King Saud University
College of Sciences
Department of Mathematics

Final Exam, S2 1438/1439
M 380 – Stochastic Processes
Time: 3 hours

Choose only 5 questions from the following:

Q1: [4+4]

(a) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \text{pr}\{X_0 = i_0\}$

(b) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.7 & 0.2 & 0.1 \\ 1 & 0 & 0.6 & 0.4 \\ 2 & 0.5 & 0 & 0.5 \end{array}$$

Find $pr\{X_1 = 1, X_2 = 1 \mid X_0 = 0\}$.

Q2: [4+4]

Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0.1 & 0.6 & 0.1 & 0.2 \\ 2 & 0.2 & 0.3 & 0.4 & 0.1 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

(a) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(b) Determine the mean time to absorption.

Q3: [5+3]

(a) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array} \right\| \end{array}$$

Every period that the process spends in state 0 incurs a cost \$2. Every period that the process spends in state 1 incurs a cost of \$5. Every period that the process spends in state 2 incurs a cost of \$3. What is the long run cost per period associated with this Markov chain?

(b) Let $X(t)$ be a Yule process that is observed at a random time U, where U is uniformly distributed over $[0,1)$. Show that $pr\{X(U)=k\} = p^k/(\beta k)$ for $k=1,2,\ldots$, with $p=1-e^{-\beta}$.

Q4: [4+4]

(a) Let $\{X_n\}$ be a Markov chain with state space $S=\{1,2\}$ has the transition probability matrix $\mathbf{P}=\left\| \begin{array}{cc} 0.5 & 0.5 \\ 1 & 0 \end{array} \right\|$, find $pr\{X_5 = 2 | X_2 = 1\}$.

(b) The probability of the thrower winning in the dice game is $p=0.4929$. Suppose player A is the thrower and begins the game with \$5, and player B, his opponent, begins with \$10. What is the probability that player A goes bankrupt before player B? Assume that the bet is \$1 per round.

Q5: [8]

Suppose that the weather on any day depends on the weather conditions for the previous 2 days. Suppose also that if it was sunny today and yesterday, then it will be sunny tomorrow with probability 0.8; if it was sunny today but cloudy yesterday, then it will be sunny tomorrow with probability 0.6; if it was cloudy today but sunny yesterday, then it will be sunny tomorrow with probability 0.4; if it was cloudy for the last 2 days, then it will be sunny tomorrow with probability 0.1. Transform this model into a Markov chain, and then find the transition probability matrix. Find also the long run fraction of days in which it is sunny.

Q6: [4+4]

(a) Using the differential equations

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - \lambda p_n(t), \text{ n=1,2,3, ...} \qquad (2)$$

where all birth parameters are the same constant λ with initial condition $X(0)=0$,

Show that $p_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!}, \ n = 0,1,2,...$

(b) Suppose that customers arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let $X(t)$ be the number of customers that have arrived up to time t. Determine the following conditional probabilities

$pr\{X(3) = 6 | X(1) = 2\}$ and $pr\{X(1) = 2 | X(3) = 6\}$.

Model Answer For

M 380 — Stochastic Processes

Q1

(a) $pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}, X_n = i_n\}$

$= pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}$

$\cdot\, pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\}$

$pr(x, y) = pr(x|y)\, pr(y)$

joint prob

$= pr\{X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\} \cdot pr\{X_n = i_n | X_{n-1} = i_{n-1}\}$

by using Markov prop.

$= pr\{X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\}\, P_{i_{n-1} i_n}$

when $P_{ij} = pr\{X_{n+1} = j | X_n = i\}$

By repeating this argument $n-1$ times, we obtain

$pr\{X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i_n\}$

$= P_{i_0} P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{n-1} i_n}$ when $P_{i_0} = pr\{X_0 = i_0\}$

b) $pr\{X_1 = 1, X_2 = 1 | X_0 = 0\}$

$= pr\{X_2 = 1 | X_1 = 1, X_0 = 0\} \cdot pr\{X_1 = 1 | X_0 = 0\}$

$= pr\{X_2 = 1 | X_1 = 1\} \cdot pr\{X_1 = 1 | X_0 = 0\}$ Markov prop.

$= P_{11} P_{01} = 0.6(0.2) = \boxed{0.12}$

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Q2

$$P = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0.1 & 0.6 & 0.1 & 0.2 \\ 2 & 0.2 & 0.3 & 0.4 & 0.1 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

0 and 3 are 2 absorbing states but 1 and 2 are not abs.

$\Rightarrow u_i = pr\{X_T = 0 \mid X_0 = i\}, \ i = 1, 2$

احتمال الوصول للحالة 0 بشرط الحالة كانت عند $X_0=1$

$$u_1 = pr\{X_T = 0 \mid X_0 = 1\}$$
$$= P_{10} + P_{11} u_1 + P_{12} u_2$$
$$u_1 = 0.1 + 0.6 u_1 + 0.1 u_2$$
$$\therefore \boxed{0.4 u_1 - 0.1 u_2 = 0.1} \quad ①$$

and

احتمال الوصول للحالة 0 بشرط الحالة كانت عند $X_0=2$

$$u_2 = pr\{X_T = 0 \mid X_0 = 2\}$$
$$= P_{20} + P_{21} u_1 + P_{22} u_2$$
$$= 0.2 + 0.3 u_1 + 0.4 u_2$$
$$\therefore \boxed{0.3 u_1 - 0.6 u_2 = -0.2} \quad ②$$

Solving ①, ②, $6 \times ① - ② \Rightarrow$

$$\therefore u_1 = u_{10} = \boxed{\frac{8}{21}}$$

$$v_i = E[T \mid X_0 = i], \ i = 1, 2$$

$$v_1 = 1 + P_{11} v_1 + P_{12} v_2$$
$$v_1 = 1 + 0.6 v_1 + 0.1 v_2$$
$$\Rightarrow \boxed{0.4 v_1 - 0.1 v_2 = 1} \quad ①$$

$$v_2 = 1 + P_{21} v_1 + P_{22} v_2$$
$$v_2 = 1 + 0.3 v_1 + 0.4 v_2$$
$$\Rightarrow 0.3 v_1 - 0.6 v_2 = -1 \quad ②$$

Solving ①, ②, $① \times 6 - ② \Rightarrow$

$$v_1 = \boxed{\frac{10}{3}}$$

Q3

(a)

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & 0.3 & 0.2 & 0.5 \\ 1 & 0.5 & 0.1 & 0.4 \\ 2 & 0.5 & 0.2 & 0.3 \end{array}$$

proce X_0, X_1, X_2

$C_0 = \$2 \quad C_1 = \$5 \quad C_2 = \$3$

long run mean cost per unit period

$$= \sum_{j=0}^{2} \pi_j C_j$$

$$= \pi_0 C_0 + \pi_1 C_1 + \pi_2 C_2 \qquad \text{I}$$

Also, we have

$$\begin{cases} \pi_0 = 0.3\pi_0 + 0.5\pi_1 + 0.5\pi_2 \\ \pi_1 = 0.2\pi_0 + 0.1\pi_1 + 0.2\pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 7\pi_0 - 5\pi_1 - 5\pi_2 = 0 & (1) \\ 2\pi_0 - 9\pi_1 + 2\pi_2 = 0 & (2) \\ \pi_0 + \pi_1 + \pi_2 = 0 & (3) \end{cases}$$

Solving (1), (2) and (3) by using Cramer's rule

$$\Delta = \begin{vmatrix} 7 & -5 & -5 \\ 2 & -9 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -132$$

& $\Delta_1 = -55$, $\Delta_2 = -24$, $\Delta_3 = -53$

$\Rightarrow$

$$\left.\begin{array}{l} \pi_0 = \frac{\Delta_1}{\Delta} = \frac{5}{12} \\ , \pi_1 = \frac{\Delta_2}{\Delta} = \frac{2}{11} \\ , \pi_2 = \frac{\Delta_3}{\Delta} = \frac{53}{132} \end{array}\right\} \quad \text{II}$$

Subs. II in I

long run mean cost per unit period

$= 389/132$

$\simeq 2.95$

(b) For Yule process

$$P_n(t) = e^{-\beta t}\left(1 - e^{-\beta t}\right)^{n-1}, \; n \geq 1$$

$$pr\{X(U) = k\} = \int_0^1 e^{-\beta u}\left(1 - e^{-\beta u}\right)^{k-1} du$$

$$, k = 1, 2, \ldots$$

$$= \frac{1}{\beta}\int_0^1 \left(1 - e^{-\beta u}\right)^{k-1} \beta e^{-\beta u}\, du$$

$$= \frac{1}{\beta}\left[\frac{\left(1 - e^{-\beta u}\right)^k}{k}\right]_0^1$$

$$= \frac{1}{\beta k}\left(1 - e^{-\beta}\right)^k$$

$$= \frac{p^k}{\beta k}, \; k = 1, 2, \ldots$$

where $p = 1 - e^{-\beta}$

#

Q 4 (a) P_{12}^{3} ?

$$P^2 = \begin{Vmatrix} 1/2 & 1/2 \\ 1 & 0 \end{Vmatrix} \begin{Vmatrix} 1/2 & 1/2 \\ 1 & 0 \end{Vmatrix} = \begin{Vmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{Vmatrix}$$

$$P^3 = \begin{array}{c} 1 \\ 2 \end{array}\begin{Vmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{Vmatrix} \begin{Vmatrix} 1/2 & 1/2 \\ 1 & 0 \end{Vmatrix}$$

$$P_{12}^{3} = 3/8$$

(b) $i = \$5$ fortune for player A

$N = \$5 + \$10 = \$15$

$p = 0.4929 \Rightarrow q = 1 - p = 0.5071$

$u_i = pr\{X_n \text{ reaches state } 0 \text{ before state } N \mid X_0 = i\}$

$$u_i = \frac{(q/p)^i - (q/p)^N}{1 - (q/p)^N}, \quad p \neq q$$

$$u_i = \frac{\left[\left(\frac{0.5071}{0.4929}\right)^5 - \left(\frac{0.5071}{0.4929}\right)^{15}\right]}{1 - \left(\frac{0.5071}{0.4929}\right)^{15}}$$

$\therefore u_i \simeq 0.71273$

\#

Q5

Weather state = $\{(S,S), (S,C), (C,S), (C,C)\}$

S → sunny & C → cloudy

⇒ The transition prob. Mx is given by

	(S,S)	(S,C)	(C,S)	(C,C)
(S,S)	0.8	0.2	0	0
(S,C)	0	0	0.4	0.6
(C,S)	0.6	0.4	0	0
(C,C)	0	0	0.1	0.9
	↓ π_0	↓ π_1	↓ π_2	↓ π_3

For long run the limiting distⁿ is

$$\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$$

$0.8\pi_0 + 0.6\pi_2 = \pi_0 \Rightarrow \pi_2 = \frac{1}{3}\pi_0$ } $\pi_1 = \pi_2 = \frac{1}{3}\pi_0$

$0.2\pi_0 + 0.4\pi_2 = \pi_1 \Rightarrow \pi_1 = \frac{1}{3}\pi_0$

$0.4\pi_1 + 0.1\pi_3 = \pi_2 \Rightarrow \pi_3 = 6\pi_1 = 2\pi_0$

$\because \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow (1 + \frac{1}{3} + \frac{1}{3} + 2)\pi_0 = 1$

$\therefore \pi_0 = \boxed{\frac{3}{11}}, \pi_1 = \pi_2 = \boxed{\frac{1}{11}}, \pi_3 = \boxed{\frac{6}{11}}$

∗ the long run fraction of days in which it is sunny

$= \pi_0 + \pi_1 = \frac{4}{11}$

Q6

(a) $X(t)$ represents the size of the population

$X(0) = 0$ is the initial condition

$$\Rightarrow P_n(0) = \begin{cases} 1 & ; n = 0 \\ 0 & ; \text{otherwise} \end{cases}$$

$$(1) \Rightarrow \frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

$$\int_0^t \frac{dP_0(u)}{P_0(u)} = -\lambda \int_0^t du$$

$$\therefore \left[\ln P_0(u)\right]_0^t = -\lambda \left[u\right]_0^t$$

$$\ln P_0(t) - \ln P(0) = -\lambda t$$

$P_0(0) = 1$ initial condition $\Rightarrow \ln(1) = 0$

$$\therefore \ln P_0(t) = -\lambda t \Rightarrow \boxed{P_0(t) = e^{-\lambda t}} \quad (3)$$

$$(2) \Rightarrow \frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t)$$

$$\therefore \frac{dP_n(t)}{dt} + \lambda P_n(t) = \lambda P_{n-1}(t) \quad , n = 1, 2, \cdots$$

Multiply both sides by $e^{\lambda t}$

$$e^{\lambda t}\left[\frac{dP_n(t)}{dt} + \lambda P_n(t)\right] = e^{\lambda t}\left[\lambda P_{n-1}(t)\right]$$

$$\therefore \frac{d}{dt}\left[P_n(t)\, e^{\lambda t}\right] = \lambda P_{n-1}(t)\, e^{\lambda t}$$

$$d[P_n(t)\, e^{\lambda t}] = \lambda P_{n-1}(t)\, e^{\lambda t}\, dt$$

$$\Rightarrow \int_0^t d[P_n(x)\, e^{\lambda x}] = \lambda \int_0^t P_{n-1}(x)\, e^{\lambda x}\, dx$$

$$\therefore \left[P_n(x) e^{\lambda x}\right]_0^t = \lambda \int_0^t P_{n-1}(x) e^{\lambda x}\, dx$$

$$P_n(t) e^{\lambda t} - \cancel{P_n(0)} = \lambda \int_0^t P_{n-1}(x) e^{\lambda x}\, dx$$

0 , $n = 1, 2, \cdots$

$$\therefore P_n(t) = e^{-\lambda t}\left[\lambda \int_0^t P_{n-1}(x) e^{\lambda x}\, dx\right]$$

$$P_n(t) = \lambda e^{-\lambda t} \int_0^t P_{n-1}(x) e^{\lambda x}\, dx \quad , n = 1, 2, \cdots \qquad (4)$$

which is a recurrence relation

$\boxed{\text{at } n=1}$ $(4) \Rightarrow P_1(t) = \lambda e^{-\lambda t} \int_0^t P_0(x) e^{\lambda x}\, dx$

$(3) \Rightarrow$

$$= \lambda e^{-\lambda t} \int_0^t e^{-\lambda x} e^{\lambda x}\, dx$$

$$= \lambda e^{-\lambda t} \int_0^t dx$$

$$\boxed{P_1(t) = \lambda t e^{-\lambda t}} \qquad (5)$$

$\boxed{\text{at } n=2}$ $(4) \Rightarrow P_2(t) = \lambda e^{-\lambda t} \int_0^t P_1(x) e^{\lambda x}\, dx$

$(5) \Rightarrow$

$$P_2(t) = \lambda e^{-\lambda t} \int_0^t \lambda x\, e^{-\lambda x} e^{\lambda x}\, dx$$

$$P_2(t) = \lambda^2 e^{-\lambda t} \int_0^t x\, dx = \frac{1}{2}\lambda^2 e^{-\lambda t} t^2$$

$$\therefore P_2(t) = \frac{1}{2}(\lambda t)^2 e^{-\lambda t} \quad \cdots \qquad (6)$$

Similarly $P_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!}$, $n = 0, 1, 2, \cdots$

which is called Poisson process #

(b) $pr\{X(3)=6 \mid X(1)=2\}$

$= pr\{X(3)=6\}$

$= pr\{X(3)-X(1)=4\}$

$= \dfrac{(\lambda t)^k e^{-\lambda t}}{k!}$

$= \dfrac{4^4 e^{-4}}{4!} = \dfrac{32 e^{-4}}{3}$

$pr(A \mid B) = pr(A)$ when A & B are independent

$\overset{2}{X(1)}$ ... $\overset{6}{X(3)}$

$t=1$ ... 4 occurrences ... $t=3$

$\lambda t = 2(2) = 4$

$pr\{X(1)=2 \mid X(3)=6\}$

Binomial dist'n

$= \binom{n}{x} p^x q^{n-x}$

$= \binom{6}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$ #