xi_2 + \dots + \xi_N$

**Q2: [4+4]**

Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{matrix}$$

(a) Starting in state 2, determine the probability that the Markov chain ends in state 0.

(b) Determine the mean time to absorption.

**Q3: [4+4]**

(a) Demands on a first aid facility in a certain location occur according to a nonhomogeneous Poisson process having the rate function

$$\lambda(t) = \begin{cases} 2t & \text{for } 0 \leq t < 1 \\ 2 & \text{for } 1 \leq t < 2 \\ 4-t & \text{for } 2 \leq t \leq 4 \end{cases}$$

where  $t$  is measured in hours from the opening time of the facility. What is the probability that two demands occur in the first 2h of operation and two in the second 2h?

(b) Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

What fraction of families are upper class in the long run?

**Q4: [4+5]**

(a) Let  $X$  and  $Y$  be independent Poisson distributed random variables with parameters  $\alpha$  and  $\beta$ , respectively. Determine the conditional distribution of  $X$ , given that  $N = X + Y = n$ .

(b) If  $X(t)$  represents a size of a population where  $X(0) = 1$ , using the following differential equations

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \tag{1}$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \quad n=1,2,3, \dots \tag{2}$$

Prove that:  $X(t) \sim geom(p)$ ,  $p = e^{-\lambda t}$  when  $\lambda_0 = 0$  and  $\lambda_n = n\lambda$ , and then find the mean and variance of this process.

**Q5: [7]**

A pure death process starting from  $X(0) = 3$  has death parameters  $\mu_0 = 0$ ,  $\mu_1 = 3$ ,  $\mu_2 = 2$  and  $\mu_3 = 5$ . Determine  $P_n(t)$  for  $n=0, 1, 2, 3$ .

## Model Answer

Q1: [8]

We have  $f_X(z) = \sum_{n=1}^{\infty} f^n(z) P_N(n)$

$\therefore$  The  $n$ -fold convolution of  $f(z)$  is the Gamma density function,  $n \geq 1$

$$\therefore f^n(z) = \begin{cases} \frac{\lambda^n}{\Gamma(n)} z^{n-1} e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$\Rightarrow$

$$f^n(z) = \begin{cases} \frac{\lambda^n}{(n-1)!} z^{n-1} e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$$\begin{aligned} \therefore f_X(z) &= \lambda \beta e^{-\lambda z} \sum_{n=1}^{\infty} \frac{[\lambda(1-\beta)z]^{n-1}}{(n-1)!} \\ &= \lambda \beta e^{-\lambda z} e^{\lambda(1-\beta)z} \\ &= \lambda \beta e^{-\lambda \beta z}, \quad z \geq 0 \end{aligned}$$

$\therefore X$  has an exponential distribution with parameter  $\lambda \beta$ .

Q2: [4+4]

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{matrix}$$

$u_i = pr\{X_T = 0 | X_0 = i\}$  for  $i=1,2$ ,

and  $v_i = E[T | X_0 = i]$  for  $i=1,2$ .

(a)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$

$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.6u_1 + 0.1u_2$$

$$u_2 = 0.2 + 0.3u_1 + 0.4u_2$$

$\Rightarrow$

$$4u_1 - u_2 = 1 \quad (1)$$

$$3u_1 - 6u_2 = -2 \quad (2)$$

Solving (1) and (2), we get

$$u_1 = \frac{8}{21} \text{ and } u_2 = \frac{11}{21}$$

Starting in state 2, the probability that the Markov chain ends in state 0 is

$$u_2 = u_{20} = \frac{11}{21}$$

(b) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$

$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.6v_1 + 0.1v_2$$

$$v_2 = 1 + 0.3v_1 + 0.4v_2$$

$\Rightarrow$

$$4v_1 - v_2 = 10 \quad (1)$$

$$3v_1 - 6v_2 = -10 \quad (2)$$

Solving (1) and (2), we get  $v_2 = v_{20} = \frac{10}{3}$

**Q3: [4+4]**

(a)

i)

$$\begin{aligned}\mu_1 &= \int_0^2 \lambda(u) du \\ &= \int_0^1 2t dt + \int_1^2 2 dt \\ &= [t^2]_0^1 + 2[t]_1^2 \\ &= 3\end{aligned}$$

The prob. that two demands occur in the first 2h of operation is

$$\begin{aligned}\Pr\{X(2) = 2\} &= \Pr\{X(2) - X(0) = 2\} \\ &= \frac{e^{-\mu_1} \mu_1^k}{k!} \\ &= \frac{e^{-3} \cdot 3^2}{2!} \\ &= 0.2240\end{aligned}$$

ii)

$$\begin{aligned}
\mu_2 &= \int_2^4 \lambda(u) du \\
&= \int_2^4 (4-t) dt \\
&= \left[ 4t - \frac{t^2}{2} \right]_2^4 \\
&= 2
\end{aligned}$$

The prob. that two demands occur in the second 2h of operation is

$$\begin{aligned}
\Pr\{X(4) - X(2) = 2\} \\
&= \frac{e^{-\mu_2} \mu_2^k}{k!} \\
&= \frac{e^{-2} \cdot 2^2}{2!} \\
&= 0.2707
\end{aligned}$$

(b)

Let  $\pi = (\pi_0, \pi_1, \pi_2)$  be the limiting distribution

$\Rightarrow$

$$\pi_0 = 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2$$

$$\pi_1 = 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2$$

$$\pi_2 = 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

Solving the following equations

$$3\pi_0 - 2\pi_1 - \pi_2 = 0 \quad (1)$$

$$\pi_0 + 2\pi_1 - 5\pi_2 = 0 \quad (2)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \quad (3)$$

We get  $\pi_0 = \frac{6}{17}$ ,  $\pi_1 = \frac{7}{17}$ ,  $\pi_2 = \frac{4}{17}$

In the long run, approximately 23.5% of families are upper class.

**Q4: [4+5]**

(a)

$$X \sim \text{Poisson}(\alpha), \quad Y \sim \text{Poisson}(\beta)$$

$$\therefore X+Y \sim \text{Poisson}(\alpha + \beta)$$

$\Rightarrow$

$$\begin{aligned} \Pr\{X = k | N = n\} &= \Pr\{X = k | X + Y = n\} \\ &= \frac{\Pr\{X = k\} \cap \Pr\{X + Y = n\}}{\Pr\{X + Y = n\}} \\ &= \frac{\Pr\{X = k\} \cap \Pr\{Y = n - k\}}{\Pr\{X + Y = n\}} \\ &= \frac{e^{-\alpha} \alpha^k / k! \cdot e^{-\beta} \beta^{n-k} / (n-k)!}{e^{-(\alpha+\beta)} (\alpha + \beta)^n / n!} \\ &= \alpha^k \beta^{n-k} \left( \frac{1}{\alpha + \beta} \right)^n \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k} \left( \frac{\alpha}{\alpha + \beta} \right)^k \left( \frac{\beta}{\alpha + \beta} \right)^{n-k} \\ &= \binom{n}{k} p^k (1-p)^{n-k} \quad \text{Binomial distribution} \end{aligned}$$

$$\text{where } p = \frac{\alpha}{\alpha + \beta} \text{ and } 1-p = \frac{\beta}{\alpha + \beta}$$

(b)

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \quad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \quad n=1,2,3, \dots, 7 \quad (2)$$

The initial condition is  $X(0) = 1 \Rightarrow p_1(0) = 1$

$$\Rightarrow p_n(0) = \begin{cases} 1 & , n=1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\begin{aligned} \lambda_0 = 0 \quad (1) &\Rightarrow \frac{dp_0(t)}{dt} = 0 \\ &\Rightarrow p_0(t) = 0 \quad (3) \end{aligned}$$

$$\begin{aligned} (2) &\Rightarrow \frac{dp_n(t)}{dt} = \lambda_{n-1}p_{n-1}(t) - \lambda_n p_n(t) \\ &\Rightarrow \frac{dp_n(t)}{dt} + \lambda_n p_n(t) = \lambda_{n-1}p_{n-1}(t), \quad n = 1, 2, \dots \end{aligned}$$

$$\because \lambda_n = n\lambda, \quad \lambda_{n-1} = (n-1)\lambda$$

$$\therefore \frac{dp_n(t)}{dt} + n\lambda p_n(t) = (n-1)\lambda p_{n-1}(t), \quad n=1, 2, \dots$$

Multiply both sides by  $e^{n\lambda t}$

$$\begin{aligned} e^{n\lambda t} \left[ \frac{dp_n(t)}{dt} + n\lambda p_n(t) \right] &= (n-1)\lambda p_{n-1}(t) e^{n\lambda t} \\ \therefore \frac{d}{dt} [p_n(t) e^{n\lambda t}] &= (n-1)\lambda p_{n-1}(t) e^{n\lambda t} \\ \Rightarrow \int_0^t d [p_n(x) e^{n\lambda x}] &= (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx \\ \therefore [p_n(x) e^{n\lambda x}]_0^t &= (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx \\ \Rightarrow p_n(t) &= e^{-n\lambda t} \left[ p_n(0) + (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx \right], \quad n = 1, 2, \dots \quad (4) \end{aligned}$$

which is a recurrence relation.

at  $n = 1$

$$p_1(t) = e^{-\lambda t} [p_1(0) + 0] = e^{-\lambda t} \quad (5)$$

at  $n = 2$



$$p_2(t) = e^{-2\lambda t} \left[ p_2(0) + \lambda \int_0^t p_1(x) e^{2\lambda x} dx \right]$$

$$(5) \Rightarrow p_1(x) = e^{-\lambda x}$$

$$\therefore p_2(t) = e^{-2\lambda t} \left[ \lambda \int_0^t e^{-\lambda x} e^{2\lambda x} dx \right]$$

$$\begin{aligned} \therefore p_2(t) &= \lambda e^{-2\lambda t} \int_0^t e^{\lambda x} dx \\ &= e^{-\lambda t} (1 - e^{-\lambda t}) \end{aligned}$$

Similarly

$$\begin{aligned} p_n(t) &= e^{-\lambda t} (1 - e^{-\lambda t})^{n-1} \\ &= p(1-p)^{n-1}, \quad p = e^{-\lambda t}, \quad n = 1, 2, \dots \end{aligned}$$

$$\therefore X(t) \sim \text{geom}(p), \quad p = e^{-\lambda t}$$

$$\text{Mean}[X(t)] = 1/p = e^{\lambda t},$$

$$\text{Variance}[X(t)] = \frac{1-p}{p^2} = \frac{1-e^{-\lambda t}}{e^{-2\lambda t}}$$

### Q5: [7]

The transition probabilities are given by

$$p_N(t) = e^{-\mu_N t} \quad (1)$$

and for  $n < N$

$$\begin{aligned} p_n(t) &= \text{pr} \{ X(t) = n \mid X(0) = N \} \\ &= \mu_{n+1} \mu_{n+2} \dots \mu_N \left[ A_{n,n} e^{-\mu_n t} + \dots + A_{k,n} e^{-\mu_k t} + \dots + A_{N,n} e^{-\mu_N t} \right] \quad (2) \end{aligned}$$

$$\text{Where } A_{k,n} = \prod_{i=N}^n \frac{1}{(\mu_i - \mu_k)}, \quad i \neq k, \quad n \leq k \leq N, \quad i = N, N-1, \dots, n \quad (3)$$

$$\text{For } N=3 \quad (1) \Rightarrow p_3(t) = e^{-\mu_3 t}$$

$$\therefore p_3(t) = e^{-5t} \quad (I)$$

$$\text{For } n=2 \quad (2) \Rightarrow p_2(t) = \mu_3 \left[ A_{2,2} e^{-\mu_2 t} + A_{3,2} e^{-\mu_3 t} \right]$$

$$\begin{aligned} (3) \Rightarrow A_{2,2} &= \prod_{i=3}^2 \frac{1}{(\mu_i - \mu_2)}, \quad i \neq 2 \\ &= \frac{1}{\mu_3 - \mu_2} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned}
, A_{3,2} &= \prod_{i=3}^2 \frac{1}{(\mu_i - \mu_3)}, \quad i \neq 3 \\
&= \frac{1}{\mu_2 - \mu_3} = \frac{-1}{3}
\end{aligned}$$

$$\therefore p_2(t) = 5 \left[ \frac{1}{3} e^{-2t} - \frac{1}{3} e^{-5t} \right] \quad (\text{II})$$

$$\text{For } n=1 \quad (2) \Rightarrow p_1(t) = \mu_2 \mu_3 \left[ A_{1,1} e^{-\mu_1 t} + A_{2,1} e^{-\mu_2 t} + A_{3,1} e^{-\mu_3 t} \right]$$

$$\begin{aligned}
(3) \Rightarrow A_{1,1} &= \prod_{i=3}^1 \frac{1}{(\mu_i - \mu_1)}, \quad i \neq 1 \\
&= \frac{1}{(\mu_3 - \mu_1)(\mu_2 - \mu_1)} = -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
A_{2,1} &= \prod_{i=3}^1 \frac{1}{(\mu_i - \mu_2)}, \quad i \neq 2 \\
&= \frac{1}{(\mu_3 - \mu_2)(\mu_1 - \mu_2)} = \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
, A_{3,1} &= \prod_{i=3}^1 \frac{1}{(\mu_i - \mu_3)}, \quad i \neq 3 \\
&= \frac{1}{(\mu_2 - \mu_3)(\mu_1 - \mu_3)} = \frac{1}{6}
\end{aligned}$$

$$\therefore p_1(t) = 10 \left[ -\frac{1}{2} e^{-3t} + \frac{1}{3} e^{-2t} + \frac{1}{6} e^{-5t} \right] \quad (\text{III})$$

Using (I), (II) and (III) we can get  $p_0(t)$  as follows

$$\begin{aligned}
\therefore p_0(t) &= 1 - [p_1(t) + p_2(t) + p_3(t)] \\
&= 1 - \left[ -5e^{-3t} + \frac{10}{3}e^{-2t} + \frac{5}{3}e^{-5t} + \frac{5}{3}e^{-2t} - \frac{5}{3}e^{-5t} + e^{-5t} \right] \\
&= 1 + 5e^{-3t} - 5e^{-2t} - e^{-5t} \quad (\text{IV})
\end{aligned}$$