Final Exam, Semester I, 1446

Department of Mathematics, College of Science, KSU

Course: Math 481 Marks: 40 Duration: 3 Hours

Question 1 [2+3+3 points]

(a) Prove that if f is continuous on $[a, b]$, then $f \in \mathcal{R}(a, b)$.

(b) Provide an example of a function that is Riemann integrable but not continuous.

(c) Suppose (f_n) is a sequence of Riemann integrable functions that converges uniformly on $[a, b]$ to a function f. Prove that f is Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

Question 2 [4+4 points]

(a) The functions f_n on $[-1, 1]$ are defined by $f_n(x) = \dfrac{x}{1 + n^2x^2}$. Show that the pointwise limit of f_n is differentiable, but the equality $f'(x) = \lim_{n\to\infty} f_n'(x)$ does not hold for all $x \in [-1, 1]$.

(b) Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n^2}{n^3}$$

converges uniformly on any bounded interval, but does not converge uniformly on $\mathbb{R}$.

Question 3 [2+2+4 points]

(a) Define a measurable set and a measurable function.

(b) Given E such that $m^*(E) = 0$, prove that E is a measurable set.

(c) Consider the function:

$$f(x) = \begin{cases} \dfrac{1}{x^2}, & \text{if } x > 1, \\ 0, & \text{if } x \leq 1. \end{cases}$$

(i) Is this function Riemann integrable on the interval $[1, a]$ for all $a > 1$?

(ii) Show that this function is Lebesgue integrable on $[0, \infty$ and

$$\int_{[0,\infty)} f(x)\, dm = 1.$$

Question 4 **[4+4 points]**

(a) Provide an example of a function that is Lebesgue integrable but not Riemann integrable.

(b) Evaluate the limit of the following integral:

$$\lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2}\, dx.$$

Question 5 **[4+2+2+2 points]**

(a) Prove that

$$\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^n e^{-2x}\, dx = 1.$$

(b) If $p, q > 0$, prove that

$$\int_0^1 \frac{x^{p-1}}{1+x^q}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{p+nq}.$$

Then conclude that:

(i) $\log 2 = \sum_{n=0}^\infty \dfrac{(-1)^n}{1+n}$,

(ii) $\frac{\pi}{4} = \sum_{n=0}^\infty \frac{(-1)^n}{2n+1}$.