

# Model Solution

Final Exam, Semester II, 1447

Department of Mathematics, College of Science, KSU

Course: Math 209 Maximum Marks: 40 Duration: 3 Hours

## Question 1

### 1(a)

We have

$$a_n = \frac{3n^2 + 1}{n^2 + n}.$$

Dividing numerator and denominator by  $n^2$ , we get

$$a_n = \frac{3 + \frac{1}{n^2}}{1 + \frac{1}{n}}.$$

Since

$$\frac{1}{n^2} \rightarrow 0 \quad \text{and} \quad \frac{1}{n} \rightarrow 0$$

as  $n \rightarrow \infty$ , it follows that

$$\lim_{n \rightarrow \infty} a_n = \frac{3}{1} = 3.$$

Therefore, the sequence converges and

$$\boxed{\lim_{n \rightarrow \infty} a_n = 3.}$$

### 1(b)

We have

$$b_n = (-1)^n \frac{n}{n+1}.$$

Since

$$\frac{n}{n+1} \rightarrow 1,$$

the sign of  $b_n$  depends on  $(-1)^n$ . For even  $n$ ,

$$b_n \rightarrow 1,$$

while for odd  $n$ ,

$$b_n \rightarrow -1.$$

Thus the even and odd subsequences have different limits. Therefore, the sequence diverges.

$$\boxed{\text{The sequence } b_n \text{ diverges.}}$$

## 2(a)

Consider

$$\sum_{n=1}^{\infty} \left( \frac{1}{n^2} + \frac{1}{3^n} \right).$$

We write

$$\sum_{n=1}^{\infty} \left( \frac{1}{n^2} + \frac{1}{3^n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{3^n}.$$

The first series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a  $p$ -series with  $p = 2 > 1$ , so it converges. The second series

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

is a geometric series with ratio  $r = \frac{1}{3}$ , and since  $|r| < 1$ , it converges. Therefore,

$$\boxed{\sum_{n=1}^{\infty} \left( \frac{1}{n^2} + \frac{1}{3^n} \right) \text{ converges.}}$$

## 2(b)

Consider

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}.$$

We compare it with the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Using the limit comparison test,

$$\lim_{n \rightarrow \infty} \frac{\frac{n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3+1} = 1.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, the given series also diverges.

$$\boxed{\sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \text{ diverges.}}$$

## Question 2

### 1. Interval and radius of convergence

Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n+2} (x-2)^n.$$

Using the root test, we consider

$$\left| \frac{(-1)^n 3^n}{n+2} (x-2)^n \right|^{1/n} = 3|x-2| \left( \frac{1}{n+2} \right)^{1/n}.$$

Since

$$\left( \frac{1}{n+2} \right)^{1/n} \rightarrow 1,$$

the series converges when

$$3|x-2| < 1.$$

Thus

$$|x-2| < \frac{1}{3}.$$

Hence the radius of convergence is

$$\boxed{R = \frac{1}{3}}.$$

Now we check the endpoints.

The left endpoint is

$$x = 2 - \frac{1}{3} = \frac{5}{3}.$$

Substituting  $x = \frac{5}{3}$ , we get  $x-2 = -\frac{1}{3}$ . Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n+2} \left(-\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n+2}.$$

This is a harmonic-type series, so it diverges. Thus  $x = \frac{5}{3}$  is not included.

The right endpoint is

$$x = 2 + \frac{1}{3} = \frac{7}{3}.$$

Substituting  $x = \frac{7}{3}$ , we get  $x-2 = \frac{1}{3}$ . Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n+2} \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+2}.$$

This is an alternating series whose terms decrease to 0. Therefore, it converges by the alternating series test. Thus  $x = \frac{7}{3}$  is included.

Hence the interval of convergence is

$$\boxed{\left(\frac{5}{3}, \frac{7}{3}\right]}.$$

## 2. Power series representation

We want to find the power series representation of

$$f(x) = \frac{x}{9 - x^2}.$$

Write

$$f(x) = \frac{x}{9\left(1 - \frac{x^2}{9}\right)} = \frac{x}{9} \cdot \frac{1}{1 - \frac{x^2}{9}}.$$

Using the geometric series formula

$$\frac{1}{1 - r} = \sum_{n=0}^{\infty} r^n, \quad |r| < 1,$$

with  $r = \frac{x^2}{9}$ , we obtain

$$\frac{1}{1 - \frac{x^2}{9}} = \sum_{n=0}^{\infty} \left(\frac{x^2}{9}\right)^n.$$

Therefore,

$$\frac{x}{9 - x^2} = \frac{x}{9} \sum_{n=0}^{\infty} \left(\frac{x^2}{9}\right)^n.$$

Hence

$$\boxed{\frac{x}{9 - x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{9^{n+1}}.}$$

The convergence condition is

$$\left|\frac{x^2}{9}\right| < 1.$$

Thus  $x^2 < 9$ , which gives  $|x| < 3$ . Therefore, the interval of convergence is

$$\boxed{(-3, 3).}$$

### Question 3

#### 1(a). Fourier series of $f(x) = x^2$

Let

$$f(x) = x^2, \quad -\pi \leq x \leq \pi.$$

Since  $x^2$  is an even function, its Fourier series contains only cosine terms. Thus

$$x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx).$$

We compute

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx.$$

Since  $x^2$  is even,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx.$$

Thus

$$a_0 = \frac{2}{\pi} \left[ \frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

Therefore,

$$\frac{a_0}{2} = \frac{\pi^2}{3}.$$

Now compute  $a_n$ :

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx.$$

Again, the integrand is even, so

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) dx.$$

Let

$$I_n = \int_0^{\pi} x^2 \cos(nx) dx.$$

Using integration by parts, take

$$u = x^2, \quad dv = \cos(nx) dx.$$

Then

$$du = 2x dx, \quad v = \frac{\sin(nx)}{n}.$$

Hence

$$I_n = \left[ \frac{x^2 \sin(nx)}{n} \right]_0^{\pi} - \frac{2}{n} \int_0^{\pi} x \sin(nx) dx.$$

Since  $\sin(n\pi) = 0$ , the boundary term is zero. Thus

$$I_n = -\frac{2}{n} \int_0^{\pi} x \sin(nx) dx.$$

Now compute

$$J_n = \int_0^\pi x \sin(nx) dx.$$

Using integration by parts again, take

$$u = x, \quad dv = \sin(nx) dx.$$

Then

$$du = dx, \quad v = -\frac{\cos(nx)}{n}.$$

So

$$J_n = \left[ -\frac{x \cos(nx)}{n} \right]_0^\pi + \frac{1}{n} \int_0^\pi \cos(nx) dx.$$

But

$$\int_0^\pi \cos(nx) dx = \left[ \frac{\sin(nx)}{n} \right]_0^\pi = 0.$$

Hence

$$J_n = -\frac{\pi \cos(n\pi)}{n} = -\frac{\pi(-1)^n}{n}.$$

Therefore,

$$I_n = -\frac{2}{n} \left( -\frac{\pi(-1)^n}{n} \right) = \frac{2\pi(-1)^n}{n^2}.$$

Thus

$$a_n = \frac{2}{\pi} I_n = \frac{2}{\pi} \cdot \frac{2\pi(-1)^n}{n^2} = \frac{4(-1)^n}{n^2}.$$

Since  $x^2$  is even,  $b_n = 0$ . Therefore, the Fourier series is

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

### 1(b). Deduction of $\sum_{n=1}^{\infty} \frac{1}{n^2}$

From the Fourier series,

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

Put  $x = \pi$ . Then  $\cos(n\pi) = (-1)^n$ . Hence

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}.$$

Since  $(-1)^n(-1)^n = 1$ , we obtain

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore,

$$4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \cdot \frac{2\pi^2}{3} = \frac{\pi^2}{6}.$$

Hence

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

## 2(a). Fourier integral

Let

$$f(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

The function  $f$  is even. Hence its Fourier integral has the form

$$f(x) = \int_0^{\infty} A(\lambda) \cos(\lambda x) d\lambda.$$

For an even function,

$$A(\lambda) = \frac{2}{\pi} \int_0^{\infty} f(t) \cos(\lambda t) dt.$$

Since  $f(t) = 1$  for  $0 \leq t \leq 1$  and  $f(t) = 0$  for  $t > 1$ , we get

$$A(\lambda) = \frac{2}{\pi} \int_0^1 \cos(\lambda t) dt.$$

Therefore,

$$A(\lambda) = \frac{2}{\pi} \left[ \frac{\sin(\lambda t)}{\lambda} \right]_0^1 = \frac{2 \sin \lambda}{\pi \lambda}.$$

Thus the Fourier integral is

$$\boxed{f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda}{\lambda} \cos(\lambda x) d\lambda.}$$

At points of continuity, this integral represents  $f(x)$ . At the discontinuity points  $x = \pm 1$ , it gives the average value

$$\frac{1+0}{2} = \frac{1}{2}.$$

Hence

$$\frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda}{\lambda} \cos(\lambda x) d\lambda = \begin{cases} 1, & |x| < 1, \\ \frac{1}{2}, & |x| = 1, \\ 0, & |x| > 1. \end{cases}$$

## Question 4

(a)

Solve

$$(1 + y^2) \frac{dy}{dx} = 2x - 3.$$

Separating variables gives

$$(1 + y^2) dy = (2x - 3) dx.$$

Integrating both sides,

$$\int (1 + y^2) dy = \int (2x - 3) dx.$$

Thus

$$y + \frac{y^3}{3} = x^2 - 3x + C.$$

Therefore, the solution is

$$\boxed{y + \frac{y^3}{3} = x^2 - 3x + C.}$$

(b)

Solve

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}.$$

We rewrite the right-hand side:

$$\frac{x^2 + y^2}{xy} = \frac{x}{y} + \frac{y}{x}.$$

This is a homogeneous differential equation. Put

$$y = vx.$$

Then

$$\frac{dy}{dx} = v + x \frac{dv}{dx}.$$

Also,

$$\frac{x}{y} + \frac{y}{x} = \frac{x}{vx} + \frac{vx}{x} = \frac{1}{v} + v.$$

Therefore,

$$v + x \frac{dv}{dx} = v + \frac{1}{v}.$$

Cancelling  $v$ , we get

$$x \frac{dv}{dx} = \frac{1}{v}.$$

Hence

$$v dv = \frac{dx}{x}.$$

Integrating,

$$\frac{v^2}{2} = \ln |x| + C.$$

Thus

$$v^2 = 2 \ln |x| + C.$$

Since  $v = \frac{y}{x}$ , we obtain

$$\left(\frac{y}{x}\right)^2 = 2 \ln |x| + C.$$

Therefore,

$$\boxed{y^2 = x^2 (2 \ln |x| + C)}.$$

(c)

Solve

$$y' + \frac{2}{1+x}y = 3x.$$

This is a first-order linear differential equation of the form

$$y' + P(x)y = Q(x),$$

where

$$P(x) = \frac{2}{1+x}.$$

The integrating factor is

$$\mu(x) = e^{\int P(x) dx} = e^{\int \frac{2}{1+x} dx}.$$

Thus

$$\mu(x) = e^{2 \ln |1+x|} = (1+x)^2.$$

Multiplying the differential equation by  $(1+x)^2$ , we get

$$(1+x)^2 y' + 2(1+x)y = 3x(1+x)^2.$$

The left-hand side is

$$\frac{d}{dx} \left( (1+x)^2 y \right).$$

Hence

$$\frac{d}{dx} \left( (1+x)^2 y \right) = 3x(1+x)^2.$$

Integrating both sides,

$$(1+x)^2 y = \int 3x(1+x)^2 dx.$$

Now

$$3x(1+x)^2 = 3x(1+2x+x^2) = 3x+6x^2+3x^3.$$

Therefore,

$$(1+x)^2 y = \int (3x+6x^2+3x^3) dx.$$

Thus

$$(1+x)^2 y = \frac{3x^2}{2} + 2x^3 + \frac{3x^4}{4} + C.$$

Hence

$$y = \frac{\frac{3x^2}{2} + 2x^3 + \frac{3x^4}{4} + C}{(1+x)^2}.$$

(d)

Solve

$$y' + \frac{2}{x}y = x^3y^2.$$

This is a Bernoulli differential equation. Divide by  $y^2$ :

$$\frac{y'}{y^2} + \frac{2}{x} \frac{1}{y} = x^3.$$

Let

$$u = \frac{1}{y}.$$

Then

$$u' = -\frac{y'}{y^2}.$$

Thus

$$\frac{y'}{y^2} = -u'.$$

Substituting into the equation gives

$$-u' + \frac{2}{x}u = x^3.$$

Multiplying by  $-1$ , we obtain

$$u' - \frac{2}{x}u = -x^3.$$

This is a linear differential equation in  $u$ . The integrating factor is

$$\mu(x) = e^{\int -\frac{2}{x} dx} = e^{-2 \ln|x|} = x^{-2}.$$

Multiplying by  $x^{-2}$ , we get

$$x^{-2}u' - 2x^{-3}u = -x.$$

The left-hand side is

$$\frac{d}{dx} (x^{-2}u).$$

Thus

$$\frac{d}{dx} (x^{-2}u) = -x.$$

Integrating both sides,

$$x^{-2}u = -\frac{x^2}{2} + C.$$

Therefore,

$$u = x^2 \left( C - \frac{x^2}{2} \right).$$

Since  $u = \frac{1}{y}$ , we get

$$\frac{1}{y} = Cx^2 - \frac{x^4}{2}.$$

Hence

$$y = \frac{1}{Cx^2 - \frac{x^4}{2}}.$$

Also, the zero function

$$y = 0$$

is a solution of the original differential equation.