KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

MATH-244 (Linear Algebra); Final Exam; Semester 1 (1443 H)

Max. Marks: 40 **Max. Time: 3 hours**

Note: ***Attempt all the five questions!***

Question 1 [4+2+2 marks]:

a) Find adjoint of the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 2 \\ -2 & 3 & 6 \end{bmatrix}$ and then find A^{-1}.

b) Evaluate $det(det(A)\, B^2 A^{-1})$, where **A** and **B** are square matrices of order 3 with $det(\mathbf{A}) = \mathbf{3}$ and $det(\mathbf{B}) = \mathbf{2}$.

c) Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 6 & 3 \\ 0 & 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ -1 & 0 & 8 \end{bmatrix}$. Explain why the matrices A and B are not row equivalent to each other?

Question 2 [5+3 marks]:

a) Find the values of α and β such that the following linear system:

$$\begin{aligned} x - 2y + 3z &= 4 \\ 2x - 3y + \alpha z &= 5 \\ 3x - 4y + 5z &= \beta \end{aligned}$$

has:

i) No solution;

ii) Infinitely many solutions.

b) Let $s_1 = 3 - 2x$, $s_2 = 2 + x$, $s_3 = 1 + x - x^2$, $s_4 = x + x^2 - x^3$. Find the values of a, b, c and d such that $1 - 6x - 3x^2 - 4x^3 = as_1 + bs_2 + cs_3 + ds_4$.

Question 3 [4+4 marks]:

a) Let $F = span\{u_1 = (1,1,1,1), u_2 = (0,1,2,1), u_3 = (1,0,-2,3),\ u_4 = (1,1,2,-2)\}$ in the Euclidean space $\mathbb{R}^4$. Then:

i) Find $dim(F)$

ii) Show that $(1,1,0,1) \notin F$.

b) Let $B = \{v_1 = (1,1,2), v_2 = (3,2,1), v_3 = (2,1,5)\}$ and $C = \{u_1, u_2, u_3\}$ be two bases for $\mathbb{R}^3$ such that

$${}_BP_C = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

is the transition matrix from C to B. Find the vectors u_1, u_2 and u_3.

Question 4 [4+2+2 marks]:

a) Let $w_1 = (0,0,1)$, $w_2 = (0,1,1)$, $w_3 = (1,1,1)$ be vectors in the Euclidean space $\mathbb{R}^3$. Then:

   i) Find the angle between w_1 and w_3.

   ii) By applying the Gram-Schmidt process on $\{w_1, w_2, w_3\}$ to find an orthonormal basis of the Euclidean space $\mathbb{R}^3$.

b) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x, y) = (x + 4y, 2x + 3y)$. Find:

   i) $Ker(T)$ ii) $dim\ \mathrm{Im}(T)$

c) Let the linear transformation $\mathrm{T} : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$\mathrm{T}(x, y) = (x + 2y, x - y, 3x + y).$$

Find matrix of the transformation $[\mathrm{T}]_{\mathrm{B}}^{\mathrm{C}}$, where B and C are the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Question 5 [4 + 4 marks]:

a) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find eigenvalue/s of the matrix A and determine one basis of the corresponding eigenspace/s. Then, give reason for the non-diagonalizability of A.

b) Show that the matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$ diagonalizes the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$ and then use this fact to compute A^{-1}.

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Math-244 (Linear Algebra); Mid-term Exam; Semester 2 (1442)
Max. Marks: 30 **Time: 2 hours**

Note: ***Attempt all the five questions***!

Question 1: [Marks: 3+3]

a) **Let A** $= \begin{bmatrix} -1 & 1 & 3 & 0 \\ 1 & 2 & 3 & -2 \\ 0 & -1 & -2 & 7 \\ 2 & 1 & 0 & 6 \end{bmatrix}$. Then:

i) **Find** the **reduced row echelon form** of the **matrix A**.
ii) Use the reduced row echelon form to **show** that the **matrix A** is **not invertible**.

b) **Let** $\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$. **Find** the value of λ such that $\mathbf{X}^8 - 4\lambda\mathbf{I} = \mathbf{O}$.

Question 2: [Marks: 3+3]

a) **Let** $X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. **Find** the **matrix Y** such that $(2X + Y)^{-1} = adj(X)$.

b) **Let A** $= \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$ and **B** $= \begin{bmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{bmatrix}$. **Show** that $det(B) = -det(A)$.

Question 3: [Marks: 3+3].

a) **Find** the value/s of α such that the following **linear system**

$$\begin{aligned} x + y + \tfrac{\alpha}{3}z &= 1 \\ x + y + z &= 1 \\ x + \alpha y + z &= 2 \end{aligned}$$

has: **(i) no solution** **(ii) unique solution** **(iii) infinitely many solutions.**

b) **Solve** the following **homogeneous linear system**. **Why** this system cannot be solved by Cramer's Rule?

$$\begin{aligned} x - 2y + 3z &= 0 \\ 3x + y - 2z &= 0 \\ 2x - 4y + 6z &= 0. \end{aligned}$$

Question 4: [Marks: 3+3]

a) **Show** that $\{1 - x,\ 1 - x^2,\ 1 + x + x^2\}$ is a **basis** of the **vector space** P_2 of all polynomials in real variable x with **degree** ≤ 2.

b) **Let** $S = \{(1,0,1,1), (1,-1,2,1), (1,-2,3,1)\}$ **generates** the **vector subspace** F of the Euclidean space $\mathbb{R}^4$. **Find** a **basis** of F contained in S and **show** that $(0,-2,7,6) \notin F$.

Question 5: [Marks: 3+3]

a) **Let** B = {(2, 1), (1, 0)} and C = {(1, -2), (0, 1)} be **bases** of the Euclidean space $\mathbb{R}^2$ and $v = (1,2)$. **Find** the **coordinate vector** $[v]_B$ and the **transition matrix** ${}_C P_B$. Then use the transition matrix to **find** $[v]_C$.

b) **Let** $A = \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 & 1 \\ -2 & -2 & 0 & 1 & 4 \end{bmatrix}$. **Find:**

(i) a **basis** of *col* (A) (ii) *rank* (A) (iii) *nullity* (A).

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Max. Marks: 30 **Time: 2 hours**

Note: ***Attempt all the five questions***!

Question 1: [Marks: 2+3]

a) **Let A** = $\begin{bmatrix} 1 & -1 & 4 & 5 \\ -2 & 1 & -11 & -8 \\ -1 & 2 & 2 & 2 \end{bmatrix}$ and **B** = $\begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 8 & 6 \end{bmatrix}$. Then show that the matrices **A** and **B** are row equivalent to each other.

b) Give any two matrices **A** and **B** that satisfy:

$trace\,(\mathbf{A}+\mathbf{B}) = trace\,(\mathbf{A}) + trace\,(\mathbf{B})$ and $trace\,(\mathbf{AB}) = trace\,(\mathbf{A})\, trace\,(\mathbf{B})$.

Question 2: [Marks: 2+3]

a) Let **A**, **B** $\in \mathbf{M_2}(\mathbb{R})$ with $|\mathbf{A}| = 3$ and $|\mathbf{B}| = 6$. Then evaluate $\left|\,|\mathbf{A}|A^tB^2\,\boldsymbol{adj}(A^2)\right|$.

b) Let **A** = $\begin{bmatrix} 1 & 0 & \delta \\ 2 & 1 & 2+\delta \\ 2 & 3 & \delta^2 \end{bmatrix}$. Find the values of δ if the matrix **A** is not invertible.

Question 3: [Marks: 2+4]

a) Find the values of x and y if **A** = $\begin{bmatrix} - & 2 & - \\ - & x & - \\ - & y & - \end{bmatrix}$ and $\mathbf{A}^{-1}$ = $\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ - & - & - \end{bmatrix}$.

b) Find the value/s of α such that the following linear system:

$$\begin{aligned} x + 2y - z &= 2 \\ x - 2y + 3z &= 1 \\ x + 2y - (\alpha^2 - 3)z &= \alpha \end{aligned}$$

has:

(i) no solution (ii) unique solution (iii) infinitely many solutions.

Question 4: [Marks: 2+3+3]

a) Let $\mathbf{S} = \{(1,1,1,0), (1,2,3,1), (2,0,1,1)\}$ generates the subspace **F** of Euclidean space $\mathbb{R}^4$. Show that $(1, 1, 1, 1) \notin \mathbf{F}$.

b) Let **B** $=\{(1,0,0), (0,1,0), ((0,0,1)\}$ and **C** $=\{(1,1,1), (1,2,2), (1,1,2)\}$ be bases of the Euclidean space $\mathbb{R}^3$ and $[v]_B = [1 \;\; 2 \;\; 3]^T$. Find the transition matrix ${}_C P_B$ and $[v]_C$.

c) Let $\mathbf{A}^T = \begin{bmatrix} 1 & 2 & 2 & 0 & 0 \\ 2 & 4 & 5 & 2 & 2 \\ 1 & 2 & 3 & 2 & 2 \\ 3 & 6 & 4 & -3 & -4 \end{bmatrix}$. Then find:

(i) a basis of *col*(**A**) (ii) *rank*(**A**) (iii) *nullity*(**A**).

Question 5: [Marks: 2+1+3]

Let $\mathbf{S} = \{v_1 = (1,-1,0,1), v_2 = (1,1,1,0), v_3 = (0,1,1,1)\}$ generates the subspace **W** of the Euclidean space $\mathbb{R}^4$. Then:

a) Show that **S** is a basis of **W**.

b) Find the angle θ between the vectors v_1 and v_2.

c) Apply the Gram-Schmidt process on **S** to obtain an orthonormal basis of **W**.

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Max. Marks: 40 **Max. Time: 3 hours**

Note: ***Attempt all the five questions!***

Question 1 [3+2+3 marks]:

a) If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 3 \end{bmatrix}$, then find A^{-1}.

b) Evaluate $\det(\det(\det(\det(A)\, A^2)\, A)\, A^{-1})$, where **A** is a square matrix of order 3 with $\det(A) = 3$.

c) Let $\begin{bmatrix} 1 & 0 & 2 & 0 & b_1 \\ 0 & 1 & 5 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \end{bmatrix}$ be reduced row echelon form of the augmented matrix of linear system $AX = B$. Explain! Why this system has a solution for any $B \in \mathbb{R}^3$?

Question 2 [5+3 marks]:

a) Find the values of α such that the following linear system:

$$\begin{aligned} x + y + z &= 0 \\ x + \alpha y + z &= 1 \\ x + y + (\alpha-2)^2 z &= 0 \end{aligned}$$

has:

i) No solution;
ii) Unique solution;
iii) Infinitely many solutions.

b) Let $v_1 = (1, 2, 0, 3, -1)$, $v_2 = (2, 4, 3, 0, 7)$, $v_3 = (1, 2, 2, 0, 9)$, $v_4 = (-2, -4, -2, -2, -3)$. **Find** a basis of the Euclidean space $\mathbb{R}^5$ which includes the vectors v_1, v_2, v_3, v_4.

Question 3 [2+3+3 marks]:

a) Let $\{x, y\}$ be linearly independent set of vectors in vector space **V**. Determine whether the set $\{2x,\ x + y\}$ is linearly independent or not?

b) Suppose G is a subspace of the Euclidean space $\mathbb{R}^{15}$ of dimension 3, $S = \{u, v, w\}$ and **Q** are two bases of the space G and ${}_Q P_S = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ be the transition matrix from the basis **S** to the basis **Q**. Find $[g]_Q$ where $g = 3v - 5u + 7w$.

c) Let P_2 be the vector space of polynomials of degree ≤ 2 with the inner product: $< p, q > = aa_1 + 2bb_1 + cc_1$ for all $p = a + bx + cx^2$, $q = a_1 + b_1 x + c_1 x^2 \in P_2$. Find $\cos\theta$, where θ is the angle between the polynomials $1 + x + x^2$ and $1 - x + 2x^2$.

Question 4 [3+1+4 marks]:

a) Find an orthonormal basis for the subspace **F**= **span(A)** of Euclidean space $\mathbb{R}^4$, where **A** = $\{x_1 = (1, 2, 3, 0),\ x_2 = (1, 2, 0, 0),\ x_3 = (1, 0, 0, 1)\}$.

b) Let **S, T**: $\mathbb{R}^n \to \mathbb{R}^n$ be the linear transformations such that:

$$\mathbf{S}(u) = \mathbf{T}(u),\ \mathbf{S}(v) = \mathbf{T}(v) \text{ and } \mathbf{S}(w) = \mathbf{T}(w).$$

Show that $\mathbf{S}(x) = \mathbf{T}(x)$ for all $x \in \mathbf{span}(\{u, v, w\})$.

c) Let the linear transformation $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$\mathrm{T}(x, y) = (x + 2y, x - y, 3x + y)$$

for all $v = (x, y) \in \mathbb{R}^2$. Find $[\mathrm{T}]_{\mathrm{B}}^{\mathrm{C}}$, $[v]_{\mathrm{B}}$ and $[\mathrm{T}(v)]_{\mathrm{C}}$, where **B** = {(1,-2), (2,3)} and **C** = {(1,1,1), (2,1,-1), (3,1,2)} are bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Question 5 [2× 4 marks]:

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ -3 & 0 & -2 \end{bmatrix}$. Then:

i) Show that 1 and -1 are the eigenvalues of A and find their algebraic and geometric multiplicities.

ii) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

iii) Show that A^{-1} exists and it is also diagonalizable.

iv) Compute the matrix A^{2020}.

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