Math 316

Final Exam 1440/1441, 1st semester
Exam Time: 3 hours

Q1. (a) Prove that if L is a self-adjoint differential operator of second order, then the eigenvalues of the equation

$$Lu + \lambda u = 0$$

are all real.

(b) Consider the eigenvalue problem

$$5x^3u'' + 15x^2u + \lambda u = 0, \qquad x \in [1,2], \tag{1}$$
$$u(1) = 0, \qquad u(2) = 0.$$

i. Put equation (1) in the standard form: $Lu + \lambda u = 0$.

ii. Show that L in part (i) is a self-adjoint operator.

Q2. Consider the function

$$f(x) = \begin{cases} 1, & -\pi \leq x \leq 0 \\ -1, & 0 < x < \pi \end{cases}$$

and

$$f(x + 2\pi) = f(x), \; x \in \mathbb{R}.$$

(a) Find the Fourier series representation for f.

(b) Find a series representation for $\frac{\pi}{4}$.

(c) Redefine the function f so that the Fourier series converges to f at every $x \in \mathbb{R}$.

Q3. (a) Consider the Laguerre Polynomials

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right), n \in \mathbb{N}_0, x \in (0,\infty).$$

i. Show that

$$\langle x^m, L_n\rangle_{e^{-x}} = \begin{cases} 0, & m < n \\ (-1)^n n!, & m = n \end{cases}.$$

ii. Using part (i), show that for all $n \in \mathbb{N}_0$,

$$\|L_n\|^2_{e^{-x}} = 1.$$

(b) Prove that

$$\int_0^x tJ_0(t)\,dt = xJ_1(x),$$

for all $x > 0$, where J_0, J_1 are Bessel functions of the first kind of orders zero and one, respectively.

Q4. (a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & -\pi < x \leq 0 \\ -1, & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$$

i. Show that $f \in \mathcal{L}^1(\mathbb{R})$.

ii. Find the Fourier integral of f.

iii. Redefine the function f so that the Fourier integral converges to f at every $x \in \mathbb{R}$.

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{-x^2}.$$

i. Find the derivative of the Fourier transform $\hat{f}$ of f.

ii. Find a closed form of the Fourier transform $\hat{f}$.

Good Luck
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