King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1 1441
M 380 - Stochastic Processes
Time: 3 hours

Answer the following questions:

Q1: [4+4]

(a)Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P}=\begin{array}{c} \\ 0\\ 1\\ 2\\ 3\end{array}\begin{array}{c}\begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ \left\|\begin{array}{cccc}1 & 0 & 0 & 0\\ 0.1 & 0.4 & 0.1 & 0.4\\ 0.2 & 0.1 & 0.6 & 0.1\\ 0 & 0 & 0 & 1\end{array}\right\|\end{array}$$

(i) Starting in state 2, determine the probability that the Markov chain ends in state 0.

(ii) Determine the mean time to absorption.

b) Let X_n denote the quality of the nth item that produced in a certain factory with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective". Suppose that $\{X_n\}$ be a Markov chain whose transition matrix is

$$P=\begin{array}{c} \\ 0\\ 1\end{array}\begin{array}{c}\begin{array}{cc}0 & 1\end{array}\\ \left\|\begin{array}{cc}0.99 & 0.01\\ 0.12 & 0.88\end{array}\right\|\end{array}$$

i) What is the probability that the fourth item is defective given that the first item is defective?

ii) In the long run, what is the probability that an item produced by this system is good?

Q2: [4+4]

(a) The following experiment is performed: An observation is made of a Poisson random variable N with parameter λ. Then N independent Bernoulli trials are performed, each with probability p of success. Let Z be the total number of successes observed in the N trials.

i) Formulate Z as a random sum and thereby determine its mean and variance.

ii) What is the distribution of Z?

(b) Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a good item is followed by another good item with probability α and is followed by a defective item with probability $1-\alpha$. Similarly, a defective item is followed by another defective item with probability β and is followed by a good item with probability $1-\beta$. Answer each of the following:

i) If the first item is good, what is the probability that the first defective item to appear is the fifth item?

ii) If the first item is bad, what is the probability that the first good item to appear is the fifth item?

Q3: [5+4]

(a) Suppose that the weather on any day depends on the weather conditions for the previous 2 days. Suppose also that if it was sunny today but cloudy yesterday, then it will be sunny tomorrow with probability 0.4; if it was cloudy today but sunny yesterday, then it will be sunny tomorrow with probability 0.6; if it was sunny today and yesterday, then it will be sunny tomorrow with probability 0.2; if it was cloudy for the last 2 days, then it will be sunny tomorrow with probability 0.9. Transform this model into a Markov chain, and then find the transition probability matrix. Find also the long run fraction of days in which it is sunny.

(b) Suppose that customers arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let $X(t)$ be the number of customers that have arrived up to time t. Determine the following:

i) $pr\{X(1)=2 \text{ and } X(3)=6\}$

ii) $pr\{X(3)=6 | X(1)=2\}$

Q4: [5+4]

(a) If $X(t)$ represents a size of a population where $X(0)=1$, using the following differential equations

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \quad n=1,2,3, \ldots \qquad (2)$$

Prove that: $X(t) \sim geom\ (p)$, $p = e^{-\lambda t}$ when $\lambda_0 = 0$ and $\lambda_n = n\lambda$, and then find the mean and variance of this process.

(b) Let $X(t)$ be a Yule process that is observed at a random time U, where U is uniformly distributed over $[0,1)$. Show that $pr\{X(U)=k\} = p^k / (\beta k)$ for $k = 1, 2, \ldots$, with $p = 1 - e^{-\beta}$.

Q5: [6]

A pure birth process starting from $X(0) = 0$ has birth parameters $\lambda_0 = 1$, $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = 5$. Determine $P_n(t)$ for $n = 0,\ 1,\ 2$.

Model Answer

Q1: [4+4]

(a)

$$\mathbf{P}=\begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0.1 & 0.4 & 0.1 & 0.4 \\ 2 & 0.2 & 0.1 & 0.6 & 0.1 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$u_i = pr\{X_T = 0 | X_0 = i\}$ for i=1,2,

and v_i =E[$T | X_0 = i$] for i=1,2.

(i)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.4u_1 + 0.1u_2$$
$$u_2 = 0.2 + 0.1u_1 + 0.6u_2$$

$\Rightarrow$

$$6u_1 - u_2 = 1 \qquad (1)$$
$$u_1 - 4u_2 = -2 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \tfrac{6}{23} \text{ and } u_2 = \tfrac{13}{23}$$

Starting in state 2, the probability that the Markov chain ends in state 0 is

$$u_2 = u_{20} = \tfrac{13}{23}$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.4v_1 + 0.1v_2$$
$$v_2 = 1 + 0.1v_1 + 0.6v_2$$

$\Rightarrow$

$$6v_1 - \ v_2 = 10 \qquad (1)$$
$$v_1 - 4v_2 = -10 \qquad (2)$$

Solving (1) and (2), we get $v_2 = v_{20} = \frac{70}{23}$

(b)

i)

$$P^3 = \begin{bmatrix} 0.9737 & 0.0263 \\ 0.3152 & 0.6848 \end{bmatrix}$$
$$pr\left\{X_3 = 1 \middle| X_0 = 1\right\} = p_{11}^3 = 0.6848$$

ii)

In the long run, the probability that an item produced by this system is good is given by:

$$b/(a+b) = \frac{0.12}{0.01+0.12}$$
$$= \frac{12}{13} = 92.13\ \%\ ,$$

where $\lim\limits_{n\to\infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

Q2: [4+4]

(a)

i) $Z = \xi_1 + \xi_2 + ... + \xi_N,\ N > 0$

$$E(\xi_k) = \mu = p,\ \text{Var}(\xi_k) = \sigma^2 = p(1-p)$$
$$E(N) = v = \lambda,\ \ \text{Var}(N) = \tau^2 = \lambda$$
$$\because E(Z) = \mu v$$
$$\therefore E(Z) = \lambda p$$
$$\because\ \text{Var(Z)} = v\sigma^2 + \mu^2\tau^2$$
$$\therefore\ \text{Var(Z)} = \lambda p(1-p) + p^2\lambda$$
$$= \lambda p$$

ii) $Z \sim$ Poisson(λp)

(b)

i)

$$\Pr\left\{X_2 = G,\ X_3 = G,\ X_4 = G,\ X_5 = D \middle| X_1 = G\right\}$$
$$= \Pr\left\{X_5 = D,\ X_4 = G,\ X_3 = G,\ X_2 = G \middle| X_1 = G\right\}$$
$$= \Pr\left\{X_5 = D \middle| X_4 = G\right\} \cdot \Pr\left\{X_4 = G \middle| X_3 = G\right\} \cdot \Pr\left\{X_3 = G \middle| X_2 = G\right\} \cdot \Pr\left\{X_2 = G \middle| X_1 = G\right\}$$
$$= p_{GD} p_{GG}^3$$
$$= (1-\alpha)\alpha^3$$
$$= \alpha^3(1-\alpha)$$

Also, you can solve it as follows.

$$p_1 p_{12} p_{23} p_{34} p_{45},\ p_1 = \Pr(X_1 = G) = 1$$
$$= p_G p_{GG}^3 p_{GD}$$
$$= \alpha^3(1-\alpha)$$

ii)

Similarly,

$$\Pr\{X_2 = D,\ X_3 = D, X_4 = D, X_5 = G | X_1 = D\}$$
$$= \Pr\{X_5 = G,\ X_4 = D, X_3 = D, X_2 = D | X_1 = D\}$$
$$= \Pr\{X_5 = G | X_4 = D\} \cdot \Pr\{X_4 = D |\ X_3 = D\} \cdot \Pr\{X_3 = D |\ X_2 = D\} \cdot \Pr\{X_2 = D | X_1 = D\ \}$$
$$= p_{DG} p_{DD}^3$$
$$= (1-\beta)\beta^3$$
$$= \beta^3 (1-\beta)$$

Also, you can solve it as follows.

$$p_1 p_{12} p_{23} p_{34} p_{45},\ p_1 = \Pr(X_1 = D) = 1$$
$$= p_D p_{DD}^3 p_{DG}$$
$$= \beta^3 (1-\beta)$$

Q3: [5+4]

(a)

	(S,S)	(S,C)	(C,S)	(C,C)
(S,S)	0.2	0.8	0	0
(S,C)	0	0	0.6	0.4
(C,S)	0.4	0.6	0	0
(C,C)	0	0	0.9	0.1

In the long run, the limiting distribution is $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$

$$0.2\pi_0 + 0.4\pi_2 = \pi_0 \Rightarrow \pi_2 = 2\pi_0 \qquad (1)$$
$$0.8\pi_0 + 0.6\pi_1 = \pi_1 \Rightarrow \pi_1 = 2\pi_0 \qquad (2)$$
$$0.4\pi_1 + 0.1\pi_3 = \pi_3 \Rightarrow \pi_3 = \frac{8}{9}\pi_0 \qquad (3)$$

And $\because\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \qquad (4)$

$$\therefore\ \pi_0 = \frac{9}{53} = 0.1698$$
$$\Rightarrow \pi = \left(\frac{9}{53}, \frac{18}{53}, \frac{18}{53}, \frac{8}{53}\right)$$

The long run fraction of days in which it is sunny is

$$\pi_0 + \pi_1 = \frac{9}{53} + \frac{18}{53}$$
$$= \frac{27}{53} = 0.5094$$

(b)

i)

$$pr\{X(1) - X(0) = 2,\ X(3) - X(1) = 4\}$$

indpendent r.v$_s$

$$= \frac{2^2 e^{-2}}{2!} \cdot \frac{4^4 e^{-4}}{4!}$$
$$= \frac{64}{3} e^{-6} = 0.05288$$

ii)

$$pr\{X(3) = 6 \mid X(1) = 2\}$$
$$= pr\{X(3) - X(1) = 4 \mid X(1) - X(0) = 2\}$$

indpendent r.v$_s$

$$= pr\{X(3) - X(1) = 4\}$$
$$= \frac{4^4 e^{-4}}{4!}$$
$$= \frac{64}{6} e^{-4} = 0.1953668$$

Q4: [5+4]

(a)
$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$
$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t),\ n=1,2,3,\ \ldots \qquad (2)$$

The initial condition is $X(0) = 1 \;\Rightarrow\; p_1(0) = 1$

$$\Rightarrow\ p_n(0) = \begin{cases} 1 & ,\ n=1 \\ 0 & ,\ \text{otherwise} \end{cases}$$

$$\lambda_0 = 0 \qquad (1) \Rightarrow \frac{dp_0(t)}{dt} = 0$$

$$\Rightarrow \; p_0(t) = 0 \qquad (3)$$

$$(2) \Rightarrow \; \frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t)$$

$$\Rightarrow \; \frac{dp_n(t)}{dt} + \lambda_n p_n(t) = \lambda_{n-1} p_{n-1}(t), \; n = 1, 2, \; ...$$

$$\because \; \lambda_n = n\lambda, \quad \lambda_{n-1} = (n-1)\lambda$$

$$\therefore \; \frac{dp_n(t)}{dt} + n\lambda p_n(t) = (n-1)\lambda p_{n-1}(t), \; \text{n=1,2, ...}$$

Multiply both sides by $e^{n\lambda t}$

$$e^{n\lambda t}\left[\frac{dp_n(t)}{dt} + n\lambda p_n(t)\right] = (n-1)\lambda p_{n-1}(t) e^{n\lambda t}$$

$$\therefore \; \frac{d}{dt}\left[p_n(t) e^{n\lambda t}\right] = (n-1)\lambda p_{n-1}(t) e^{n\lambda t}$$

$$\Rightarrow \int_0^t d\left[p_n(x) e^{n\lambda x}\right] = (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx$$

$$\therefore \; \left[p_n(x) e^{n\lambda x}\right]_0^t = (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx$$

$$\Rightarrow \; p_n(t) = e^{-n\lambda t}\left[p_n(0) + (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx\right], \; n = 1, 2, \; ... \quad (4)$$

which is a recurrence relation.

at $n = 1$

$$p_1(t) = e^{-\lambda t}\left[p_1(0) + 0\right] = e^{-\lambda t} \qquad (5)$$

at $n = 2$

$$p_2(t) = e^{-2\lambda t}\left[p_2(0) + \lambda \int_0^t p_1(x) e^{2\lambda x} dx\right]$$

$(5) \Rightarrow \; p_1(x) = e^{-\lambda x}$

$\therefore \; p_2(t) = e^{-2\lambda t}\left[\lambda \int_0^t e^{-\lambda x} e^{2\lambda x} dx\right]$

$\therefore \; p_2(t) = \lambda e^{-2\lambda t} \int_0^t e^{\lambda x} dx$

$\quad\quad = e^{-\lambda t}(1 - e^{-\lambda t})^1 \qquad (6)$

Similarly as (5) and (6), we deduce that

$p_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$

$\quad\quad = p(1\text{-}p)^{n-1}, \;\; p = e^{-\lambda t}, \; n = 1, 2, \; \ldots$

$\therefore X(t) \sim geom(p), \; p = e^{-\lambda t}$

$Mean\left[X(t)\right] = 1/p = e^{\lambda t},$

$Variance\left[X(t)\right] = \frac{1-p}{p^2} = \frac{1-e^{-\lambda t}}{e^{-2\lambda t}}$

(b) For Yule process,

$p_n(t) = e^{-\beta t}(1 - e^{-\beta t})^{n-1}, \quad n \geq 1$

$\Rightarrow$

$$\because pr\left\{X(U) = k\right\} = \int_0^1 e^{-\beta u}(1 - e^{-\beta u})^{k-1} du$$

$$= \frac{1}{\beta}\int_0^1 (1 - e^{-\beta u})^{k-1}.\, \beta e^{-\beta u} du$$

$$= \frac{1}{\beta}\left[\frac{(1 - e^{-\beta u})^k}{k}\right]_0^1$$

$$= \frac{1}{\beta k}\left[(1 - e^{-\beta})^k\right]$$

$\therefore pr\left\{X(U) = k\right\} = \frac{p^k}{\beta k}, \; k = 1, 2, \; \ldots$ where $p = 1 - e^{-\beta}$

Q5: [6]

For pure birth process,

$$p_0(t) = e^{-\lambda_0 t}, \quad (1)$$

$$p_1(t) = \lambda_0 \left[\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1 t} \right], \quad (2)$$

and $p_n(t) = pr\left\{ X(t) = n \middle| X(0) = 0 \right\}$

$$= \lambda_0 \lambda_1 \ldots \lambda_{n-1} \left[B_{0,n} e^{-\lambda_0 t} + \ldots + B_{k,n} e^{-\lambda_k t} + \ldots + B_{n,n} e^{-\lambda_n t} \right], \ n > 1, \quad (3)$$

where

$$B_{k,n} = \prod_{i=0}^{n} \left(\frac{1}{\lambda_i - \lambda_k} \right) i \neq k, \ 0 < k < n,$$

$$B_{0,n} = \prod_{i=1}^{n} \left(\frac{1}{\lambda_i - \lambda_0} \right)$$

and

$$B_{n,n} = \prod_{i=0}^{n-1} \left(\frac{1}{\lambda_i - \lambda_n} \right)$$

at $n = 0$ $(1) \Rightarrow p_0(t) = e^{-\lambda_0 t}, \quad \lambda_0 = 1$

$\therefore \ p_0(t) = e^{-t}$

at $n = 1$ $(2) \Rightarrow p_1(t) = \frac{1}{2}\left[e^{-t} - e^{-3t} \right]$

at $n = 2$ $(3) \Rightarrow p_2(t) = \lambda_0 \lambda_1 \left[B_{0,2} e^{-\lambda_0 t} + B_{1,2} e^{-\lambda_1 t} + B_{2,2} e^{-\lambda_2 t} \right],$

where, $B_{0,2} = \dfrac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)}$

$= \frac{1}{2},$

$$B_{1,2} = \frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)}$$

$= \frac{1}{2}$

and

$$B_{2,2} = \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}$$

$= -1$

$\therefore \ p_2(t) = 3\left[\frac{1}{2} e^{-t} + \frac{1}{2} e^{-3t} - e^{-2t} \right]$
