4) (c) $\lim_{x\to 0} \frac{1}{x^2-1} = \frac{1}{0-1} = \frac{1}{-1} = -1$

Given $\varepsilon>0$, want $\delta>0$ s.t

$0<|x-0|<\delta \Rightarrow \left|\frac{1}{x^2-1} - (-1)\right| < \varepsilon$

$\left|\frac{1}{x^2-1}+1\right| < \varepsilon \Rightarrow \frac{x^2}{|x^2-1|} < \varepsilon$

Now $\lim_{x\to 0}(x^2-1) = -1$

$\forall \varepsilon>0 \ \exists \ \delta_2>0$ s.t

$0<|x-0|<\delta_2 \Rightarrow |x^2-1+1| < \varepsilon$

$\cancel{|x^2|<\varepsilon}$

$\cancel{|x|<\sqrt{\varepsilon}}$

$\exists \ \delta_3>0 \Rightarrow 0<|x-0|<\delta_3 \Rightarrow$ Take $\varepsilon = \frac{|-1|}{2} = \frac{1}{2}$

$\big|\,|x^2-1| - (-1)\big| < \frac{1}{2}$

$-\frac{1}{2} < |x^2-1| - (-1)$

$\frac{1}{2} = -\frac{1}{2}+1 < |x^2-1|$

$2 > \frac{1}{|x^2-1|} \Rightarrow \sqrt{2} > \frac{1}{\sqrt{|x^2-1|}}$

Take $\delta = \min\{\delta_2, \delta_3\}$

If $0<|x-0|<\delta \Rightarrow \frac{1}{\sqrt{|x^2-1|}} < \sqrt{2}$ & $|x|<\sqrt{\varepsilon}$

$\Rightarrow \frac{x^2}{|x^2-1|} < 2\varepsilon$

19)

(a) $\lim_{x \to x_0^-} f(x) = -\infty$

$\forall M \in \mathbb{R} : \exists\ \delta > 0$ s.t:

If $x_0 - \delta < x < x_0 \Rightarrow f(x) < M$

(b) $\lim_{x \to x_0^+} f(x) = \infty$

$\forall M \in \mathbb{R} : \exists\ \delta > 0$ s.t:

If $x_0 < x < x_0 + \delta \Rightarrow f(x) > M$

(c) $\lim_{x \to x_0^+} f(x) = -\infty$ $\quad \forall M \in \mathbb{R} : \exists\ \delta > 0$ s.t

If $x_0 < x < x_0 + \delta \Rightarrow f(x) < M$

21) (a) $\lim_{x \to x_0} f(x) = \infty$

$\forall M \in \mathbb{R} : \exists\ \delta > 0$ s.t:

$0 < |x - x_0| < \delta \Rightarrow f(x) > M$

(b) $\lim_{x \to x_0} f(x) = -\infty$

$\forall M \in \mathbb{R} : \exists\ \delta > 0$ s.t:

$0 < |x - x_0| < \delta \Rightarrow f(x) < M$