M.S. THESIS ABSTRACT

Modern potential theory is about the study of functions with a mean value property on a topological space , particularly a locally compact space.

For example , the convex functions on the real line , the analytic functions on the complex plane , the solutions of the Laplacian in the open sets of $\Re^n$ and the Newotonian potential in $\Re^3$ all have a certain mean-value property which plays an important role in characterizing such functions.

Some of the important results like the absolute minimum of convex functions, the maximum principle of analytic functions , the convergence property of a sequence of Harmonic functions, and the uniqueness of Dirichlet solution in a bounded domain are derived as immediate consequence of the mean-value property.

To these examples , we can add another exciting one from the theory of Probability. Actually , the theory of Martingales and the theory of Brownian Motion are developed based on notions which can be expressed as mean-value properties of fundamental measurable functions.

In a fascinating work in 1944 Kakutani investigated the connection between classical potential theory and Brownian Motion in $\Re^2$.

Later Doob studied the martingales on the lines of Potential Theory , using a mean-value property as the basis for the study of martingales . Thus developing of the martingales theory. Doob showed how the Brownian Motion and the Martingales could be used to obtain some of the basis results of Potential theory.

This rekindled the interest in potential theory leading to an important discovery that some the basis results of potential theory could be proved by probabilistic methods.
This is turn led to the application of potential theoretic techniques in the study of probability theory.

This interplay of potential theory and probability theory is the subject matter of this thesis.

In chapter (0) ,we collect some definitions and theorems from potential theory and probability theory.

In chapter (1) , we give an examples of SuperHarmonic functions and Potentials and study their properties.

In chapter (2) , we give an examples of SuperMartingales and SuperPotentials and study their properties.

In chapter (3) , the connection between potential theory and probability theory will appear by using the concept of Capacity and Harmonic Measuers.

Definition of superharmonic function:

A lower semi-continuous function u defined on an open set W in $\Re^n$ ($n \geq 2$) is called superharmonic if the following conditions are satisfied:

(i) $\mathbf{u} \not\equiv \infty$

(ii) $-\infty < \mathbf{u(x)} \leq \infty \quad \forall \mathbf{x} \varepsilon \mathbf{W}$.

(iii) $\mathbf{u(x)} \geq \dfrac{1}{\sigma_n \delta^{n-1}} \displaystyle\int_{\partial B} \mathbf{u}\, d\sigma$, where σ the surface area on the boundary of the ball $B_{x,\delta}$, and $\sigma_n = \begin{cases} \dfrac{\pi^{\frac{n}{2}} . n}{\left(\frac{n}{2}\right)!} & \text{if n is even} \\ \dfrac{2^{\frac{n+1}{2}} . \pi^{\frac{n-1}{2}}}{1.3 \ldots (n-2)} & \text{if n is odd}, > 1 \end{cases}$

u is called subharmonic if -u is superharmonic

u is called harmonic if u and -u are superharmonic.

The probability counterparts of the superharmonic functions are the supermartingale:

Definition of supermartingale:

The real-valued process $(X(t))$ relative to a family of an increasing right continuous sub-σ-algebras $\{\Im(t),\ t \geq 0\}$ supermartingale if the following conditions are satisfied:

(i) For each t, $(X(t))$ is integrable and $(\Im(t))$-measurable.

(ii) For $t > s$, $E(X(t) \mid \Im(s)) \leq X(s)$ a.s.

$(X(t))$ is called submartingale if $E(X(t) \mid \Im(s)) \geq X(s)$ a.s.

$(X(t))$ is called martingale if $(X(t))$ and $(-X(t))$ are supermartingales.

An interesting connection between superharmonic functions and supermartingales is the following fact:

Let u be a positive superharmonic function, and let B(t) a Brownian Motion (a special kind of supermartinglae). Then u[B(t)] is a supermartingale.

The inequalities in the definitions of two types of functions suggest that there are similarities between them in the examples and properties.

The similarity between the two theorems

	Potential Theory Context		Probability Theory Context
(1)	$-u$ is superharmonic on W, ϕ is increasing convex function. Then $-\phi(u)$ is superharmonic.	(1)	$-X(t)$ is supermartingale. ϕ is increasing convex function. Then $-\phi(X(t))$ is supermartingale, provided that $-\phi(X(t))$ is measurable with respect to $(\Im(t))$.
(2)	u is harmonic on W, ϕ is convex. Then $-\phi(u)$ is superharmonic on W.	(2)	$X(t)$ is martingale ϕ is convex. Then $-\phi(X(t))$ is supermartingale, provided that $-\phi(X(t))$ is measurable w.r.t. $(\Im(t))$.
(3)	(u_n) decreasing seqauence of superharmonic functions on W which is locally bounded on W, $u = \lim u_n$. Then $u^* = \lim\inf u$ is superharmonic on W, and $u^*=u$ except on a polar set.	(3)	$X(t) > 0$, supermartingale, $\Im(t)$ is right-continuous and $\Im(0)$ contain the null sets. Then $X^*(t) = \liminf_{s \to t} X(s)$ is right-continuous supermartingale, and $X=X^*$ except on a polar set.

The Similarity in the Properties

	Potential Theory Context		Probability Theory Context
(1)	(u_n) increasing seq. of superharmonic on W. Then $u=\sup u_n$ is either superharmonic or $u\equiv\infty$.	(1)	$X(t) \geq 0$ $\forall t$, right continuous supermartingale $\sup_t E(\overline{X}(t)) < \infty$. Then $(X(t), \Im(t), 0 \leq t \leq \infty)$ is supermartingale where $\Im(\infty) = \bigcup_t \Im(t)$.
(2)	u are v are superharmonic on W and $a, b \geq 0$. Then $au + bv$ and $\min(u,v)$ are superharmonic.	(2)	$X(t)$, $Y(t)$ are supermartingales and $a, b \geq 0$. Then $aX(t)+bY(t)$ and $\min(X(t), Y(t)$ are supermartingale.

Definition of Potential:

u is superharmonic function on $W \subseteq \Re^n$, $u > 0$. u is potential if

$\sup\{v: v \text{ subharmonic on } W, v \leq u\} = 0$

A special kind of supermartingales which corresponds in definition and properties to potential is: supermartingale Potential.

Definition of Supermartingale Potential:

$(X(t))$ is supermartingale, $X(t) > 0$. $X(t)$ is supermartingale potential if $\sup\{Y{:}Y \text{ is submartingale}, Y(t) \leq X(t) \text{ a.e. } \forall t\} = 0$.

The Similarity in the Properties

Potential Theory Context	Probability Theory Context
(1) u_1, u_2, are potentials and a, b>0. Then $au_1 + bu_2$ is potential.	(1) X(t), Y(t) are supermartingale potentials and a,b > 0. Then aX(t) + bY(t) is supermartingale potential.
(2) u superharmonic on W, $u \geq 0$ and P is potential such that $u \leq P$. Then u is potential	(2) X(t) is supermartingale, $X(t) \to 0$ and Y(t) is supermartingale potential such that $X(t) \leq Y(t)$. Then X(t) is supermartinglae potential.
(3) Riesz Decomposition Theorem: u is superharmonic, $u > 0$. u is represented uniquely in the form u = P+h where P potential, h harmonic	(3) Doob Decomposition Theorem: X(t) is supermartingale. X(t) is represented uniquely in the form X(t)=Y(t) + Z(t) where Y(t) supermartingale potential, Z(t) martingale.
(4) $\{P_n\}$ is a sequence of potentials in $\Re^n$. Then $\sum_n P_n \equiv \infty$ or potential.	(4) $(X(t,s);\ s \geq 0)$ family of supermartingale potential. Then $\sup_t X(t,s)$ is supermartingale potential.