King Saud University: Mathematics Department Math-254
Summer Semester 1437-38 H Second Midterm Exam.
Maximum Marks = 25 Time: 90 mins.

Question 1: Use the simple Gaussian elimination method, find all values of a and b for which the following linear system is consistent or inconsistent. Find the solutions when the system is consistent. [5 Marks]

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ 2x_1 & - & 3x_2 & + & ax_3 & = & 5 \\ 3x_1 & - & 4x_2 & + & 5x_3 & = & b \end{array}$$

Question 2: Find determinant of the coefficient matrix A of the following linear system using LU decomposition by Crout's method. If $\det(A) = |A| = -1$, then find the unique solution of the following system. [5 Marks]

$$\begin{array}{rcrcrcl} x_1 & & & - & x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 1 \\ -x_1 & + & x_2 & + & \alpha x_3 & = & 1 \end{array}$$

Question 3: Consider the following nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 0 & -1 & 4 \\ 5 & 0 & -1 \\ -1 & 3 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}.$$

Rearrangement the given system such that the convergence of Gauss-Seidel iterative method is guaranteed. Then find upper bound for the error $\|x - x^{(5)}\|$ using Gauss-Seidel iterative method if $x^{(0)} = [0.3, 0.5, 1.0]^T$. [5 Marks]

Question 4: Let $f(x) = \sqrt{x - x^2}$ and $p_2(x)$ be quadratic Lagrange interpolating polynomial on $x_0 = 0, x_1 = \alpha$, and $x_2 = 1$. Find the value of α in $(0, 1)$ for which [5 Marks]

$$f(0.5) - p_2(0.5) = -0.25.$$

Question 5: Construct the table for $(\alpha, Q(\alpha))$ by evaluating the integral

$$Q(\alpha) = \int_1^2 (\alpha - 1)\ dx,$$

at $\alpha = 1, 1.5, 2.5, 3.5, 4$. Then use the constructed table to find the best approximation of $Q(3.4)$ by using quadratic Lagrange polynomial. Compute the absolute error. [5 Marks]

Solution of the Midterm II Examination

King Saud University: **Mathematics Department** **Math-254**
Summer Semester **1437-38 H** **Second Midterm Exam.**
Maximum Marks = 25 **Time: 90 mins.**

Question 1: Use the simple Gaussian elimination method, find all values of a and b for which the following linear system is consistent or inconsistent. Find the solutions when the system is consistent. [5 Marks]

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ 2x_1 & - & 3x_2 & + & ax_3 & = & 5 \\ 3x_1 & - & 4x_2 & + & 5x_3 & = & b \end{array}$$

Solution. Writing the given system in the augmented matrix form

$$[A|b] = \begin{pmatrix} 1 & -2 & 3 & 4 \\ 2 & -3 & a & 5 \\ 3 & -4 & 5 & b \end{pmatrix} \approx \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 2 & -4 & b-12 \end{pmatrix} \approx \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & a-6 & -3 \\ 0 & 0 & -2a+8 & b-6 \end{pmatrix}.$$

CASE I. Inconsistent system (no solution), if we take $a = 4$ and $b \neq 6$, gives

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & (a-6)x_3 & = & -3 \\ & & & & (-2a+8)x_3 & = & (b-6) \end{array}$$

CASE II. Consistent system (infinitely many solutions), if we take $a = 4$ and $b = 6$, gives

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & (a-6)x_3 & = & -3 \\ & & & & (-2a+8)x_3 & = & (b-6) \end{array}$$

gives

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & (a-6)x_3 & = & -3 \\ & & & & 0x_3 & = & 0 \end{array}$$

Thus the infinitely many solutions

$$x_1 = -2 + t, \quad x_2 = -3 + 2t, \quad x_3 = t, \quad t \in R.$$

CASE III. Consistent system (exactly one solution), if we take $a \neq 4$ and $b \in R$, gives

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ & & x_2 & + & (a-6)x_3 & = & -3 \\ & & & & (-2a+8)x_3 & = & (b-6) \end{array}$$

$x_1 = \dfrac{-20a - 7b + 122}{-2a+8}, \quad x_2 = \dfrac{6a + 2b - 18}{-2a+8}, \quad x_3 = \dfrac{b-6}{-2a+8}$, the required unique solution. •

Question 2: Find determinant of the coefficient matrix A of the following linear system using LU decomposition by Crout's method. If $\det(A) = |A| = -1$, then find the unique solution of the following system. [5 Marks]

solution. The Crout's method makes LU factorization a byproduct of Gaussian elimination. The process begins with the product matrices form

$$A = \mathbf{I}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & \alpha \end{pmatrix}.$$

$$A = \mathbf{I}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & \alpha - 1 \end{pmatrix}.$$

$$A = \mathbf{I}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & \alpha - 2 \end{pmatrix}.$$

$$A = \mathbf{I}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & \alpha - 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = LU.$$

$$\det(A) = \det(L)\det(U) = (\alpha - 2)(1) = \alpha - 2.$$

Given

$$\det(A) = -1 = (\alpha - 2), \quad \text{gives} \quad \alpha = 1.$$

Now solving the first system $L\mathbf{y} = \mathbf{b}$ for unknown vector $\mathbf{y}$, that is

$$L\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & \alpha - 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{b}.$$

Performing forward substitution yields

$$y_1 = 1, \qquad y_2 = 1, \qquad y_3 = 1.$$

Then solving the second system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$U\mathbf{x} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \mathbf{y}.$$

Performing backward substitution yields

$$x_1 = 0, \qquad x_2 = 2, \qquad x_3 = -1.$$

Thus $\mathbf{x}^* = [0, 2, -1]^T$ is the approximate solution of the given system. •

Question 3: Consider the following nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$, where [5 Marks]

$$A = \begin{pmatrix} 0 & -1 & 4 \\ 5 & 0 & -1 \\ -1 & 3 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}.$$

Rearrangement the given system such that the convergence of Gauss-Seidel iterative method is guaranteed. Then find upper bound for the error $\|x - x^{(5)}\|$ using Gauss-Seidel iterative method if $x^{(0)} = [0.3, 0.5, 1.0]^T$.

Solution. The rearrangement linear system $A\mathbf{x} = \mathbf{b}$,is

$$A = \begin{pmatrix} 5 & 0 & -1 \\ -1 & 3 & 0 \\ 0 & -1 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

The Gauss-Seidel iteration matrix T_G is defined as

$$T_G = -(D+L)^{-1}U = -\begin{pmatrix} 5 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & -1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and it gives

$$T_G = -\begin{pmatrix} \frac{1}{5} & 0 & 0 \\ \frac{1}{15} & \frac{1}{3} & 0 \\ \frac{1}{60} & \frac{1}{15} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{15} \\ 0 & 0 & \frac{1}{60} \end{pmatrix}.$$

Now to find the first approximation using Gauss-Seidel method, we will the following formula

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{5}\Big[1 \qquad\qquad + \; x_3^{(k)}\Big] \\ x_2^{(k+1)} &= \frac{1}{3}\Big[2 \; + \; x_1^{(k+1)} \qquad\Big] \\ x_3^{(k+1)} &= \frac{1}{4}\Big[4 \qquad\qquad + \; x_2^{(k+1)}\Big] \end{aligned}$$

Starting with initial approximation $x_1^{(0)} = 0.3, x_2^{(0)} = 0.5, x_3^{(0)} = 1$ and for $k = 0$, we obtain the first approximation as

$$\mathbf{x}^{(1)} = [0.4, 0.8, 1.2]^T, \qquad \|T_G\| = 0.2, \qquad \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| = 0.3.$$

Thus

$$\|\mathbf{x} - \mathbf{x}^{(5)}\| \leq \frac{(0.2)^5}{0.8}(0.3) = 1.2 \times 10^{-4},$$

the required an error bound. •

Question 4: Let $f(x) = \sqrt{x - x^2}$ and $p_2(x)$ be quadratic Lagrange interpolating polynomial on $x_0 = 0, x_1 = \alpha$, and $x_2 = 1$. Find the value of α in $(0, 1)$ for which [5 Marks]

$$f(0.5) - p_2(0.5) = -0.25.$$

Solution. Consider the quadratic Lagrange interpolating polynomial as follows:

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2).$$

At the given values of $x_0 = 0, x_1 = \alpha, x_2 = 1$, we have, $f(0) = 0, f(\alpha) = \sqrt{\alpha - \alpha^2}$ and $f(1) = 0$, gives

$$f(x) = p_2(x) = L_0(x)(0) + L_1(x)(f(x_1\sqrt{\alpha - \alpha^2})) + L_2(x)(0),$$

where

$$L_1(x) = \frac{(x-0)(x-1)}{(\alpha-0)(\alpha-1)} = \frac{x^2 - x}{\alpha^2 - \alpha}.$$

Thus

$$f(x) = p_2(x) = \frac{x^2 - x}{\alpha^2 - \alpha}\sqrt{\alpha - \alpha^2} \quad \text{and} \quad p_2(0.5) = \frac{-0.25}{\alpha^2 - \alpha}\sqrt{\alpha - \alpha^2}.$$

Given

$$f(0.5) - p_2(0.5) = -0.25, \quad \text{gives} \quad p_2(0.5) = f(0.5) + 0.25 = 0.5 + 0.25 = 0.75,$$

so

$$-0.25\frac{\sqrt{\alpha - \alpha^2}}{(\alpha - \alpha^2)} = 0.75, \quad \text{or} \quad \sqrt{\alpha - \alpha^2} = -3(\alpha - \alpha^2).$$

Thus, taking square on both sides, we get

$$\alpha - \alpha^2 = 9(\alpha - \alpha^2)^2, \quad \text{or} \quad (\alpha - \alpha^2)[1 - 9(\alpha - \alpha^2)] = 0,$$

which can be also written as

$$\alpha(1 - \alpha)[9\alpha^2 - 9\alpha + 1] = 0.$$

Solving this equation for α, we get

$$\alpha = 0, \quad \text{or} \quad \alpha = 1, \quad \text{or} \quad \alpha = 0.127322, \quad \text{or} \quad \alpha = 0.872678.$$

Thus $\alpha = 0.872678$, the required value in the given interval $(0, 1)$. •

Question 5: Construct the table for $(\alpha, Q(\alpha))$ by evaluating the integral

$$Q(\alpha) = \int_1^2 (\alpha - 1)\ dx,$$

at $\alpha = 1, 1.5, 2.5, 3.5, 4$. Then use the constructed table to find the best approximation of $Q(3.4)$ by using quadratic Lagrange polynomial. Compute the absolute error. [5 Marks]

Solution. Since

$$Q(\alpha) = \int_1^2 (\alpha - 1)\ dx = (\alpha x - x)\Big|_1^2 = \alpha - 1,$$

so by using the given values of α, we get

$$Q(1) = 0.0, \quad Q(1.5) = 0.5, \quad Q(2.5) = 1.5, \quad Q(3.5) = 2.5, \quad Q(4) = 3.0.$$

Thus we have the following table

α	1.00	1.5	2.5	3.5	4.0
$Q(\alpha)$	0.0	0.5	1.5	2.5	3.0

Since a quadratic polynomial can be determined so that it passes through the three points, let us consider the best form of the constructed table for the quadratic Lagrange interpolating polynomial to approximate $Q(3.4)$ as

α	2.5	3.5	4.0
$Q(\alpha)$	1.5	2.5	3.0

So using the quadratic Lagrange interpolating polynomial

$$Q(\alpha) = p_2(\alpha) = L_0(\alpha)f(\alpha_0) + L_1(\alpha)f(\alpha_1) + L_2(\alpha)f(\alpha_2),$$

to get the approximation of $Q(3.4)$, we have

$$Q(3.4) \approx p_2(3.4) = (1.5)L_0(3.4) + (2.5)L_1(3.4) + (3.0)L_2(3.4).$$

$$\begin{aligned}
L_0(4) &= \frac{(3.4-3.5)(3.4-4.0)}{(2.5-3.5)(2.5-4.0)} = \frac{1}{25}, \\
L_1(4) &= \frac{(3.4-2.5)(3.4-4.0)}{(3.5-2.5)(3.5-4.0)} = \frac{27}{25}, \\
L_2(4) &= \frac{(3.4-2.5)(3.4-3.5)}{(4.0-2.5)(4.0-3.5)} = -\frac{3}{25}.
\end{aligned}$$

$$Q(3.4) \approx p_2(3.4) = \frac{1}{25}(1.5) + \frac{27}{25}(2.5) - \frac{3}{25}(3.0) = 2.40,$$

which is the required approximation of $Q(3.4)$ by the quadratic interpolating polynomial. From the given integral we can obtained the exact value as follows

$$Q(3.4) = \int_1^2 (3.4 - 1)\ dx = (2.4x)\Big|_1^2 = 2.40,$$

so

$$|Q(3.4) - p_2(3.4)| = |2.40 - 2.40| = 0.0000,$$

is the required absolute error in our approximation. •