King Saud University: **Math. Dept.** **M-254** **Semester I** **1437-1438 H**
Second Midterm Exam **Time: 90 mins.** **Max. Marks: 25**

Question 1: (5 + 5 + 5 + 5 + 5)

(Q1:) Solve the following nonlinear system

$$\begin{aligned} 0.1x^2 + 0.1y^2 &= -0.8 \\ 0.1x + 0.1xy &= -0.8 \end{aligned}$$

using Newton's method, starting with $(x_0, y_0)^T = (0.5, 0.5)^T$, find $(x_1, y_1)^T$.

(Q2:) Solve the following linear system using the Gaussian elimination with the partial pivoting.

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & x_3 & = & 4 \\ 5x_1 & - & 2x_2 & - & x_3 & = & -2 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 9 \end{array}$$

(Q3:) Use LU-factorization method with Doolittle's method ($l_{ii} = 1$) to find values of α for which the following linear system has unique solution and infinitely many solutions. Write down the solution for both cases.

$$\begin{array}{rcrcrcr} x_1 & + & 0.5x_2 & + & \alpha x_3 & = & 0.5 \\ 2x_1 & - & 3x_2 & + & x_3 & = & -1 \\ -x_1 & - & 1.5x_2 & + & 2.5x_3 & = & -1 \end{array}$$

(Q4:) Rearrange the given linear system such that the convergence of the Jacobi method is guaranteed.

$$\begin{array}{rcrcrcr} x_1 & + & 6x_2 & - & 3x_3 & = & 4 \\ 2x_1 & + & 2x_2 & + & 6x_3 & = & 7 \\ 5x_1 & + & 2x_2 & - & x_3 & = & 6 \end{array}$$

Use the Jacobi method to find the first two iterations, using the initial approximation $x^{(0)} = [0, 0, 0]^T$. Compute an error bound for the approximation.

(Q5:) Find the condition number of the following matrix (for $n = 2, 3, \ldots$)

$$A_n = \begin{bmatrix} 1 & 1 \\ 1 & (1 - 1/n) \end{bmatrix}.$$

If $n = 2$ and $x^* = [-1.99, 2.99]^T$ be the approximate solution of the linear system $A\mathbf{x} = [1, -0.5]^T$, then find the relative error.

Solution of second Midterm Exam

Q1)

$f_1(x,y) = 0.1x^2 + 0.1y^2 + 0.8 \quad ; \quad \frac{\partial f_1}{\partial x} = 0.2x \quad ; \quad \frac{\partial f_1}{\partial y} = 0.2y$

$f_2(x,y) = 0.1x + 0.1xy + 0.8 \quad ; \quad \frac{\partial f_2}{\partial x} = 0.1 + 0.1y \quad ; \quad \frac{\partial f_2}{\partial y} = 0.1x$

$f_1(0.5, 0.5) = \frac{17}{20} \qquad \frac{\partial f_1}{\partial x}(0.5,0.5) = \frac{1}{10} \qquad \frac{\partial f_1}{\partial y}(0.5,0.5) = \frac{1}{10}$

$f_2(0.5,0.5) = \frac{7}{8} \qquad \frac{\partial f_2}{\partial x}(0.5,0.5) = \frac{3}{20} \qquad \frac{\partial f_2}{\partial y}(0.5,0.5) = \frac{1}{20}$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\Bigg|_{(0.5,0.5)} = \begin{pmatrix} \frac{1}{10} & \frac{1}{10} \\ \frac{3}{20} & \frac{1}{20} \end{pmatrix}$$

$$J^{-1} = \frac{1}{-\frac{1}{100}} \begin{pmatrix} \frac{1}{20} & -\frac{1}{10} \\ -\frac{3}{20} & \frac{1}{10} \end{pmatrix} = \begin{pmatrix} -5 & 10 \\ 15 & -10 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} + \begin{pmatrix} -5 & 10 \\ 15 & -10 \end{pmatrix} \begin{pmatrix} 17/20 \\ 7/8 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 9/2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 4.5 \end{pmatrix}$$

Q2)

$E_1 \leftrightarrow E_2$
$$\begin{cases} 5x_1 - 2x_2 - x_3 = -2 \\ x_1 + 3x_2 - x_3 = 4 \\ 2x_1 + 2x_2 + x_3 = 9 \end{cases} \qquad \begin{matrix} E_2 - \frac{1}{5}E_1 \\ E_3 - \frac{2}{5}E_1 \end{matrix} \qquad \begin{cases} 5x_1 - 2x_2 - x_3 = -2 \\ \frac{17}{5}x_2 - \frac{4}{5}x_3 = \frac{22}{5} \\ \frac{14}{5}x_2 + \frac{7}{5}x_3 = \frac{49}{5} \end{cases}$$

$$E_3 - \frac{14}{17}E_2 \qquad \begin{cases} 5x_1 - 2x_2 - x_3 = -2 \\ \frac{17}{5}x_2 - \frac{4}{5}x_3 = \frac{22}{5} \\ \frac{35}{17}x_3 = \frac{105}{17} \end{cases} \rightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

Q3)

$$\begin{pmatrix} 1 & 0.5 & \alpha \\ 2 & -3 & 1 \\ -1 & -1.5 & 2.5 \end{pmatrix} \qquad \begin{matrix} m_{21} = 2 \\ m_{31} = -1 \end{matrix} \qquad \begin{pmatrix} 1 & 0.5 & \alpha \\ 0 & -4 & 1-2\alpha \\ 0 & -1 & 2.5+\alpha \end{pmatrix}$$

$$m_{32} = \frac{1}{4} \qquad \begin{pmatrix} 1 & 0.5 & \alpha \\ 0 & -4 & 1-2\alpha \\ 0 & 0 & 2.25 + \frac{3}{2}\alpha \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & \frac{1}{4} & 1 \end{pmatrix}$$

2

$$Ax = b = \begin{pmatrix} 0.5 \\ -1 \\ -1 \end{pmatrix} \iff L.Ux = b \iff \begin{matrix} L.y = b \\ U.x = y \end{matrix}$$

$$L.y = b \to y = \begin{pmatrix} 0.5 \\ -2 \\ 0 \end{pmatrix}$$

$$U.x = y \to \iff \begin{pmatrix} 1 & 0.5 & \alpha \\ 0 & -4 & 1-2\alpha \\ 0 & 0 & 1.5\alpha + 2.25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -2 \\ 0 \end{pmatrix}$$

From 3rd equation $(1.5\alpha + 2.25)x_3 = 0$

if $1.5\alpha + 2.25 \neq 0 \iff \alpha \neq \frac{2.25}{1.5} = 1.5$

$\to x_3 = 0$, $x_2 = 0.5$ and $x_1 = 0.25$

if $1.5\alpha + 2.25 = 0 \iff \alpha = 1.5$ we have infinity many solution

we choose $x_3 = m \in \mathbb{R}$, then $x_2 = 0.5 + m$ and $x_1 = 0.25 + m$

Q4) To have convergence, the system must be S.D.D, by rearranging the given system we obtain

$$5x_1 + 2x_2 - x_3 = 6$$
$$x_1 + 6x_2 - 3x_3 = 4$$
$$2x_1 + 2x_2 + 6x_3 = 7$$

Jacobi Method:

$$x_1^{(k+1)} = \frac{1}{5}\left[6 - 2x_2^{(k)} + x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{6}\left[4 - x_1^{(k)} + 3x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{6}\left[7 - 2x_1^{(k)} - 2x_2^{(k)}\right]$$

starting with $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

we obtain $x^{(1)} = \begin{pmatrix} 1.2 \\ 0.467 \\ 1.033 \end{pmatrix}$, $x^{(2)} = \begin{pmatrix} 1.22 \\ 0.98 \\ 0.895 \end{pmatrix}$

Error bound : $\|x^{(2)} - x\| \leq \dfrac{\|T_J\|^2}{1 - \|T_J\|} \|x^{(1)} - x^{(0)}\|$

$$T_J = -D^{-1}(L+U) = \begin{pmatrix} 0 & -\frac{2}{5} & \frac{1}{5} \\ -\frac{1}{6} & 0 & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & 0 \end{pmatrix}$$

$$\|T_J\|_\infty = \max\left(\frac{3}{5}, \frac{4}{6}, \frac{2}{3}\right) = \frac{2}{3}$$

$$\|x^{(1)} - x^{(0)}\| = \left\| \begin{pmatrix} 1.2 \\ 0.467 \\ 1.033 \end{pmatrix} \right\| = 1.2$$

$$\rightarrow \|x^{(2)} - x\| \leq \frac{(2/3)^2}{1 - 2/3} \, 1.2 = \frac{8}{5}$$

Q5)

$$A_n = \begin{pmatrix} 1 & 1 \\ 1 & 1 - \frac{1}{n} \end{pmatrix} \longrightarrow \|A_n\|_\infty = 2$$

$$A_n^{-1} = \begin{pmatrix} 1-n & n \\ n & -n \end{pmatrix} \longrightarrow \|A_n\|_\infty = 2n$$

$$\kappa(A) = 4n$$

$$r = b - A_2 x^* = \begin{pmatrix} 0 \\ -0.005 \end{pmatrix} \longrightarrow \|r\|_\infty = 0.005$$

$$\frac{\|x - x^*\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|b\|} = 8 \cdot \frac{0.005}{1} = 0.04$$