Introduction to Partial Differential Equations

First order equations.

Differential equations in three variables. From our previous study for the differential equations (ordinary) we know, there are two variables one dependent and the other independent, for example $x\frac{dy}{dx}+2y=x^2$. This equation is linear and from first order, and the method to find its solution is well known.

However we may increase the number of variables to be two or more such that $P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz=0$ (1)

where three variables are involved. Now suppose we considered:

$$\phi(x,y,z)=c \quad (2)$$

which represents the most general equation for surfaces. In this case we find

$$d\phi=0=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz \quad (3)$$

By making comparison with equation (1) yield

$$\frac{\partial\phi}{\partial x}=\mu P \qquad \frac{\partial\phi}{\partial y}=\mu Q\,, \qquad \frac{\partial\phi}{\partial z}=\mu R$$

where $\mu\neq 1$ is known as integration factor.

If we consider the case when $\mu = 1$ then

$$\frac{\partial \phi}{\partial x} = P, \quad \frac{\partial \phi}{\partial y} = Q, \quad \frac{\partial \phi}{\partial z} = R \Longrightarrow$$

$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

For the case when $\mu \neq 1$ and $\mu \neq 0$ we have

$\mu P\,dx + \mu Q\,dy + \mu R\,dz = 0$. This leads to the following equations

$$\frac{\partial \mu Q}{\partial z} = \frac{\partial \mu R}{\partial y}, \quad \frac{\partial \mu R}{\partial x} = \frac{\partial(\mu P)}{\partial z}$$

$$\frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}$$ or in other word

$$\mu \frac{\partial Q}{\partial z} + Q\frac{\partial \mu}{\partial z} = \mu \frac{\partial R}{\partial y} + R\frac{\partial \mu}{\partial y} \qquad P$$

$$\mu \frac{\partial R}{\partial x} + R\frac{\partial \mu}{\partial x} = \mu \frac{\partial P}{\partial z} + P\frac{\partial \mu}{\partial z} \qquad -Q$$

$$\mu \frac{\partial P}{\partial y} + P\frac{\partial \mu}{\partial y} = \mu \frac{\partial Q}{\partial x} + Q\frac{\partial \mu}{\partial x} \qquad R$$

Hence

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) =$$

$$\begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0$$

This equation represents the necessary condition to find the integrating factor for equation, and it can be proved, this is also sufficient condition.

one of the method which may be used to find the solution for differential equations with more than two variables, is to consider one of the variable to be constant, and then solving the equation with two variables and hence taking the constant of integration to be the other function to see that, let us consider the differential equation

$$2yz\,dx + zx\,dy - xy(1+z)\,dz = 0.$$

Before we start to find the solution, let us check if this equation can be solved or not. To do so, we have to show the following determinant is zero.

$$\begin{vmatrix} 2yz & xz & -xy(1+z) \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & xz & -xy(1+z) \end{vmatrix} = 0$$

$$= 2yz\left[-x(1+z)-x\right] - xz\left[-y(1+z)-2y\right] - xy(1+z)(-z) = 0$$

From which the equation can be solved.

By setting $z = \text{const.}$ $\quad 2yz\,dx + zx\,dy = 0 \Longrightarrow x^2y = c$

Then $f(z) = x^2y$ $\quad \therefore 2xy\,dx + x^2dy = f'(z)\,dz$

$$\therefore \frac{2xy}{2yz} = \frac{x^2}{xz} = \frac{-f'(z)}{-xy(1+z)} \qquad \therefore x^2y = z\frac{f'(z)}{1+z}$$

$$\therefore f(z) = z\frac{f'(z)}{1+z} \Longrightarrow \ln x^2y = z + \ln z + c$$

Ex: Find the value of $f(z)$ which makes the equation

$(2x^2 + 2xy + 2xz^2 + 1)\,dx + dy + f(z)\,dz = 0$ integrable, and then find the solution.

$$\begin{vmatrix} 2x^2+2xy+2xz^2+1 & 1 & f(z) \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2+2xy+2xz^2+1 & 1 & f(z) \end{vmatrix} = 0 \Longrightarrow f(z) = 2z$$

Now let us $x = \text{const.}$ $\quad dy + 2z\,dz = 0 \quad \therefore y + z^2 = c = f(x)$

$$\therefore \frac{f'(x)}{2x^2+2xy+2xz^2+1} = -1 \qquad \therefore -f'(x) = 2x^2 + 2x(y+z^2) + 1$$

$$\therefore -f'(x) - 2xf = (1+2x^2), \qquad \frac{df}{dx} + 2xf = -(2x^2+1)$$

$$\therefore f(x) = -x + c_1e^{-x^2} \Longrightarrow y + z^2 = -x + c_1e^{-x^2}$$

The Homogeneous differential Equations in three variables

Suppose $P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0$

which homogeneous in x, y and z, in this case we have to set $x = uz$

$y = vz$. Now $P(uz,vz,z)(u\,dz + z\,du) + Q(uz,vz,z)(v\,dz + z\,dv)$

$+ R(uz,vz,z)\,dz = 0$. Now if R, Q, P homogeneous function from degree n then

$$z\left[P(u,v,1)\,du + Q(u,v,1)\,dv\right] + \left[u\,P(u,v,1) + v\,Q(u,v,1) + R(u,v,1)\right]dz = 0$$

or

$$\frac{P_1}{uP_1+vQ_1+R_1}\,du + \frac{Q_1}{uP_1+vQ_1+R_1}\,dv + \frac{1}{z}\,dz = 0$$

where $P_1 = P(u,v,1)$, $Q_1 = Q(u,v,1)$, $R_1 = R(u,v,1)$

whence $$\frac{\partial}{\partial v}\left(\frac{P_1}{uP_1+vQ_1+R_1}\right) = \frac{\partial}{\partial u}\left(\frac{Q_1}{uP_1+vQ_1+R_1}\right)$$

Example:- Find the solution of the equation

$$y(y+z)\,dx + z(x+z)\,dy + y(y-x)\,dz = 0$$

$$\begin{vmatrix} y(y+z) & z(x+z) & y(y-x) \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y(y+z) & z(x+z) & y(y-x) \end{vmatrix} = 0$$

$$= y(y+z)(2y-2x-2z) + 2yz(x+z) - 2y^2(y-x) = 0$$

If we set $x = uz,\ y = vz$

$$\therefore\ dx = u\,dz + z\,du, \quad dy = v\,dz + z\,dv$$

$$\therefore\ y(y+z)(u\,dz + z\,du) + z(x+z)(v\,dz + z\,dv) + y(y-x)dz$$

Thus $v z^3 (v+1)\,du + z^3 (u+1)\,dv$

$$+ \left[uv(v+1) + v(u+1) + v(v-u)\right] z^2\,dz = 0$$

$$\therefore\ \frac{v(v+1)}{v(u+1)(v+1)}\,du + \frac{u+1}{v(u+1)(v+1)}\,dv + \frac{dz}{z} = 0$$

$$\therefore\ \ln(u+1) + \ln\frac{v}{v+1} + \ln z = \ln c$$

$$\therefore\ \frac{y(x+z)}{y+z} = c$$

Ex: $(y^2+z^2+2xy+2xz)\,dx + (x^2+z^2+2xy+2yz)\,dy + (x^2+y^2+2xz+2yz)\,dz = 0$

(2) $2(y+z)\,dx - (x+z)\,dy + (2y-x+z)\,dz = 0$

(3) $(e^x y + e^z)\,dx + (e^y z + e^x)\,dy + (e^y - e^x y - e^y z)\,dz = 0$

Simultaneous Total differential equations

If we consider the differential equations

$$P_1\,dx + Q_1\,dy + R_1\,dz = 0$$

$$P_2\,dx + Q_2\,dy + R_2\,dz = 0$$

$$\Longrightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$ subsidiary equation

where $P = \begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix}$, $Q = \begin{vmatrix} R_1 & P_1 \\ R_2 & P_2 \end{vmatrix}$, $R = \begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}$

The solution of the above equations contains two arbitrary constants.

$$\phi_1(x, y, z, c_1) = 0 \qquad \phi_2(x, y, z, c_2) = 0$$

which represents a collection of curves, where their directions may be determined at (x, y, z) from $dx : dy : dz$. This means that the direction cosine of their tangent are proportional with $P : Q : R$

Ex: Find the Solution of $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$

$$\frac{dx}{z} = \frac{dz}{x}, \quad \frac{dy}{z} = \frac{dz}{y} \qquad \therefore x^2 = z^2 + c_1, \quad y^2 = z^2 + c_2.$$

Ex: Find the Solution $\frac{dx}{y} = \frac{dy}{x+z} = \frac{dz}{y}$

$$\frac{dx}{y} = \frac{dz}{y} \qquad \therefore x = z + c_1 \qquad \frac{dy}{x+z} = \frac{dz}{y}$$

$$\therefore \tfrac{1}{2} y^2 = z^2 + c_1 z + c_2$$

$$\therefore \tfrac{1}{2} y^2 = z^2 + (x - z) z + c_2 \qquad \therefore \tfrac{1}{2} y^2 = xz + c_2$$

Ex: Find the solution

$$dx = \frac{dy}{2} = \frac{dz}{3z + \cos(y - 2x)}$$

$$y - 2x = c_1 \qquad \therefore dx = \frac{dz}{3z + \cos c_1}$$

$\therefore x = \frac{1}{3} \ln(3z + c.5c_1) + c_2$

whence $x = \frac{1}{3} \ln[3z + c.5(y-2x)] + c_2$.

Another method to deal with the differential equation with three variables.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

There are two cases: ① If we manage to choose l, m, and n such that $lP + mQ + nR = 0$ and $l\,dx + m\,dy + n\,dz$ is exact differential

② If we manage to choose l, m, and n such that $lP + mQ + nR \neq$ and $l\,dx + m\,dy + n\,dz$ is exact differential.

Examples:- ① Find the solution of the equation

$$\frac{dx}{y+2z} = \frac{dy}{x+y+2z} = \frac{2\,dz}{-x}$$

$$\frac{dx}{y+2z} = \frac{dy}{x+y+2z} = \frac{2}{-x}dz = \frac{dx - dy - 2dz}{0}$$

Then $dx - dy - 2dz = 0 \Rightarrow x - y - 2z = c_1$

$\therefore \frac{dx}{x - c_1} = 2\frac{dz}{-x} \qquad \therefore c_2 - 2z = x + c_1 \ln(x - c_1)$

$\therefore c_2 - 2z = x + (x - y - 2z)\ln(y + 2z)$ #

Example: Find the Solution of $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dx-dz}{z-x}$$

This leads to $\quad y-x = c_1(z-y) \; , \; (z-y) = c_2(z-x)$

example: Find the solution of

$$\frac{dx}{4y-3z} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x} \qquad (1)$$

let us find l, m, and n such that

$$l(4y-3z) + m(4x-2z) + n(2y-3x) = 0 \qquad (2)$$

In this case we have $x(4m-3n) + y(4l+2n) - z(3l+2m) = 0$

Hence $4m-3n=0 \; , \; 4l+2n=0 \; , \; 3l+2m=0$

Thus $l:m:n = 2:-3:-4$

$\therefore \; 2dx - 3dy - 4dz = 0 \implies 2x-3y-4z = c_1 \qquad (3)$

Also from (2) we may have

$$4(ly+mx) + 3(-lz-nx) + 2(ny-mz) = 0$$

This will be satisfied when

$$ly+mx=0 \; , \; -lz-nx=0 \; , \; ny-mz=0$$

$\therefore\ l : m : n :: x : -y : -z$

$\therefore\ x\,dx - y\,dy - z\,dz = 0 \Rightarrow x^2 - y^2 - z^2 = c_2$ (4)

Then the general Solution ~~are~~ is equation (3) and (4) together.

The geometrical meaning of the differential equations.

$$P\,dx + Q\,dy + R\,dz = 0 \qquad (1)$$

Suppose $\phi(x,y,z) = c$ (2)

$$\therefore \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0 \qquad (3)$$

$$\frac{\partial\phi}{\partial x} : \frac{\partial\phi}{\partial y} : \frac{\partial\phi}{\partial z} :: P : Q : R$$

But $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$ are direction cosines for the $\phi(x,y,z) = c$ and hence we may regard R, Q, P are the direction of the normal on ϕ.

Ex: Find the set of normal ~~cures~~ curves with the surface.

$$x^2 + 2y^2 + 4z^2 = c$$

$$\therefore 2x\,dx + 4y\,dy + 8z\,dz = 0.$$

$$\therefore \frac{dx}{x} = \frac{dy}{2y} = \frac{dz}{4z} \Rightarrow y = c_1 x^2,\ z = c_2 y^2$$

Ex: Find the equation of the surfaces which are normal with the curves $\frac{dx}{x} = \frac{dy}{z} = \frac{dz}{y+2z}$

and hence find the equation of the surface which pass with the point $(0,1,1)$.

$$x\,dx + z\,dy + (y+2z)\,dz = 0.$$

$$x\,dx + (z\,dy + y\,dz) + 2z\,dz = 0$$

$$\tfrac{1}{2}x^2 + zy + z^2 = c^2/2$$

or $\quad x^2 + 2zy + 2z^2 = c^2 \quad$ setting $x=0,\ y=1,\ z=1$

$\therefore\ c^2 = 4 \qquad \therefore\ x^2 + 2yz + 2z^2 = 4.$

<u>Ex:</u> Find the differential equation of the family of all tangent planes to the ellipsoid $x^2+4y^2+4z^2=4$ which are not perpendicular to the xy plane. If $P_0(x_0,y_0,z_0)$ is a point on the ellipsoid, $z_0 \neq 0$ the equation of the tangent plane to the surface at P_0 is

$$x_0(x-x_0) + 4y_0(y-y_0) + 4z_0(z-z_0) = 0 \quad \text{or}$$

$$x_0x + 4y_0y + 4z_0z = 4.$$

As P_0 varies over the ellipsoid, this equation yields all tangent plane in the family. observe that only two arbitrary constants are present, S.

P_0 must lie on the surface. Differentiation gives

$$x_0 + 4z_0 \frac{\partial z}{\partial x} = 0, \qquad 4y_0 + 4z_0 \frac{\partial z}{\partial y} = 0$$

substituting $-4z_0 \frac{\partial z}{\partial x}$ for x_0 and $-z_0 \frac{\partial z}{\partial y}$ for y_0 in the

equation of the family of tangent planes The result is

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = -\frac{1}{z_0}, \text{ Now } \left(\frac{x_0}{z_0}\right)^2 + 4\left(\frac{y_0}{z_0}\right)^2 + 4 = \frac{4}{z_0^2}$$

Hence

$$16\left(\frac{\partial z}{\partial x}\right)^2 + 4\left(\frac{\partial z}{\partial y}\right)^2 + 4 = \frac{4}{z_0^2}$$, The differential equation

of the family of tangent planes is

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = -\left[1 + 4\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2}$$

linear First order partial differential equation:-

A linear first-order partial differential equation in two independent

variables x, y and dependent variable z has the form

$$A(x,y)\frac{\partial z}{\partial x} + B(x,y)\frac{\partial z}{\partial y} + C(x,y)\,z = G(x,y)$$

where A, B, C and G have continuous first derivatives

with respect to x and y in some region $\mathcal{R}$ of the xy-plane

Example: $\frac{\partial z}{\partial x} + z = x$, the solution is $z = e^{-x} f(y)$

Example: $A\frac{\partial z}{\partial x} + B\frac{\partial z}{\partial y} + Cz = G$. A, B, and C are const such that $A^2 + B^2 \neq 0$, and G is a given continuously differentiable function of x and y.

To find the solution, let's set $G = 0$. $\therefore A\frac{\partial z}{\partial x} + B\frac{\partial z}{\partial y} + Cz = 0$

Let $\xi = \alpha x + \beta y$, $\eta = \gamma x + \delta y$ $\therefore z(x,y) \longrightarrow \phi(\xi, \eta)$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{\partial \phi}{\partial \xi}d\xi + \frac{\partial \phi}{\partial \eta}d\eta$$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{\partial \phi}{\partial \xi}(\alpha dx + \beta dy) + \frac{\partial \phi}{\partial \eta}(\gamma dx + \delta dy)$$

$$\therefore \frac{\partial z}{\partial x} = \alpha\frac{\partial \phi}{\partial \xi} + \gamma\frac{\partial \phi}{\partial \eta}, \quad \frac{\partial z}{\partial y} = \beta\frac{\partial \phi}{\partial \xi} + \delta\frac{\partial \phi}{\partial \eta}$$

$$\therefore A\left(\alpha\frac{\partial \phi}{\partial \xi} + \gamma\frac{\partial \phi}{\partial \eta}\right) + B\left(\beta\frac{\partial \phi}{\partial \xi} + \delta\frac{\partial \phi}{\partial \eta}\right) + C\phi = 0$$

$$\therefore (A\alpha + B\beta)\frac{\partial \phi}{\partial \xi} + (A\gamma + \delta B)\frac{\partial \phi}{\partial \eta} + C\phi = 0$$

Assume $A \neq 0$. $\alpha = 1$, $\beta = 0$, $\gamma = B$, $\delta = -A$. Then

$$A\frac{\partial \phi}{\partial \xi} + C\phi = 0 \quad \therefore \phi = e^{-C\xi/A} f(\eta)$$

where f is an arbitrary function. Thus the general solution of the homogeneous equation is $z = e^{-C\frac{x}{A}} f(Bx - Ay)$

If $A \neq 0$, and if z_p is a particular solution of the original diff. equations, then the general solution of the inhomogeneous equation is

$z = e^{-cx/A} f(Bx - Ay) + z_p$.

Ex:- Find the solution of the differential equation $x^2 \frac{\partial z}{\partial x} - xy \frac{\partial z}{\partial y} + yz = 0$. The result is $z = e^{y/2x} f(xy)$

Now let $z \xrightarrow{(x,y)} \phi(\xi, \eta)$ $\quad \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{\partial \phi}{\partial \xi} d\xi + \frac{\partial \phi}{\partial \eta} d\eta$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial y}$$

$\frac{\partial z}{\partial \xi} + \frac{y}{x^2 \frac{\partial \xi}{\partial x} - xy \frac{\partial \xi}{\partial y}} z = 0$.

Let, $d\eta = \partial x$ $\quad \eta = xy$

$\xi = x \quad \therefore \frac{\eta}{\xi} = y \quad \therefore \frac{\partial z}{\partial \xi} + \frac{\eta}{\xi^3} z =$

$\therefore z = f(\eta) \exp \frac{\eta}{2\xi^2} \quad \therefore z = f(xy)\, e^{\frac{y}{2x}}$

$$\therefore x^2 \left(\frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \right) - xy \left(\frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right) + yz$$

$$\therefore \left(x^2 \frac{\partial \xi}{\partial x} - xy \frac{\partial \xi}{\partial y} \right) \frac{\partial z}{\partial \xi} + \left(x^2 \frac{\partial \eta}{\partial x} - xy \frac{\partial \eta}{\partial y} \right) \frac{\partial z}{\partial \eta} + yz =$$

$$x^2 \frac{\partial \xi}{\partial x} - xy \frac{\partial \xi}{\partial y} = 0. \qquad \therefore x \frac{\partial \xi}{\partial x} - y \frac{\partial \xi}{\partial y} = 0 \qquad (*)$$

Now if we consider a function $\phi(x, y) = C$

$$\therefore \quad d\phi = 0 = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0. \qquad (*)$$

making a comparison between the last two equations yield

$\frac{dy}{dx} = -\frac{y}{x}$. whence $x\,dy + y\,dx = 0$ or $xy = c$

The appropriate transformation is $\xi = x$, $\eta = xy$. The coefficient $A(\xi,\eta) = \xi^2$ and $C(\xi,\eta) = \eta/\xi$. The general solution of the partial differential equation is $z = e^{y/2x} f(xy)$ where f is an arbitrary function.

To derive the form of the general solution of the linear first-order homogeneous equation

$$A(x,y)\frac{\partial z}{\partial x} + B(x,y)\frac{\partial z}{\partial y} + C(x,y)\,z = 0$$

let $\xi = \xi(x,y)$ $\eta = \eta(x,y)$, be a transformation on $\mathcal{R}$ with Jacobian $\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$. Since $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$, and $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$

$$\therefore \left(A\frac{\partial \xi}{\partial x} + B\frac{\partial \xi}{\partial y}\right)\frac{\partial z}{\partial \xi} + \left(A\frac{\partial \eta}{\partial x} + B\frac{\partial \eta}{\partial y}\right)\frac{\partial z}{\partial \eta} + Cz = 0.$$

Let us choose η such that $A\eta_x + B\eta_y = 0$. Assume $A(x,y) \neq 0$ and consider the ordinary differential equation $\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)}$

let the general solution of this equation $\eta(x,y) = c$, where $\eta_y \neq 0$ and c is an arbitrary constant. Then the function $\eta(x,y)$ determined in this manner

Since $\eta_x\,dx + \eta_y\,dy = 0$ and so $\frac{B}{A} = -\frac{\eta_x}{\eta_y}$

Now choose $\xi(x,y) = x$. Then $\xi_x \eta_y - \xi_y \eta_x \neq 0$ and the transformation constructed in this manner is invertible, and hence $A \frac{\partial z}{\partial \xi} + Cz = 0$.

$\therefore z(\xi,\eta) = f(\eta) \exp\left[-\int \frac{C}{A} d\xi\right]$ where f is an arbitrary function

Quasilinear first-order equations; Method of Lagrange

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z)$$

in the two independent variables x, y and the dependent variable z are considered. Method of Lagrange: The system of first-order ordinary differential equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Called the subsidiary equations

Ex: Solve $xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = -(x^2+y^2)$, Here $P(x,y,z) = xz$, $Q(x,y,z) = yz$, $R(x,y,z) = -(x^2+y^2)$.

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{-(x^2+y^2)} \qquad \therefore \quad x = y + c \quad \text{or} \quad \frac{y}{x} = c_1$$

$$\therefore \frac{dx}{xz} = -\frac{dz}{x^2+y^2}. \text{ Integration gives } (1+c_1^2)x^2 + z^2 = c_2$$

Now replace c_1 by y/x. The second integral of the subsidiary equation is $v(x,y,z) = x^2+y^2+z^2 = c_2$. Thus the general solution can be written as $F(y/x, x^2+y^2+z^2) = 0$

$(y-x)\frac{\partial z}{\partial x}+(y+x)\frac{\partial z}{\partial y}=\frac{x^2+y^2}{z}$. The subsidiary equations are

$$\frac{dx}{y-x}=\frac{dy}{y+x}=\frac{z\,dz}{x^2+y^2} \qquad \frac{dx+dy}{2y}=\frac{dy}{y+x} \quad \therefore (y+x)(dx+dy)=2y\,dy$$

$$\therefore \frac{1}{2}(y+x)^2=y^2+c_1 \qquad \therefore x^2+2xy+y^2=2y^2+2c_1$$

$$\boxed{x^2+2xy-y^2=2c_1}\ , \text{ again } \frac{x\,dx}{-x^2+xy}=\frac{-y\,dy}{-(y^2+xy)}=\frac{z\,dz}{x^2+y^2}$$

$$\therefore x\,dx-y\,dy+z\,dz=0 \implies x^2-y^2+z^2=c_2$$

$\therefore F(x^2+2xy-y^2, x^2-y^2+z^2)=0$ is the general solution of the partial differential equation.

Cauchy problem for quasilinear First-order equations

Find an integral surface of

$$(xz+y)\frac{\partial z}{\partial x}+(x+yz)\frac{\partial z}{\partial y}=z^2-1$$ which passes through the

parabola $x=t,\ y=1,\ z=t^2$. The subsidiary equations are

$$\frac{dx}{y+xz}=\frac{dy}{x+yz}=\frac{dz}{z^2-1}, \quad \text{Now } \frac{dx+dy}{(z+1)(x+y)}=\frac{dz}{z^2-1}$$

$$\therefore \ln(x+y)=\ln(z-1)+\ln c_1 \quad \text{or} \quad (x+y)=c_1(z-1)$$

and $(x-y)=c_2(z+1)$ whence $c_1=\frac{1}{t-1}$ and $c_2=\frac{t-1}{t^2+1}$ ##.

Linear second-order Equations

Introduction: Any differential equation containing partial derivati[ves] is called a partial differential equation, the order of the equations i[s] equal to the order of the highest partial differential coefficient occuring in it. The dependent variable (the unknown function) [of] any partial differential equation must be a function of at least two independent variables otherwise partial derivatives would n[ot] arise, and in general may be a function of $n\,(\geq 2)$ independent variables. For example, the equations

$$3y^2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=2u \qquad (1)$$

$$\frac{\partial^2 u}{\partial x^2}+f(x,y)\frac{\partial^2 u}{\partial y^2}=0 \qquad (2)$$

$$u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial y}\right)^2=u^2 \qquad (3)$$

The last equation represent a non-linear equations in two indepen[dent] variable. In general the solution of the partial differential equati[on] represent a much more difficult problem than the solution of ordinary differential equations, and except for certain special ty[pes] of linear partial differential equations, no general method of solution is available. We shall concentrate therefore on the solution of particular types of linear equations.

The most general linear homogeneous equation of the second order

$$a\frac{\partial^2 u}{\partial x^2} + 2h\frac{\partial^2 u}{\partial x \partial y} + b\frac{\partial^2 u}{\partial y^2} + 2f\frac{\partial u}{\partial x} + 2g\frac{\partial u}{\partial y} + eu = 0.$$

where a, b, h, f, g and e are constants or variables in x and y.

Note that if we choose $h = f = g = e = 0$.

$$\therefore \frac{\partial^2 u}{\partial x^2} = -b/a \frac{\partial^2 u}{\partial y^2}, \quad \text{when } y \rightarrow ict \;\&\; b = a$$

Then we get $\quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

which represents the wave equation.

The type of the partial differential equation can be determined as follows:

elliptic	$ab - h^2 > 0$
parabolic	$ab - h^2 = 0$
hyperbolic	$ab - h^2 < 0$

For the wave equation $a = 1,\ h = 0,\ b = -\frac{1}{c^2}$

Since $-\frac{1}{c^2} < 0 \Longrightarrow$ The wave equation is hyperbolic.

On the other hand, the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

is elliptic type where $a=1=b$, $h=0$.

Finally, the heat equation $\frac{\partial^2 u}{\partial x^2}=\frac{1}{k}\frac{\partial u}{\partial t}$ is of parabolic type.

Now let us derive the most general equation, from which we can deduce, the wave, the heat ~~of~~ or the Laplace's equation:

Let us consider the Maxwell's equation

$$\underline{\nabla}\wedge\underline{H}=\frac{\partial \underline{D}}{\partial t}+\underline{J} \qquad \underline{\nabla}\wedge\underline{E}=-\frac{\partial \underline{B}}{\partial t}$$

$$\underline{\nabla}\cdot\underline{B}=0, \qquad \underline{\nabla}\cdot\underline{D}=\rho$$

where $\underline{H}$ and $\underline{E}$ are the magnetic and electric field respectively, while $\underline{J}$ is the current of the electric field given by $\underline{J}=\delta\underline{E}$, δ being the conductivity factor. The quantities $\underline{D}=\epsilon_0\underline{E}$, and $\underline{B}=\mu_0\underline{H}$, where ϵ_0 is called the susceptibility and μ_0 is known as
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permeability where $\epsilon_0\mu_0=\frac{1}{c^2}$, c is the velocity of light.

Now $\underline{\nabla}\wedge(\underline{\nabla}\wedge\underline{H})=\frac{\partial}{\partial t}(\underline{\nabla}\wedge\underline{D})+\underline{\nabla}\wedge\underline{J}$

$$=-\frac{\partial}{\partial t}\left(\epsilon_0\frac{\partial \underline{B}}{\partial t}\right)+\delta\cdot\frac{\partial \underline{B}}{\partial t}$$

$$-\nabla^2\underline{H}=-\epsilon_0\mu_0\frac{\partial^2 \underline{H}}{\partial t^2}-\delta\mu_0\frac{\partial \underline{H}}{\partial t}$$

$$\therefore \nabla^2 H = \epsilon_0 \mu_0 \frac{\partial^2 H}{\partial t^2} + \sigma \mu_0 \frac{\partial H}{\partial t}$$

which can be rewritten as $\nabla^2 \psi = \epsilon_0 \mu_0 \frac{\partial^2 \psi}{\partial t^2} + \sigma \mu_0 \frac{\partial \psi}{\partial t}$

when $\mu_0 = 0$. $\nabla^2 \psi = 0$ gives Laplace's equation

when $\sigma = 0$ $\quad \nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$ The wave equation

when $\epsilon_0 = 0$ $\quad \nabla^2 \psi = \frac{1}{k} \frac{\partial \psi}{\partial t}$ The heat equation,

where $k = \frac{1}{\sigma \mu_0}$.

<u>Eular's equation</u>:

$$a \frac{\partial^2 u}{\partial x^2} + 2h \frac{\partial^2 u}{\partial x \partial y} + b \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

$u(x,y) \longrightarrow \psi(\eta, \xi)$ $\quad \therefore du \equiv d\psi$ Then $\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \frac{\partial \psi}{\partial \eta} d\eta + \frac{\partial \psi}{\partial \xi} d\xi$

where $\xi = px + qy$ & $\eta = rx + sy$ $\quad (2)$

$$\frac{\partial u}{\partial x} = r \frac{\partial \psi}{\partial \eta} + p \frac{\partial \psi}{\partial \xi}, \qquad \frac{\partial u}{\partial y} = s \frac{\partial \psi}{\partial \eta} + q \frac{\partial \psi}{\partial \xi}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = r^2 \frac{\partial^2 \psi}{\partial \eta^2} + 2rp \frac{\partial^2 \psi}{\partial \eta \partial \xi} + p^2 \frac{\partial^2 \psi}{\partial \xi^2} \qquad (3)$$

$$\frac{\partial^2 u}{\partial y^2} = s^2 \frac{\partial^2 \psi}{\partial \eta^2} + 2sq \frac{\partial^2 \psi}{\partial \eta \partial \xi} + q^2 \frac{\partial^2 \psi}{\partial \xi^2} \qquad (4)$$

$$\frac{\partial^2 u}{\partial x \partial y} = rs \frac{\partial^2 u}{\partial \eta^2} + pq \frac{\partial^2 \psi}{\partial \xi^2} + (rq + sp) \frac{\partial^2 \psi}{\partial \xi \partial \eta} \qquad (5)$$

$$\frac{\partial^2 \psi}{\partial \eta^2} \left[ar^2 + 2hrs + bs^2\right] + \frac{\partial^2 \psi}{\partial \xi^2} \left[bq^2 + ap^2 + 2hpq\right]$$

$$+ \frac{\partial^2 \psi}{\partial \eta \partial \xi} \left[2arp + 2sqb + 2h(rq + sp)\right] = 0 \qquad (6)$$

Now if we assume $ar^2 + 2hrs + s^2 b = 0 \Longrightarrow b\lambda^2 + 2h\lambda + a =$

where $\lambda = s/r$ $\therefore \lambda = \frac{-h \pm \sqrt{h^2 - ab}}{b}$, then $\lambda_1 = \frac{-h + \sqrt{h^2 - ab}}{b}$

$\lambda_2 = \frac{-h - \sqrt{h^2 - ab}}{b}$, Let us take $p = r = 1$

$$\therefore \frac{\partial^2 u}{\partial \eta \partial \xi}\left[a + b\left(\frac{h^2 - h^2 + ab}{b^2}\right) + h\left(\frac{-2h}{b}\right)\right] = 0 \Rightarrow \frac{\partial^2 u}{\partial \eta \partial \xi} = 0$$

where $ab \neq h^2$

$$\therefore u = f(\xi) + g(\eta)$$

$$\therefore u(x,y) = F(px + qy) + G(rx + sy)$$

$$u(x,y) = F(x + \lambda_1 y) + G(x + \lambda_2 y)$$

D'Alembert's Solution of the wave equation :-

There are three main types of boundary conditions which ari[se] frequently in the description of physical phenomena. These a[re]

a) Dirichlet Conditions, where u is specified at each point of a boundary of a region (for example, the bounding curve of a pl[ane] region, or the surface of a three-dimensional domain). The pro[blem] of solving Laplace's equation $\nabla^2 u = 0$ inside a region with prescribed values of u on the boundary is called the Dirichle[t] problem.

b) Neumann conditions, where values of the normal derivative $\frac{\partial u}{\partial n}$ of the function are prescribed on the boundary.

c) Cauchy conditions. Here, if one of the independent variables is t (time, say) and the values of both u and $\frac{\partial u}{\partial t}$ on a boundary $t=0$ (that is, the initial values of u and $\frac{\partial u}{\partial t}$) are given, then the boundary conditions are of Cauchy type with respect to the variable t.

The wave equation $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$

The general solution is

$$\psi(x,t) = F(x+ct) + G(x-ct)$$

this solution is called D'Alembert's solution of the wave equation where $\psi(x,t)$ subject to the Cauchy initial condition

$$\psi(x,0) = f(x) \quad , \quad \left(\frac{\partial \psi}{\partial t}\right)_{t=0} = g(x)$$

$$F(x) + G(x) = f(x) \quad , \quad -\frac{1}{c}\, g(x) = \left(G'(x) - F'(x)\right)$$

$$\therefore \; -G(x) + F(x) = \frac{1}{c}\int_a^x g(x')\,dx'$$

$$\therefore \; G(x) = -\frac{1}{2c}\int_a^x g(x')\,dx' + \tfrac{1}{2} f(x)$$

$$F(x) = \frac{1}{2c}\int_0^x g(x')\,dx' + \tfrac{1}{2} f(x)$$

$$\therefore G(x-ct) = \frac{1}{2c}\int_0^{x-ct} g(x')\,dx' + \tfrac{1}{2} f(x-ct)$$

$$F(x+ct) = \frac{1}{2c}\int_{x+ct}^{a} g(x')\,dx' + \tfrac{1}{2} f(x+ct)$$

$$\therefore u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')\,dx'$$

Problems:

1- Determine the nature of each of the following equations (i.e. whe[ther] elliptic, parabolic or hyperbolic) and obtain the general solution in each case:

a) $3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = 0,$

b) $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

c) $4\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

d) $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 0$

e) $\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial x^2} = 0$

2) A function $u(r,t)$ satisfies the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where c is a constant. By introducing the new dependent variable $v(r,t) = r\,u(r,t)$ and writing $\xi = r+ct$, $\eta = r-ct$, reduce this equ[ation] to $\frac{\partial^2 v}{\partial \xi \partial \eta} = 0$, Hence show that the general solution $u(r,t)$ has t[he] form $u(r,t) = \frac{1}{r}\left[f(r+ct) + g(r-ct)\right]$, where f and g are arbitrary (twice differentiable) functions.

3- Solve the following boundary value problems by first obtaining the general solutions of the partial differential equations

a) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$, given $u(x,0)=0$, $\left(\frac{\partial u}{\partial t}\right)_{t=0} = \frac{1}{1+x^2}$

b) $\frac{\partial^2 u}{\partial x^2} = 2xy$, given $u(0,y) = y^2$, and $\left(\frac{\partial u}{\partial x}\right)_{x=0} = y$

c) $\frac{\partial^2 u}{\partial x \partial y} = 1$, given $u=0$, $\frac{\partial u}{\partial x} = 0$ on $x+y=0$

Ex: Given the Laplace's equation

$$\nabla^2 \psi = 0$$

where $\psi(x,y) \equiv \psi$. write $\nabla^2\psi$ in terms of (r and θ), and hence find the general solution, $\nabla^2\psi = 0 \Longrightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

$$\psi(x,y) \longrightarrow \phi(r,\theta) \quad \therefore \ d\psi = d\phi$$

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = \frac{\partial \phi}{\partial r}dr + \frac{\partial \phi}{\partial \theta}d\theta$$

$x = r\cos\theta$, $y = r\sin\theta$

$$dx = (\cos\theta\, dr - r\sin\theta\, d\theta),$$

$$dy = (\sin\theta\, dr + r\cos\theta\, d\theta),$$

$$\frac{\partial \psi}{\partial x}(\cos\theta\, dr - r\sin\theta\, d\theta) + \frac{\partial \psi}{\partial y}(\sin\theta\, dr + r\cos\theta\, d\theta) = \frac{\partial \phi}{\partial r}dr + \frac{\partial \phi}{\partial \theta}d\theta$$

$$\therefore \frac{\partial \phi}{\partial r} = \cos\theta \frac{\partial \psi}{\partial x} + \sin\theta \frac{\partial \psi}{\partial y}$$

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \cos\theta \frac{\partial \psi}{\partial y} - \sin\theta \frac{\partial \psi}{\partial x}$$

$$\frac{\partial \psi}{\partial x} = \cos\theta \frac{\partial \phi}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial \phi}{\partial \theta}, \quad \frac{\partial \psi}{\partial y} = \sin\theta \frac{\partial \phi}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial \phi}{\partial \theta}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial \theta}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial \phi}{\partial \theta}\right)$$

$$= \cos^2\theta \frac{\partial^2 \phi}{\partial r^2} - \cos\theta\sin\theta\left(-\frac{1}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial r}\right)$$

$$- \frac{1}{r}\sin\theta\left(\cos\theta \frac{\partial^2 \phi}{\partial r \partial \theta} - \sin\theta \frac{\partial \phi}{\partial r}\right)$$

$$+ \frac{1}{r^2}\sin\theta\left(\cos\theta \frac{\partial \phi}{\partial \theta} + \sin\theta \frac{\partial^2 \phi}{\partial \theta^2}\right)$$

Similarly

$$\frac{\partial^2 \psi}{\partial y^2} = \left(\sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial \phi}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial \phi}{\partial \theta}\right)$$

$$\therefore \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \cdots = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0.$$

To solve this equation, let us take $\phi(r,\theta) = R(r)\,F(\theta)$.

$$\therefore \frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{r}\frac{1}{R}\frac{dR}{dr} - \frac{m^2}{r^2} = 0. \quad \text{where } \frac{1}{F}\frac{d^2F}{d\theta^2} + m^2 = 0.$$

$$\therefore F = (A\cos m\theta + B\sin m\theta), \quad r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - m^2R = 0$$

$$\therefore R = C r^m + D r^{-m} \quad \therefore \phi(r,\theta) = (Cr^m + Dr^{-m})(A\cos m\theta$$

problem: If $\nabla^2 \psi(x,y,z) = 0 \Longrightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0$

Find $\nabla^2 \psi(r,\phi,z) = 0$

In the complex plane we may find the solution of Laplace's equation as follows.

let $\psi(x,y) \longrightarrow \phi(z,\bar{z})$, where $z = x+iy$, $\bar{z} = x-iy$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = \frac{\partial \phi}{\partial \bar{z}} d\bar{z} + \frac{\partial \phi}{\partial z} dz \quad , \quad dz = dx + i dy$$
$$d\bar{z} = dx - i dy$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial \bar{z}} (dx - i dy) + \frac{\partial \phi}{\partial z} (dx + i dy)$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial \bar{z}} \quad , \quad \frac{\partial \psi}{\partial y} = i\left(\frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial \bar{z}}\right)$$

$$\therefore \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 4 \frac{\partial^2 \phi}{\partial z \partial \bar{z}} = 0 . \quad \therefore \frac{\partial^2 \phi}{\partial z \partial \bar{z}} = 0 \Longrightarrow$$

$$\phi(z,\bar{z}) = F(z) + G(\bar{z})$$

which represents the solution in the complex plane for the Laplace's equation.

It is well known the most general form of the Laplace's equation in the curvilinear coordinates is:

$$\nabla^2 \psi(q_1, q_2, q_3) = 0 .$$

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\right)\frac{\partial \psi}{\partial q_1} + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\right)\frac{\partial \psi}{\partial q_2} + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial q_3}\right)\right] = 0 .$$

From which we can deduce all the other form for example in cartesian coordinates, polar coordinates, cylindrical coordinates and so on

Now, By setting $q_1 = x$, $q_2 = y$, $q_3 = z$. $h_1 = h_2 = h_3$ yield

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0.$$

Separation of variables: To find the solution of the above equation we shall use the separation of variables.

$$\psi(x,y,z) = X(x)Y(y)Z(z), \quad \frac{\partial \psi}{\partial x} = \frac{dX}{dx} Y Z, \quad \frac{\partial \psi}{\partial y} = \frac{dY}{dy} X Z$$

$$\frac{\partial \psi}{\partial z} = X Y \frac{dZ}{dz}. \text{ Also } \frac{\partial^2 \psi}{\partial x^2} = \frac{d^2 X}{dx^2} Y Z, \quad \frac{\partial^2 \psi}{\partial y^2} = \frac{d^2 Y}{dy^2} X Z$$

$$\frac{\partial^2 \psi}{\partial z^2} = X Y \frac{d^2 Z}{dz^2} \quad \therefore \frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0.$$

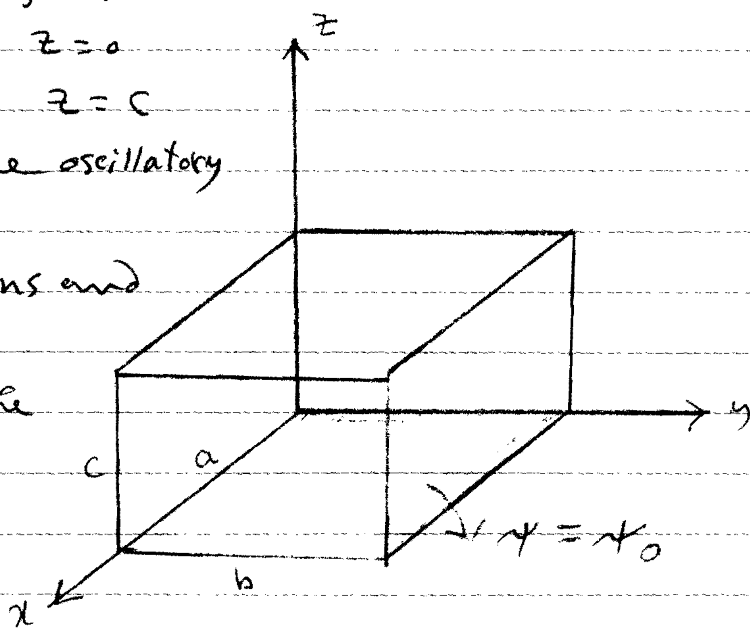
$$\frac{d^2 X}{dx^2} + \alpha^2 X = 0. \qquad \frac{d^2 Y}{dy^2} + \beta^2 Y = 0. \qquad \frac{d^2 Z}{dz^2} + \gamma^2 Z = 0$$

$$\therefore \psi(x,y,z) = (A_1 \cos\alpha x + B_1 \sin\alpha x)(A_2 \cos\beta y + B_2 \sin\beta y)(A_3 \cosh\gamma z + B_3 \sinh$$

Application for Laplace's equation in three dimensional (x, y, z).

Ex: let us consider the calculation of the potential within a conducting Box for which all of the sides except one are grounded and the remaining side is at a potential ψ_0, let the lengths of the sides in the x, y, and z directions be respectively a, b and c

$$\psi = \begin{cases} 0 & x = a, 0 \\ 0 & y = b, 0 \\ \psi_0 & z = 0 \\ 0 & z = c \end{cases}$$



In this problem, we must have oscillatory solutions in the x and y-directions and an exponential solution in the z-direction

Since

$$\psi(x,y,z) = (A_\alpha \cos\alpha x + B_\alpha \sin\alpha x)(A_\beta \cos\beta y + B_\beta \sin\beta y)(A_\gamma \cosh\gamma z + B_\gamma \sinh\gamma z)$$

with the condition $\alpha^2 + \beta^2 = \gamma^2$. In order that the potential vanish at $x = 0, a$ and at $y = 0, b$, Then

$$\psi_{rs}(x,y,z) = \sin\frac{r\pi}{a}x \, \sin\frac{s\pi}{b}y \, \sinh\gamma_{sr}(c - z)$$

where, we have identified that $\alpha_r = \frac{r\pi}{a}$, $\beta_s = \frac{s\pi}{b}$

So that there must be a γ for each pair of values for r, s i.e.

$$\gamma_{r,s} = \pi\left(r^2/a^2 + s^2/b^2\right)^{1/2}$$

Consequently, there will be particular potential functions which satisfy the conditions of the problem and which have the form

$$\psi_{rs} = \sin \frac{r\pi}{a}x \sin \frac{s\pi}{b}y \sinh \gamma_{sr}(c-z).$$

where we have chosen to write the exponential factor in terms of
hyperbolic sine function, and where we have placed $(c-z)$ in the arg
in order to insure that the boundary condition $\psi(x,y,c) = 0$ is m
The general solution to the problem will be $\psi(x,y,z) = \sum_{r,s} A_{rs}\psi$
where the coefficients A_{rs} are determined by the boundary
condition at $z=0$. $\psi(x,y,0) = \psi_0$

Thus $\psi_0 = \sum_{s,r=1}^{\infty} A_{rs} \sin \frac{r\pi}{a}x \sin \frac{s\pi}{b}y \sinh \gamma_{rs} c$

Here ψ_0 is expressed as a double fourier sine series characteristi
of the expansion of an odd function in two dimensions, and the
values of the A_{rs} are found in a manner entirely analogous t
that used for the single fourier series we find

$$A_{rs} = \frac{16}{\pi^2 rs}\psi_0 \operatorname{csch} \gamma_{rs} c$$

from which we have in general

$$\psi(x,y,z) = \frac{16\psi_0}{\pi^2} \sum_{\substack{r,s \\ odd}} \frac{1}{rs} \frac{\sinh \gamma_{rs}(c-z)}{\sinh \gamma_{rs} c} \sin \frac{r\pi}{a} x \sin \frac{s\pi}{b} y .$$

In the previous example we have used a fourier series technique which state that

$$\int_0^a \sin \frac{r\pi}{a} x \sin \frac{s\pi}{a} x \, dx = \begin{cases} a/2 & s = r \\ 0 & s \neq r \end{cases}$$

Ex: Find the solution of the two-dimensional Laplace equation $\nabla^2 \psi = 0$ within the rectangle R defined by $0 \leq x \leq a$, $0 \leq y \leq b$, where a and b are constants, which satisfies the Dirichlet conditions

$\psi(0, y) = 0$, $\psi(a,y) = 0$ $0 \leq y \leq b$

$\psi(x,0) = f(x)$ $0 < x < a$, $\psi(x,b) = 0$ $0 \leq x \leq a$

The solution:

$f(x) = e^{\gamma x}$

$$\psi(x,y) = \sum_{r=1}^{\infty} \left[\frac{2}{a} \int_0^a f(x') \sin \frac{r\pi x'}{a} dx' \right] \frac{\sin \frac{r\pi}{a} x \sinh \frac{r\pi}{a}(b-y)}{\sinh \frac{r\pi}{a} b}$$

Ex: Solve Laplace's equation $\nabla^2 \psi(x,y) = 0$ in the rectangular $0 \leq x \leq a$, $0 \leq y \leq b$ subject to the boundary condition

$$\frac{\partial \psi}{\partial x}(0,y) = 0 , \quad \frac{\partial \psi}{\partial x}(a,y) = 0 \quad \psi(x,0) = x(a-x), \ \psi(x,b) = 0$$

The general solution is

$$\psi(x,y) = a^2/6 \left(1 - \frac{y}{b}\right) + \sum_{n=1}^{\infty} \frac{2a^2}{n^2\pi^2} \frac{[1+(-)^n]}{\sinh \frac{n\pi}{a} b} \cos \frac{n\pi}{a} x \ \sinh \frac{n\pi}{a}(y-b)$$

Problems:

1) Show that if $x = a\cosh\xi\cos\eta$, $y = a\sinh\xi\sin\eta$, $z = z$

where (ξ, η, z) are called the elliptic cylinderical coordinates, show

that, the Laplace's equation $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0$ assumes the form

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} + a^2(\cosh^2\xi - \cos^2\eta)\frac{\partial^2\psi}{\partial z^2} = 0.$$

Deduce that if has solution of the form $f(\xi) f(\eta) e^{-\gamma z}$ where $f(\eta)$

Satisfies the equation

$$\frac{d^2 f}{d\eta^2} + (G + 16q\cos 2\eta) f = 0$$

G is being constant and $q = -\frac{a^2}{32}\gamma^2$.

2) Parabolic coordinates ξ, η, ϕ are defined by

$$x = \sqrt{\xi\eta}\cos\phi, \quad y = \sqrt{\xi\eta}\sin\phi, \quad z = \tfrac{1}{2}(\xi - \eta)$$

Show that in these coordinates Laplace's equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0$$

becomes

$$\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\psi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\eta\frac{\partial\psi}{\partial\eta}\right) + \frac{\xi+\eta}{4\xi\eta}\frac{\partial^2\psi}{\partial\phi^2} = 0$$

Prove that if $F_n(x)$ is a solution of the equation

$$x\frac{d^2F}{dx^2} + \frac{dF}{dx} + \left(n - \frac{m^2}{4x}\right)F = 0$$

then $F_n(\xi)\,F_{-n}(\eta)\,e^{\pm im\phi}$ is a solution of Laplace's equation. Hence deduce that, for n is zero, the function F can be written in the form

$$F_0(x) = \left[A\cosh(\ln\sqrt{x^m}) + B\sinh(\ln\sqrt{x^m})\right]$$

Show that for the case $m=0$ the function F can be generated from the relation

$$\exp -\left[\frac{t(1-t)^{-1}}{x}\right] = (1-t)\sum_{n=0}^{\infty} F_n(1/x)\,\frac{t^n}{n!}$$

Ex: The velocity potential function $\psi(r,\theta)$ for steady flow of an ideal fluid around a cylinder of radius $r=a$ satisfying the boundary value problem

$$r\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{\partial \theta^2} = 0 \qquad r>0$$

$$\left.\frac{\partial \psi}{\partial r}\right|_{r=a} = 0 \qquad \psi(r,\theta) = \psi(r,-\theta)$$

$\lim_{r\to\infty}\left(\psi(r,\theta) - \psi_0\, r\cos\theta\right) = 0$, Find the explicit form for ψ.

If $\phi(r,\theta)$ is the stream function for the flow such that $\nabla\phi \perp \nabla\psi$ show that $\phi(x,y) = \psi_0\, y\left[a^2(x^2+y^2)^{-1} - 1\right]$

The Solution: The Laplace's equation in the polar coordinates is given by

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0$$

The general solution as before is

$\psi(r,\theta) = (A\cos m\theta + B\sin m\theta)(Cr^{-m} + Dr^{m})$

This function is called circular harmonics

Now $\psi(r,\theta) = \psi(r,-\theta)$, then

$(A\cos m\theta + B\sin m\theta)(Cr^{-m}+Dr^{m}) = (A\cos m\theta - B\sin m\theta)(Cr^{-m}+Dr^{m})$

Hence $2B = 0 \Rightarrow B = 0$, Therefore.

$\psi(r,\theta) = \cos m\theta\,(\bar{C}r^{-m} + \bar{D}r^{m})$

$\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = 0$ $\quad\therefore \frac{\partial\psi}{\partial r} = \cos m\theta\,(mr^{m-1}\bar{D} - \bar{C}mr^{-m-1})$

$\therefore \quad a^{m-1}\bar{D} - \bar{C}a^{-m-1} = 0 \quad \therefore \bar{C} = a^{2m}\bar{D}$

$\therefore \psi(r,\theta) = \bar{D}\cos m\theta\left(\frac{a^{2m}}{r^{m}} + r^{m}\right)$

But $\lim\limits_{r\to\infty}(\psi(r,\theta) - \psi_0 r\cos\theta) = 0$, Then.

$\lim\limits_{r\to\infty}\left[\bar{D}\cos m\theta\left(\frac{a^{2m}}{r^{m}} + r^{m}\right) - \psi_0 r\cos\theta\right] = \lim\limits_{r\to\infty}\left[\bar{D}\cos m\theta\, r^{m} - \psi_0 r\cos\theta\right] -$

This will be hold iff $m = 1$, $D = \psi_0$.

$\therefore \psi(r,\theta) = \psi_0\cos\theta\left(\frac{a^2}{r} + r\right)$

Now $\nabla\psi = \frac{\partial\psi}{\partial x}\underline{i} + \frac{\partial\psi}{\partial y}\underline{j}$, If we take $x = r\cos\theta$

and $y = r\sin\theta$ $\quad\therefore x^2+y^2 = r^2$

$\therefore \psi(x,y) = \psi_0\left[\frac{a^2 x}{x^2+y^2} + x\right] \quad \therefore \frac{\partial \psi}{\partial x} = \psi_0\left(1 + \frac{a^2}{x^2+y^2}\right)$

$+ \psi_0 x\left[\frac{-2x a^2}{(x^2+y^2)^2}\right] \quad \therefore \frac{\partial \psi}{\partial x} = \psi_0\left[1 + \frac{a^2}{(x^2+y^2)} - \frac{2x^2 a^2}{(x^2+y^2)^2}\right]$

$$\frac{\partial \psi}{\partial x} = \psi_0\left[1 + \frac{a^2(x^2+y^2) - 2x^2a^2}{(x^2+y^2)^2}\right]$$

$$\therefore \frac{\partial \psi}{\partial x} = \psi_0\left[1 + a^2\frac{(y^2-x^2)}{(x^2+y^2)^2}\right]$$

$$\frac{\partial \psi}{\partial y} = \psi_0\left(\frac{-2xya^2}{(x^2+y^2)^2}\right)$$

$$\therefore \nabla\psi = \psi_0\left[1 + a^2\frac{(y^2-x^2)}{(x^2+y^2)^2}\right]\underline{i} - \psi_0\frac{2xya^2}{(x^2+y^2)^2}$$

To find $\phi(x,y)$ we need $\nabla\phi \cdot \nabla\psi = 0$ ... (2)

$$\left(\frac{\partial \phi}{\partial x}\underline{i} + \frac{\partial \phi}{\partial y}\underline{j}\right)\cdot\nabla\psi = 0$$

$$\therefore \left[1 + a^2\frac{(y^2-x^2)}{(x^2+y^2)^2}\right]\frac{\partial \phi}{\partial x} - \frac{2xya^2}{(x^2+y^2)^2}\frac{\partial \phi}{\partial y} = 0.$$

In order to solve this equation let us take $\phi(x,y) \to \bar{\phi}(r,\theta)$

where $x = r\cos\theta$ and $y = r\sin\theta$. $\therefore d\phi = d\bar{\phi}$ (3)

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = \frac{\partial \bar{\phi}}{\partial r}dr + \frac{\partial \bar{\phi}}{\partial \theta}d\theta$$

$$\frac{\partial \phi}{\partial x}(\cos\theta\, dr - r\sin\theta\, d\theta) + \frac{\partial \phi}{\partial y}(\sin\theta\, dr + r\cos\theta\, d\theta) = \frac{\partial \bar{\phi}}{\partial r}dr + \frac{\partial \bar{\phi}}{\partial \theta}d\theta$$

$$\therefore \cos\theta\frac{\partial \phi}{\partial x} + \sin\theta\frac{\partial \phi}{\partial y} = \frac{\partial \bar{\phi}}{\partial r}, \quad r\cos\theta\frac{\partial \phi}{\partial y} - r\sin\theta\frac{\partial \phi}{\partial x} = \frac{\partial \bar{\phi}}{\partial \theta}.$$

$$\therefore \left(1 + \frac{a^2 r^2 \cos 2\theta}{r^4}\right)\left(\cos\theta \frac{\partial \bar\phi}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial \bar\phi}{\partial \theta}\right)$$

$$- \frac{r^2 a^2 \sin 2\theta}{r^4}\left(\sin\theta \frac{\partial \bar\phi}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial \bar\phi}{\partial \theta}\right) = 0$$

$$\therefore \cos\theta \frac{\partial \bar\phi}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial \bar\phi}{\partial \theta} - \frac{a^2 \cos 2\theta \cos\theta}{r^2}\frac{\partial \bar\phi}{\partial r} + \frac{a^2}{r^3}\sin\theta \cos 2\theta \frac{\partial \bar\phi}{\partial \theta}$$

$$- \frac{a^2}{r^2}\sin 2\theta \sin\theta \frac{\partial \bar\phi}{\partial r} - \frac{a^2}{r^3}\cos\theta \sin 2\theta \frac{\partial \bar\phi}{\partial \theta} = 0$$

$$\therefore \cos\theta \frac{\partial \bar\phi}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial \bar\phi}{\partial \theta} - \frac{a^2}{r^2}\cos\theta \frac{\partial \bar\phi}{\partial r} - \frac{a^2}{r^3}\sin\theta \frac{\partial \bar\phi}{\partial \theta} = 0$$

$$\cancel{\cos\theta\, \partial\phi} \quad \left(1 - \frac{a^2}{r^2}\right)\cos\theta \frac{\partial \bar\phi}{\partial r} - \frac{1}{r}\sin\theta\left(1 + \frac{a^2}{r^2}\right)\frac{\partial \bar\phi}{\partial \theta} = 0$$

$$\therefore \frac{r(r^2 - a^2)}{(r^2 + a^2)}\frac{\partial \bar\phi}{\partial r} - \tan\theta \frac{\partial \bar\phi}{\partial \theta} = 0$$

Let $\bar\Phi = R(r)\,F(\theta)$

$$\therefore \frac{r}{R}\frac{(r^2 - a^2)}{(r^2 + a^2)}\frac{dR}{dr} - \frac{\tan\theta}{F(\theta)}\frac{dF}{d\theta} = 0$$

$$\therefore \frac{1}{R}\frac{r(r^2 - a^2)}{(r^2 + a^2)}\frac{dR}{dr} = \beta \quad , \quad \frac{\tan\theta}{F(\theta)}\frac{dF}{d\theta} = \beta$$

$$\therefore \frac{dR}{R} = \beta \frac{(r^2 + a^2)}{r(r^2 - a^2)}\,dr \quad , \quad \frac{dF}{F} = \beta \frac{\cos\theta}{\sin\theta}\,d\theta$$

$$\therefore \ln F = \beta \ln \sin\theta + \ln k. \qquad \therefore F = k(\sin\theta)^{\beta}$$

$$\frac{dR}{R} = \beta \int -\frac{1}{r} + \frac{1}{r-a} + \frac{1}{r+a}\, dr$$

$$\ln R = -\beta \ln r + \beta \ln(r^2 - a^2) + \ln c$$

$\therefore R = C\left(\frac{r^2-a^2}{r}\right)^\beta$ $\therefore \bar{\Phi}(r,\theta) = kC\left[\sin\theta\left(\frac{r^2-a^2}{r}\right)\right]^\beta$ since

β is arbitrary we can choose it to be one.

$$\therefore \bar{\Phi} = -\mu_0\left[y - \frac{a^2 y}{(x^2+y^2)}\right] = \mu_0\left(\frac{a^2 y}{x^2+y^2} - y\right)$$

where we have taken the constant $kC = -\mu_0$.

Find the solution of the Laplace's equation

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad 0 \le r \le a$$

$$u(a,\theta) = f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$$

using the separation of variables we find

$$u(r,\theta) = (Ar^n + Br^{-n})(C\cos n\theta + D\sin n\theta)$$

This function is called circular harmonics, the region of interest includes $r=0$, this implies that $B=0$. Thus

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \qquad n = 0,1,2,\ldots$$

$$f(\theta) = \sum_{n=0}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta) \qquad 0 \le \theta \le 2\pi$$

$$\therefore A_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta, \quad B_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta.$$

$$\therefore u(r,\theta) = \sum_{n=0}^{\infty} (r/a)^n (a_n \cos n\theta + b_n \sin n\theta), \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta .$$

$$\therefore\ u(r,\theta) = \sum_{n=0}^{\infty} (r/a)^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$\therefore\ u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty} (r/a)^n \left[\int_0^{2\pi} f(\phi) (\cos n\phi \cos n\theta + \sin n\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\phi) d\phi \left[1 + 2\sum_{n=1}^{\infty} (r/a)^n \cos n(\phi - \theta)\right] d\phi$$

But $$1 + 2\sum_{n=1}^{\infty} x^n \cos n\psi = \frac{1-x^2}{1-2x\cos\psi + x^2} \quad (*)$$

$$\therefore\ u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(a^2-r^2) f(\phi)}{a^2 - 2ar\cos(\phi-\theta) + r^2}\, d\phi$$

which is called Poisson's integral for the circle

(*) Let $z = x e^{i\psi}$ $\Rightarrow$ $\dfrac{1+z}{1-z} = -1 + \dfrac{2}{1-z} = -1 + 2\sum_{n=0}^{\infty} z^n$

$= 1 + 2\sum_{n=1}^{\infty} z^n$ But $\operatorname{Re} \dfrac{1+z}{1-z} = \dfrac{1-|z|^2}{1-2R(z)+|z|^2} = \dfrac{1-x^2}{1-2x\cos\psi}$

$$\therefore\ \operatorname{Re}\left(1 + 2\sum_{n=1}^{\infty} z^n\right) = 1 + 2\sum_{n=1}^{\infty} x^n \cos n\psi$$

Note that $a_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} d\phi = 1$

$a_n = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos n\theta \, d\theta = 0$, $\quad b_n = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \sin n\theta \, d\theta$

$b_n = \dfrac{1}{\pi}\left|\dfrac{\cos n\theta}{n}\right|_{\pi}^{0} = \dfrac{1}{\pi}(1 - \cos n\pi)$ $\qquad n = 1, 2, \ldots$

Accordingly the series solution is

$$u(r,\theta) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} (r/a)^{2n-1} \frac{\sin(2n-1)\theta}{(2n-1)}$$

Therefore, by using Poisson's integral

$$u(r,\theta) = 1 + \frac{1}{\pi}\tan^{-1}\left(\frac{a+r}{a-r}\tan\theta/2\right) + \frac{1}{\pi}\tan^{-1}\left(\frac{a+r}{a-r}\cot\theta/2\right)$$

Now let's consider the Laplace's equation in curvilinear coordinates

$$\nabla^2 \psi(q_1, q_2, q_3) = 0 \qquad (1)$$

By setting $\quad q_1 = r, \quad q_2 = \theta, \quad q_3 = \phi,$

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta$$

Equation (1) becomes

$$\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0 \qquad (2)$$

Suppose $\psi(r,\theta,\phi) = F(\phi)G(\theta)R(r)$, Thus

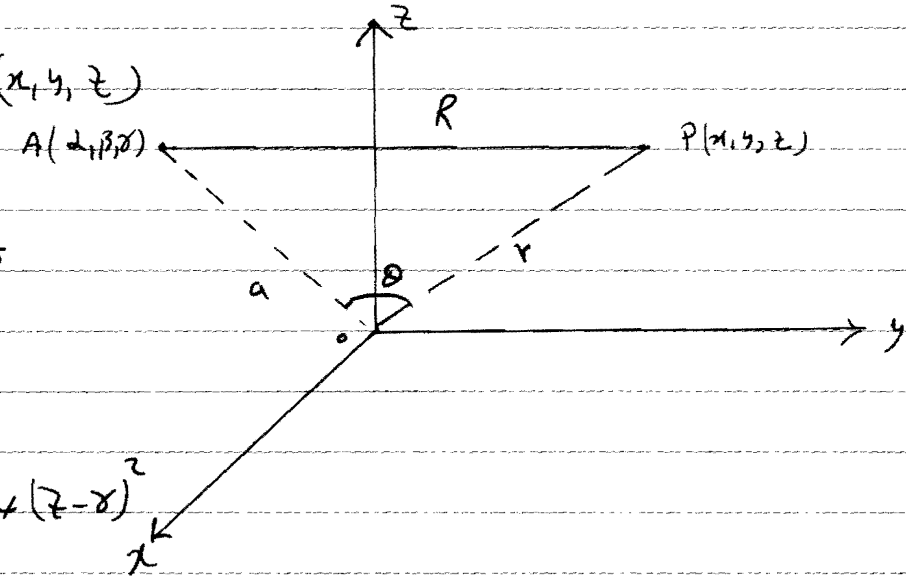
$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - n(n+1)R = 0 \qquad \frac{1}{F}\frac{d^2F}{d\phi^2} = -m^2$$

and $$\frac{1}{G\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dG}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right] = 0$$

This leads to the general solution in the form

$$\psi(r,\theta,\phi) = (A\cos m\phi + B\sin m\phi)\left[D r^{-(n+1)} + E r^{n}\right]\left[K P_n^m(\cos\theta) + L Q_n^m(\cos\theta)\right]$$

If we consider A is a fixed point with coordinates (α, β, γ) and P is a variable point (x, y, z)



The potential at P is

$$\psi \propto \frac{1}{R} \quad \therefore \psi = \frac{c}{R}$$

But $R = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$

$$\therefore R^2 = r^2 + a^2 - 2ar\cos\theta \qquad \text{Then } \psi = \frac{c}{(r^2+a^2-2ar\cos\theta)^{1/2}}$$

For $r < a$ $\quad \therefore \psi = \frac{c}{a}\sum_{n=0}^{\infty}(r/a)^n P_n(\cos\theta)$

For $r > a$ $\quad \therefore \psi = \frac{c}{r}\sum_{n=0}^{\infty}(a/r)^n P_n(\cos\theta)$

whence the general solution of the Laplace's equation in spherical polar coordinates

$$\psi(r,\theta) = \sum_{n=0}^{\infty}\left[A r^n + B r^{-(n+1)}\right] P_n(\cos\theta)$$

EX:- Compute the potential at all points in space exterior to a conducting sphere of radius a placed in a uniform electric field E_0

Suppose E_0 lies along the polar axis then the problem has azimuthal symmetry ~~about~~

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)$$

Since $E_0 = -\nabla\Phi_0 = -\frac{\partial\Phi_0}{\partial z}$ $\therefore$ $\Phi_0 = -E_0 z = -E_0 r\cos\theta$

$\Phi_0 = -E_0 r P_1(\cos\theta)$. Note that Φ_0 does not obey our usual condition that $\Phi(r\to\infty)=0$ This is because we have assumed a uniform field of infinite extent, and thus the sources of such a field must lie at infinity; Φ_0 may not then vanish as $r\to\infty$. Since our sphere is of finite extent, clearly the field at large r must be equal to E_0 and the potential there is given above.

$$\therefore \Phi(r,\theta) = -E_0 r P_1(\cos\theta) + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

which must reduce to zero at $r=a$.

$$+E_0 r P_1(\cos\theta) = \sum_{l=0}^{\infty} B_r a^{-(l+1)} P_l(\cos\theta)$$

$$E_0 a \int_{-1}^{1} P_1^2(x)\,dx = B_1\left(\frac{1}{a}\right)^2 \int_{-1}^{1} P_1^2(x)\,dx \qquad \therefore B_1 = E_0 a^3$$

$$\therefore \Phi(r,\theta) = -E_0 r\left(1-\frac{a^3}{r^3}\right) P_1(\cos\theta)$$

Problems:-

1) The temperature on the surface of a solid homogeneous sphere of unit radius is prescribed by

$$u(r,\phi) = \begin{cases} T_0 & 0<\phi<\alpha \\ 0 & \alpha<\phi<\pi \end{cases} \quad r=1$$

Show that the steady-state temperature distribution throughout the sphere is described by

$$u(r,\phi) = \frac{1}{2} T_0 \left[1-\cos\alpha - \sum_{n=1}^{\infty} r^n \left[P_{n+1}(\cos\alpha) - P_{n-1}(\cos\alpha)\right]\right] P_n(\cos\phi)$$

2) Find the electric potential inside a unit sphere where the boundary potential is prescribed by

$$u(1,\phi) = f(\phi) \begin{cases} u_0 & 0 \le \phi < \pi/2 \\ -u_0 & \pi/2 < \phi \le \pi \end{cases}$$

$$u(r,\phi) = u_0 \sum_{k=0}^{\infty} (-)^k \frac{2k!\,(4k+3)}{2^{2k+1}\,k!\,2k+1!}\, r^{2k+1}\, P_{2k+1}$$

To solve equation (1) we have

$$u(r,\phi) = \sum_{n=0}^{\infty} \left[a_n r^n + b_n r^{-(n+1)}\right] P_n(\cos\phi)$$

Since the solution is inside the sphere then $b_n = 0$.

$$\therefore \; u(r,\phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi) = a_0 + \sum_{n=1}^{\infty} A_n r^n P_n(\cos\phi)$$

$$\therefore \; T_0 = a_0 + \sum_{n=1}^{\infty} A_n P_n(\cos\phi) \qquad 0<\phi<\alpha$$

$$0 = a_0 + \sum_{n=1}^{\infty} a_n P_n(\cos\phi) \qquad \alpha<\phi<\pi$$

$$\therefore \int_0^{\alpha} T_0 \sin\phi \, d\phi = a_0 \int_0^{\alpha} \sin\phi \, d\phi + \sum_{n=1}^{\infty} a_n \int_0^{\alpha} \sin\phi \, P_n(\cos\phi)\, d\phi$$

$$\therefore \; T_0(1-\cos\alpha) = a_0(1-\cos\alpha) - \sum_{n=1}^{\infty} a_n \int_0^{\alpha} P_n(\cos\phi)\, d\cos\phi \qquad \#\#$$

But $$-a_0 \int_{\alpha}^{\pi} \sin\phi \, d\phi = -\sum_{n=1}^{\infty} a_n \int_{\alpha}^{\pi} P_n(\cos\phi)\, d\cos\phi$$

Thus $$a_0(1+\cos\alpha) = \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\pi} P_n(\cos\phi)\, d\cos\phi \qquad \#$$

From ## and # yield.

$$T_0(\cos\alpha - 1) + a_0(1-\cos\alpha) + a_0(1+\cos\alpha) = 0.$$

$$\therefore \; a_0 = \tfrac{1}{2} T_0 (1-\cos\alpha)$$

whence

$$u(r,\phi) = \tfrac{1}{2} T_0(1-\cos\alpha) + \sum_{n=1}^{\infty} A_n r^n P_n(\cos\phi)$$

Since $$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

$$\therefore \; f(\phi) = \sum_{n=0}^{\infty} a_n P_n(\cos\phi)$$

$$\int_0^{\pi} f(\phi) P_n(\cos\phi) \sin\phi \, d\phi = a_n \frac{2}{2n+1}$$

$$\therefore \; a_n = \frac{2n+1}{2} \int_0^{\pi} f(\phi) P_n(\cos\phi) \sin\phi \, d\phi$$

$$\therefore a_n = -T_0 \frac{2n+1}{2} \int_0^{\alpha} P_n(\cos\phi)\, d\cos\phi$$

$$= -T_0/2 \left[P_{n+1}(\cos\alpha) - P_{n-1}(\cos\alpha) \right]$$

$$\therefore u(r,\phi) = T_0/2 \left[(1-\cos\alpha) - \sum_{n=1}^{\infty} r^n P_n(\cos\phi) \left[P_{n+1}(\cos\alpha) - P_{n-1}(\cos\alpha) \right] \right]$$

Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

which satisfies the boundary conditions

$$u(0,t) = u(x,0) = 0 \qquad t > 0 \;, \; 0 < x < a$$

$$\frac{\partial u}{\partial x}(a,t) = f, \quad t > 0 \;, \; \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0, \; 0 < x < a$$

where a and c are given constants.

Let $u(x,t) = \cancel{V(x)+}X(x)\,T(t)$ which leads to $\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2}$

and $V''(x) = 0$. In this case the general solution becomes

$$u(x,t) = \alpha x + \beta + (A\cos c\delta t + B\sin c\delta t)(D\cos\delta x + E\sin\delta x)$$

For $u(0,t) = 0$ $\quad \therefore 0 = \beta + D(A\cos c\delta t + B\sin c\delta t)$

$\Rightarrow \beta = 0$ and $D = 0$.

$$u(x,t) = \alpha x + \sin\delta x\, (\bar{A}\cos c\delta t + \bar{B}\sin c\delta t)$$

$$\left(\frac{\partial \psi}{\partial x}\right)_{x=a} = f \qquad \therefore f = \alpha + \delta\cos\delta a\,(\bar{A}\cos c\delta t + \bar{B}\sin c\delta t)$$

This gives $f = \alpha$ and $\cos\delta a = 0 \quad \therefore \delta = (2n-1)\frac{\pi}{2a}$

$$\psi(x,t) = fx + \sum_{n=1}^{\infty} \sin(2n-1)\frac{\pi}{2a}x\,(\bar{A}\cos c\delta t + \bar{B}\sin c\delta t)$$

$\left(\frac{\partial \psi}{\partial t}\right)_{t=0} = 0 \quad \therefore 0 = \bar{B}$. Then the wave function becomes

$$\psi(x,t) = fx + \sum_{n=1}^{\infty} \bar{A}\sin(2n-1)\frac{\pi}{2a}x\cos(2n-1)\frac{\pi ct}{2a}$$

$$-fx = \sum_{n=1}^{\infty} \bar{A}\sin(2n-1)\frac{\pi}{2a}x.$$

$$\therefore -f\int_0^a x\sin(2n-1)\frac{\pi}{2a}x\,dx = \bar{A}\,a/2$$

$$\therefore \bar{A} = -\frac{2f}{a}\int_0^a x\sin(2n-1)\frac{\pi}{2a}x\,dx.$$

$$= \frac{2f}{a}\int_0^a x\,d\cos(2n-1)\frac{\pi}{2a}x. \quad \cdot\frac{2a}{\pi(2n-1)}$$

$$= \frac{4f}{\pi(2n-1)}\left[x\cos(2n-1)\frac{\pi}{2a}x\Big|_0^a - \int_0^a \cos(2n-1)\frac{\pi}{2a}x\,dx\right]$$

$$= \frac{4f}{\pi(2n-1)}\left[-\frac{2a}{\pi(2n-1)}\sin(2n-1)\frac{\pi}{2a}x\Big|_0^a\right.$$

$$\bar{A} = \frac{8af}{\pi^2(2n-1)^2}(-)^n$$

$$\therefore \psi(x,t) = fa\left[\frac{x}{a} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-)^n}{(2n-1)^2}\sin(2n-1)\frac{\pi x}{2a}\cos(2n-1)\frac{\pi ct}{2a}\right]$$

To be considered later

Problem: A uniform string of length πl is fastened at its ends $x=0$, $x=\pi l$. The point $x=\frac{1}{3}\pi l$ is drawn aside a small distance b, and the string released from rest at time $t=0$. Show that if T is the tension and ρ the density per unit length of string, the subsequent displacement of the string is

$$y(x,t)=\frac{9b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{nx}{l}\cos\frac{nct}{l}, \text{ where } c^2=T/\rho$$

see the solution later

Solve the wave equation

$$\frac{\partial^2 y}{\partial t^2}-c^2\frac{\partial^2 y}{\partial x^2}=F(x,t) \qquad 0\le x\le b,\ t\geq 0$$

subject to the initial conditions

$$y(0,t)=0 \qquad y(b,t)=0 \qquad t\geq 0$$

$$y(x,0)=f(x) \qquad \left(\frac{\partial y}{\partial t}\right)_{t=0}=g(x) \qquad 0\le x\le b$$

The solution:

let us first take $F(x,t)=0$: $\therefore \frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$

If $y=XT$ then $\frac{1}{c^2T}\frac{d^2T}{dt^2}=\frac{1}{X}\frac{d^2X}{dx^2}$, In this case we have

$$y(x,t)=TX=(A\cos kct+B\sin kct)(D\cos kx+E\sin kx)$$

For $\psi(0,t)=0$ yield $D=0$, For $\psi(b,t)=0 \Rightarrow kb=n\pi, n=1,2,\ldots$

Therefore $\psi(x,t)=\sum_{n=1}^{\infty}(A_n\cos\omega_n t+B_n\sin\omega_n t)\sin\frac{n\pi}{b}x$

where $\omega_n=\frac{n\pi c}{b}$. Now $f(x)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{b}x \Longrightarrow$

$A_n=\frac{2}{b}\int_0^b f(x)\sin\frac{n\pi}{b}x\,dx$. , Also $g(x)=\sum B_n\omega_n\sin\frac{n\pi}{b}x$

$\therefore B_n=\frac{2}{b\omega_n}\int_0^b g(x)\sin\frac{n\pi}{b}x\,dx$.

Since we have

$$\frac{\partial^2\psi}{\partial t^2}-c^2\frac{\partial^2\psi}{\partial x^2}=F(x,t) \qquad 0\le x\le b,\ t\geq 0$$

The solution of the homogeneous equation can be rewritten as

$$\psi(x,t)=\sum_{n=1}^{\infty}\phi_n(t)\sin\frac{n\pi}{b}x \quad \text{with } \phi_n(0)=0,\ \dot{\phi}_n(0)=0$$

$$\frac{\partial^2\psi}{\partial t^2}=\sum_{n=1}^{\infty}\ddot{\phi}_n(t)\sin\frac{n\pi}{b}x, \qquad \frac{\partial^2\psi}{\partial x^2}=-\sum_{n=1}^{\infty}\left(\frac{n\pi}{b}\right)^2\phi_n(t)\sin\frac{n\pi}{b}x$$

Thus

$$\sum_{n=1}^{\infty}\left(\ddot{\phi}_n(t)+\omega_n^2\phi_n(t)\right)\sin\frac{n\pi}{b}x=F(x,t)$$

$$\therefore \frac{2}{b}\int_0^b F(x,t)\sin\frac{k\pi}{b}x\,dx=\left(\ddot{\phi}_k(t)+\omega_k^2\phi_k(t)\right)=f_k(t)$$

But $\ddot{\phi}_k(t)+\omega_k^2\phi_k(t)=f_k(t)$ gives the solution

$$\phi_k(t)=\frac{1}{\omega_k}\int_0^t f_k(t')\sin\omega_k(t-t')\,dt'$$

$$\therefore \psi(x,t)=\sum_{n=1}^{\infty}\left[\frac{1}{\omega_n}\int_0^t f_k(t')\sin\omega_n(t-t')\,dt'\right]\sin\frac{n\pi}{b}x$$
$$+\sum_{n=1}^{\infty}(A_n\cos\omega_n t+B_n\sin\omega_n t)\sin\frac{n\pi}{b}x$$